Generalizations of generating functions for orthogonal polynomials in the Askey and q-Askey schemes

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Generalized hypergeometric orthogonal polynomials

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Basic hypergeometric orthogonal polynomials

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Orthogonal polynomials

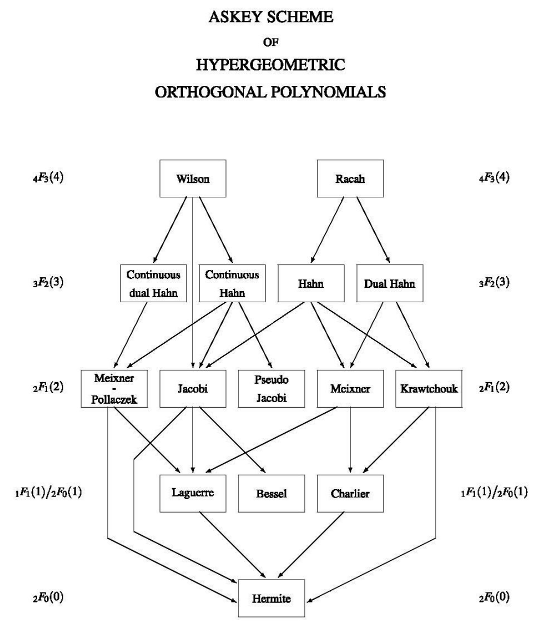
- Gabor Szegő, "Orthogonal Polynomials" (1959)
- A **polynomial** is an expression of **finite length** constructed from **variables** and **constants**, i.e.,

$$P_n^{(\alpha)}(z) = c_0(\alpha) + c_1(\alpha)z + c_2(\alpha)z^2 + c_3(\alpha)z^3 + \cdots + c_n(\alpha)z^n$$

- An **orthogonal set of polynomials** $\{P_n^{(\alpha)}(x)\}$ with support on (a, b), satisfies the following relations

$$\int_a^b P_m^{(\alpha)}(x)P_n^{(\alpha)}(x)w(x;\alpha)dx = d_n(\alpha)\delta_{m,n}.$$

- **Askey scheme** – a way of organizing **orthogonal polynomials** of **hypergeometric type** into a **hierarchy**



ASKEY SCHEME

OF

HYPERGEOMETRIC

ORTHOGONAL POLYNOMIALS

Classical continuous orthogonal polynomials

Orthog. polynomial	**Symbol**	$w(x)$	(a,b)
Wilson	W_n	$\left\lvert\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right\rvert^2$	$(0,\infty)$
Cts. dual Hahn	S_n	$\left\lvert\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right\rvert^2$	$(0,\infty)$
Continuous Hahn	p_n	$\lvert\Gamma(a+ix)\Gamma(b+ix)\rvert^2$	$(-\infty,\infty)$
Meixner-Pollaczek	$P_n^{(\lambda)}$	$e^{(2\phi-\pi)x}\lvert\Gamma(\lambda+ix)\rvert^2$	$(-\infty,\infty)$
Jacobi	$P_n^{(\alpha,\beta)}$	$(1-x)^\alpha(1+x)^\beta$	$(-1,1)$
Laguerre	L_n	$e^{-x}x^\alpha$	$(0,\infty)$
Hermite	H_n	e^{-x^2}	$(-\infty,\infty)$

Factorials and generalized hypergeometric series

Euler's gamma function and factorial for non-negative integers

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re}\, z > 0, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\Gamma(n+1) = n!, \quad n \in \mathbf{N}_0 = 0, 1, 2, 3, \ldots.$$

Pochhammer symbol: the rising factorial in the complex plane

$$(a)_n := (a)(a+1)\ldots(a+n-1), \quad (a)_0 := 1, \quad a \in \mathbf{C}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \notin -\mathbf{N}_0 = 0, -1, -2, \ldots$$

Generalized hypergeometric series

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!}$$

Example: Definition of the (continuous) Wilson polynomials

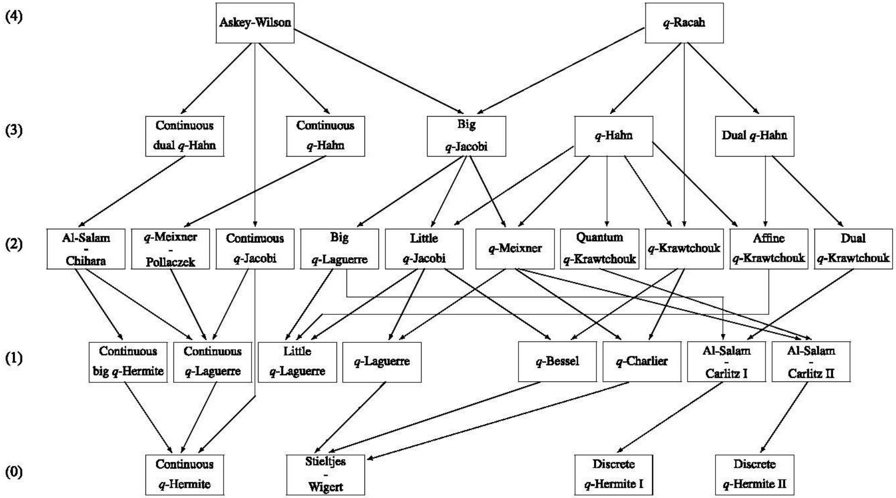
No symmetry in the **parameters**

$$W_n(x^2;a,b,c,d)$$
$$=(a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix};1\right)$$

Symmetry in the **parameters**

$$W_n(x^2;a,b,c,d)=\frac{(a-ix)_n(b-ix)_n(c-ix)_n(d-ix)_n}{(-2ix)_n}$$
$$\times\,{}_7F_6\left(\begin{matrix}-n,2ix-n,ix-\frac{1}{2}n+1,a+ix,b+ix,c+ix,d+ix\\ ix-\frac{1}{2}n,1-n-a+ix,1-n-b+ix,1-n-c+ix,1-n-d+ix\end{matrix};1\right)$$

SCHEME
OF
BASIC HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS



(4) Askey-Wilson; q-Racah

(3) Continuous dual q-Hahn; Continuous q-Hahn; Big q-Jacobi; q-Hahn; Dual q-Hahn

(2) Al-Salam - Chihara; q-Meixner - Pollaczek; Continuous q-Jacobi; Big q-Laguerre; Little q-Jacobi; q-Meixner; Quantum q-Krawtchouk; q-Krawtchouk; Affine q-Krawtchouk; Dual q-Krawtchouk

(1) Continuous big q-Hermite; Continuous q-Laguerre; Little q-Laguerre; q-Laguerre; q-Bessel; q-Charlier; Al-Salam - Carlitz I; Al-Salam - Carlitz II

(0) Continuous q-Hermite; Stieltjes - Wigert; Discrete q-Hermite I; Discrete q-Hermite II

q-calculus

q-Pochhammer symbol

$$(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1}), \quad (a;q)_0 := 1$$

Notation

$$(a_1,\dots,a_r;q)_n := (a_1;q)_n \dots (a_r;q)_n$$

Basic hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix};q,z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s,q;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n$$

$${}_{s+1}\phi_s\left(\begin{matrix} a_1,\dots,a_{s+1} \\ b_1,\dots,b_s \end{matrix};q,z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{s+1};q)_n}{(b_1,\dots,b_s;q)_n} \frac{z^n}{(q;q)_n}$$

Connection relations and coefficients

$$P_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_{n,k}(\alpha;\beta) P_k^{(\beta)}(x)$$

What are the $c_{n,k}$? This is a **problem** in **orthogonal polynomials.** In general, one can compute connection relations by using **orthogonality**

$$\int_a^b P_k^{(\alpha)}(x) P_{k'}^{(\alpha)}(x) w(x;\alpha) dx = d_k(\alpha)\delta_{k,k'}.$$

Therefore

$$c_{n,k}(\alpha,\beta) = \frac{1}{d_k(\beta)} \int_a^b P_n^{(\alpha)}(x) P_k^{(\beta)}(x) w(x;\beta) dx.$$

Generating functions

$$f(x,\rho;\alpha)=\sum_{n=0}^{\infty}c_n(\alpha)\rho^n P_n^{(\alpha)}(x)$$

Examples:

- **Hermite polynomials**

$$\exp(2x\rho-\rho^2)=\sum_{n=0}^{\infty}\frac{1}{n!}\rho^n H_n(x)$$

- **Gegenbauer polynomials**

$$\frac{1}{(1+\rho^2-2\rho x)^{\nu}}=\sum_{n=0}^{\infty}\rho^n C_n^{\nu}(x)$$

- **Jacobi polynomials**

$$2^{\alpha+\beta}\mathsf{R}^{-1}(1-\rho+\mathsf{R})^{-\alpha}(1+\rho+\mathsf{R})^{-\beta}=\sum_{n=0}^{\infty}\rho^n P_n^{(\alpha,\beta)}(x),$$

where $\mathsf{R}=\sqrt{1+\rho^2-2\rho x}$.

Ex: Laguerre polynomial connection rel. via generating fn.

Definition (orthogonal, monic):

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(n+\alpha)_{n-k}}{(n-k)!} \frac{(-x)^k}{k!}$$

Generating function (*Mourad's* trick)

$$(1-\rho)^{-\alpha-1} \exp\left(\frac{x\rho}{\rho-1}\right) = \sum_{n=0}^{\infty} \rho^n L_n^{(\alpha)}(x)$$

$$\frac{(1-\rho)^{-\alpha-1}}{(1-\rho)^{-\beta-1}} (1-\rho)^{-\beta-1} \exp\left(\frac{x\rho}{\rho-1}\right) = (1-\rho)^{\beta-\alpha} \sum_{n=0}^{\infty} \rho^n L_n^{(\beta)}(x)$$

$$(1-\rho)^{-r} = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} x^{r-k} y^k \quad \Longrightarrow \quad (1-\rho)^{\beta-\alpha} = \sum_{j=0}^{\infty} \frac{(\alpha-\beta)_j}{j!} \rho^j$$

Example: Laguerre polynomial (cont.)

$$\sum_{j=0}^{\infty}\frac{(\alpha-\beta)_j}{j!}\rho^j\sum_{k=0}^{\infty}\rho^k L_k^{(\beta)}(x)=\sum_{n=0}^{\infty}\rho^n L_n^{(\alpha)}(x)$$

$$j+k=n \quad\Longrightarrow\quad j=n-k$$

$$\sum_{n=0}^{\infty}\left\{L_n^{\alpha}(x)-\sum_{k=0}^{n}\frac{(\alpha-\beta)_{n-k}}{(n-k)!}L_k^{(\beta)}(x)\right\}=0$$

Connection relation (1 free parameter)

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}c_{n,k}(\alpha;\beta)\,L_k^{(\beta)}(x),$$

where

$$c_{n,k}(\alpha;\beta)=\frac{(\alpha-\beta)_{n-k}}{(n-k)!}$$

Generalizations of generating function families

Generalized hypergeometric orthogonal polynomials

- Laguerre polynomials
- Jacobi, Gegenbauer, Chebyshev and Legendre polynomials
- Wilson polynomials
- Continuous Hahn polynomials
- Continuous dual Hahn polynomials
- Meixner-Pollaczek polynomials

Basic hypergeometric orthogonal polynomials

- Askey-Wilson
- q-ultraspherical /Rogers polynomials
- q-Laguerre polynomials
- little q-Laguerre
- Al-Salam-Carlitz II

Example: Al-Salam-Carlitz II generating function

Definition (orthogonal, monic):

$$V_n^{(\alpha)}(x;q) = (-\alpha)^n q^{-\binom{n}{2}} \, {}_2\phi_0\left(\begin{matrix} q^{-n}, x \\ - \end{matrix} ; q, \frac{q^n}{\alpha} \right), \quad \alpha < 0$$

Connection relation (1 free parameter)

$$V_n^{(\alpha)}(x;q) = \sum_{k=0}^{n} c_{n,k}(\alpha;\beta) \, V_k^{(\beta)}(x;q).$$

where

$$c_{n,k}(\alpha;\beta) := \frac{\beta^{n-k} q^{k(k-n)} (q^{k-n+1}\alpha/\beta;q)_{n-k} (q^{n-k+1};q)_k}{(q;q)_k}$$

Generating function:

$$(\alpha\rho;q)_\infty \, {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix} ; q, \rho \right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}\rho^n}{(q;q)_n} V_n^{(\alpha)}(x;q)$$

Example: Al-Salam-Carlitz II (cont.)

Therefore:

$$(\alpha\rho;q)_\infty\ {}_1\phi_1\left(\begin{matrix}x\\ \alpha\rho\end{matrix};q,\rho\right)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}\rho^n}{(q;q)_n}$$

$$\times\sum_{k=0}^{n}\frac{\beta^{n-k}q^{k(k-n)}(q^{k-n+1}\alpha/\beta;q)_{n-k}(q^{n-k+1};q)_k}{(q;q)_k}V_k^{(\beta)}(x)$$

Reverse the **order** of **summation** and let $n\mapsto n+k$

$$(\alpha\rho;q)_\infty\ {}_1\phi_1\left(\begin{matrix}x\\ \alpha\rho\end{matrix};q,\rho\right)=\sum_{k=0}^{\infty}\frac{\rho^k V_k^{(\beta)}(x,q)}{(q;q)_k}$$

$$\times\sum_{n=0}^{\infty}\frac{\beta^n q^{-nk}(\alpha q^{1-n}/\beta;q)_n(q^{n+1};q)_k q^{(n+k)(n+k-1)}\rho^n}{(q;q)_{n+k}}$$

Example: Al-Salam-Carlitz II (cont.)

Using the **properties** of q-**Pochhammer** symbols:

$$(a;q)_{n+k} = (a;q)_n (aq^n, q)_k,$$

$$(aq^{-n};q)_n = (q/a;q)_n (-a)^n q^{-n-\binom{n}{2}}, \quad a \neq 0,$$

one obtains after **cancellation**

$$(\alpha\rho;q)_\infty \; {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{k=0}^{\infty} \frac{\rho^k q^{k(k-1)} V_k^{(\beta)}(x,q)}{(q;q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\frac{\beta}{\alpha};q)_n (-\alpha\rho q^k)^n}{(q;q)_n}$$

therefore with $\alpha, \beta < 0$, one has

$$(\alpha\rho;q)_\infty \; {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{n=0}^{\infty} \frac{\rho^n q^{n(n-1)} V_n^{(\beta)}(x,q)}{(q;q)_n} \; {}_1\phi_1\left(\begin{matrix} \beta/\alpha \\ 0 \end{matrix}; q, \alpha\rho q^n\right)$$

Classical expansions for orthogonal polynomials

Gegenbauer generating function (1874)

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^{\infty}\rho^n C_n^\nu(x)$$

Legendre generating function (1783)

$$\frac{1}{\sqrt{1+\rho^2-2\rho x}}=\sum_{n=0}^{\infty}\rho^n P_n(x)$$

Heine's reciprocal square root identity (1881)

$$\frac{1}{\sqrt{z-x}}=\frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\epsilon_n Q_{n-1/2}(z)T_n(x), \qquad \epsilon_n := \begin{cases} 1 & \text{if } n=0, \\ 2 & \text{if } n=1,2,3,\ldots \end{cases}$$

$$\frac{1}{\sqrt{1+\rho^2-2\rho x}}=\frac{1}{\pi\sqrt{\rho}}\sum_{n=0}^{\infty}\epsilon_n Q_{n-1/2}\left(\frac{1+\rho^2}{2\rho}\right)T_n(x)$$

Classical Gauss hypergeometric orthogonal polynomials

- **Jacobi polynomials**, $\alpha, \beta > -1$, $P_n^{(\alpha,\beta)} : \mathbf{C} \to \mathbf{C}$, defined as

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2}\right)$$

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx \\ = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}\delta_{m,n}$$

- **Gegenbauer polynomials**, $\mu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $C_n^{\mu} : \mathbf{C} \to \mathbf{C}$, defined as

$$C_n^{\mu}(z) := \frac{(2\mu)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+2\mu \\ \mu+\frac{1}{2} \end{matrix}; \frac{1-z}{2}\right) = \frac{(2\mu)_n}{(\mu+\frac{1}{2})_n} P_n^{(\mu-1/2,\mu-1/2)}$$

Classical Gauss hypergeometric orthogonal polynomials

- **Legendre polynomials**:

$$P_n(z) = C_n^{1/2}(z)$$

- **Chebyshev polynomials of the 1st kind**:

$$T_n(\cos\theta) = \cos(n\theta)$$

$$T_n(z) = \frac{1}{\epsilon_n} \lim_{\mu\to 0} \frac{n+\mu}{\mu} C_n^\mu(z)$$

- **Chebyshev polynomials of the 2nd kind**:

$$U_n(z) = C_n^1(z)$$

Special functions: associated Legendre functions

- **Ferrers function of the first kind:** $\mathrm{P}_\nu^\mu : (-1,1) \to \mathbf{C}$ **(associated Legendre function of the first kind on the cut)**

$$\mathrm{P}_\nu^\mu(x) := \frac{1}{\Gamma(1-\mu)} \left[\frac{1+x}{1-x}\right]^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$

- **Legendre function of the first kind**: $P_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1}\right]^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$

- **Legendre function of the second kind**, $Q_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi} e^{i\pi\mu} \Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) z^{\nu+\mu+1}} \times {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

The alg. functions $\sqrt{1+\rho^2-2\rho x}$, $\sqrt{z-x}$ from geometry

The **distance between two points** $\mathbf{x},\mathbf{x}'\in\mathbf{R}^d$ in a **polyspherical coordinate system** on $\mathbf{R}^d$ is given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{r^2+r'^2-2rr'\cos\gamma}, \tag{1}$$

where $r=\|\mathbf{x}\|$, $r'=\|\mathbf{x}'\|$, and $\cos\gamma=\dfrac{\mathbf{x}\cdot\mathbf{x}'}{rr'}$. If you define $r_{\lessgtr}:=\substack{\min\\ \max}\{r,r'\}$, then you can rewrite (1) as

$$\|\mathbf{x}-\mathbf{x}'\| = r_>\sqrt{1+\left(\frac{r_<}{r_>}\right)^2-2\frac{r_<}{r_>}\cos\gamma},$$

or with $\rho:=\dfrac{r_<}{r_>}$, and $x:=\cos\gamma$ we have

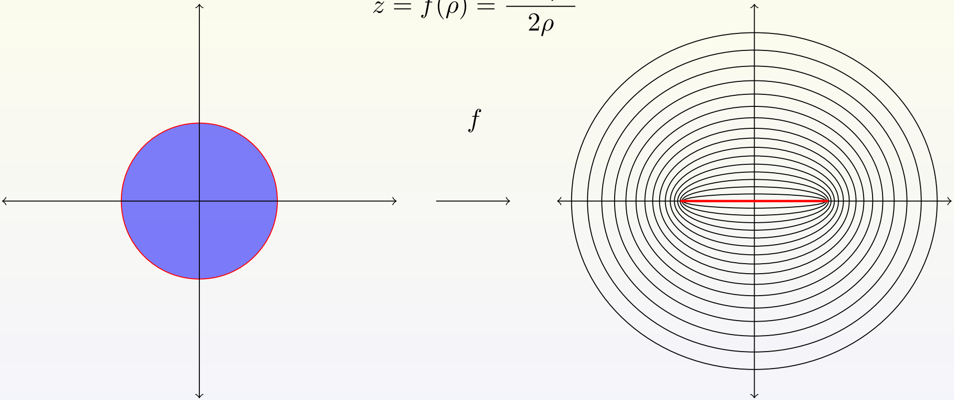
$$\|\mathbf{x}-\mathbf{x}'\| = r_>\sqrt{1+\rho^2-2\rho x},$$

where $\rho\in(0,1)$. The other option is:

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2rr'}\sqrt{z-x}, \quad \text{where} \quad z=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)=\frac{1+\rho^2}{2\rho}\in(1,\infty)$$

Szegő transformation

$$z = f(\rho) = \frac{1+\rho^2}{2\rho}$$



Expansion of an analytic function in orthogonal polynomials

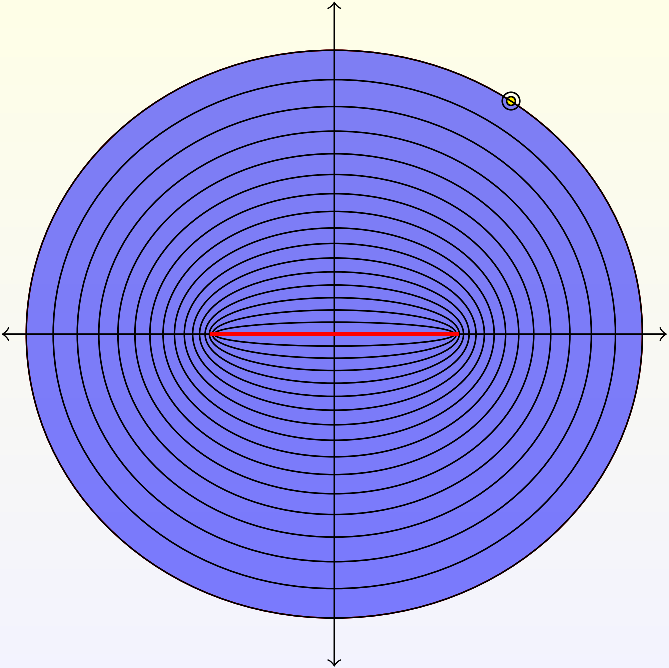
Theorem (Szegő): Let the **weight function** $w(x)$ on $[-1,1]$ have a **geometric mean,** and let $\{p_n(x)\}$ be the associated **orthonormal systems of polynomials**. Let f be an **analytic function** on $[-1,1]$ and let

$$f(x)=\sum_{n=0}^{\infty} f_n p_n(x) \tag{2}$$

and

$$f_n=\int_{-1}^{1} f(x)p_n(x)w(x)dx$$

be its **orthogonal polynomial expansion**. Let $\mathcal{E}$ be the **largest ellipse** with **foci** at ± 1 in the **interior** of which f is **regular.** Then the **orthogonal polynomial expansion** (2) is **convergent** with the sum $f(x)$ in the **interior** of $\mathcal{E}$ and **divergent** in the **exterior** of $\mathcal{E}$. The convergence is uniform on every closed set lying in the interior of the ellipse.



Obtaining generalizations: Gegenbauer generating function

Gegenbauer generating function:

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^\infty \rho^n C_n^\nu(x)$$

Connection relation:

$$C_n^\nu(x)=\frac{1}{\mu}\sum_{k=0}^{\lfloor n/2\rfloor}(\mu+n-2k)\frac{(\nu-\mu)_k\,(\nu)_{n-k}}{k!(\mu+1)_{n-k}}C^\mu_{n-2k}(x).$$

Gegenbauer expansion:

$$\frac{1}{(z-x)^\nu}=\frac{2^{\mu+\frac12}\Gamma(\mu)e^{i\pi(\mu-\nu+\frac12)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{\frac{\nu-\mu}{2}-\frac14}}\sum_{n=0}^\infty(n+\mu)Q^{\nu-\mu-\frac12}_{n+\mu-\frac12}(z)C_n^\mu(x) \qquad (3)$$

Implies **Chebyshev 1st kind expansion** (see Cohl & Dominici (2011)):

$$\frac{1}{(z-x)^\nu}=\sqrt{\frac{2}{\pi}}\frac{e^{i\pi(\frac12-\nu)}}{\Gamma(\nu)(z^2-1)^{\frac{\nu}{2}-\frac14}}\sum_{n=0}^\infty\epsilon_n Q^{\nu-\frac12}_{n-\frac12}(z)T_n(x),$$

Jacobi generalizations of Gegenbauer generating function

Gegenbauer→Jacobi generating function:

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^\infty \rho^n \frac{(2\nu)_n}{(\nu+\frac12)_n}P_n^{(\nu-\frac12,\nu-\frac12)}(x)$$

Jacobi connection relation (2 free parameters) Ismail (2005)

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^n c_{n,k}(\alpha,\beta;\gamma,\delta)P_k^{(\gamma,\delta)}(x),$$

$$c_{n,k}(\alpha,\beta;\gamma,\delta):=\frac{(\alpha+k+1)_{n-k}(n+\alpha+\beta+1)_k\Gamma(\gamma+\delta+k+1)}{(n-k)!\Gamma(\gamma+\delta+2k+1)} \times {}_3F_2\left(\begin{array}{c}-n+k,n+k+\alpha+\beta+1,\gamma+k+1\\ \alpha+k+1,\gamma+\delta+2k+2\end{array};1\right)$$

Jacobi generalization of Gegenbauer generating function

Theorem

Let $\alpha,\beta>-1$, $z\in\mathbf{C}\setminus(-\infty,1]\to\mathbf{C}$, *on any ellipse with foci at* ±1 *and* x *in the interior of that ellipse. Then*

$$\frac{1}{(z-x)^\nu}=\frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}}$$

$$\times\sum_{n=0}^\infty\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z)P_n^{(\alpha,\beta)}(x)$$

Jacobi function of the second kind

Jacobi function of the 2nd kind $Q_\gamma^{(\alpha,\beta)}:\mathbf{C}\setminus(-\infty,1]\to\mathbf{C}$ defined by

$$Q_\gamma^{(\alpha,\beta)}(z):=\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+2\gamma+2)(z-1)^{\alpha+\gamma+1}(z+1)^{\beta}}$$

$$\times\,{}_2F_1\left(\begin{array}{c}\gamma+1,\alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{array};\frac{2}{1-z}\right),$$

where $\alpha+\gamma,\beta+\gamma\notin-\mathbf{N}$.

Lemma. Let $n\in\mathbf{N}_0$, $\mu\in\mathbf{C}\setminus\{-\frac12,-\frac32,-\frac52,\ldots\}$, $\nu\in\mathbf{C}\setminus-\mathbf{N}_0$, $z\in\mathbf{C}\setminus(-\infty,1]$,

$$Q_{n+\nu-1}^{(\mu-\nu+\frac12,\mu-\nu+\frac12)}(z)=\frac{2^{\mu-\nu+\frac12}\Gamma\left(\mu+n+\frac12\right)e^{i\pi(\mu-\nu+\frac12)}}{\Gamma(\nu+n)(z^2-1)^{(\mu-\nu)/2+1/4}}Q_{n+\mu-\frac12}^{\nu-\mu-\frac12}(z).$$

Tom Koornwinder mention

The above **expansion** is consistent with the **Jacobi binomial expansion** given in Koekoek & Koekoek (2007), namely for $n \in \mathbf{N}_0$,

$$(z-x)^n = (-1)^n \, 2^n \, n! \, \Gamma(\alpha+\beta+1) \times \sum_{k=0}^{n} \frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{\Gamma(\alpha+\beta+n+k+2)} P_{n-k}^{(-\alpha-n-1,-\beta-n-1)}(z) P_k^{(\alpha,\beta)}(x)$$

The **consistency** is established through the formula

$$P_{n-k}^{(-\alpha-n-1,-\beta-n-1)}(z) = \frac{(-1)^{n+k}\Gamma(\alpha+\beta+n+k+2)}{2^{\alpha+\beta+2n+1}(n-k)!\,\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times (z-1)^{\alpha+n+1}(z+1)^{\beta+n+1}\, Q_{k-n-1}^{(\alpha+n+1,\beta+n+1)}(z)$$

Jacobi polynomials: $P_n^{(\alpha,\beta)}:\mathbf{C}\to\mathbf{C}$

DLMF (18.12.3) **generating function**

$$
\begin{aligned}
&\frac{1}{(1+\rho)^{\alpha+\beta+1}}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2}\\ \beta+1\end{matrix};\frac{2\rho(1+x)}{(1+\rho)^2}\right)\\
&\qquad=\left(\frac{2}{\rho(1+x)}\right)^{\beta/2}\frac{\Gamma(\beta+1)}{\mathsf{R}^{\alpha+1}}P_\alpha^{-\beta}\left(\frac{1+\rho}{\mathsf{R}}\right)\\
&\qquad\qquad=\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^nP_n^{(\alpha,\beta)}(x),
\end{aligned}
$$

Ismail (2005) (4.3.2) **generating function**

$$
\begin{aligned}
&\frac{(\alpha+\beta+1)(1+\rho)}{(1-\rho)^{\alpha+\beta+2}}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2}\\ \alpha+1\end{matrix};\frac{-2\rho(1-x)}{(1-\rho)^2}\right)\\
&\qquad=\left(\frac{2}{\rho(1-x)}\right)^{\alpha/2}\frac{(\alpha+\beta+1)(1+\rho)\Gamma(\alpha+1)}{\mathsf{R}^{\beta+2}}\mathrm{P}_{\beta+1}^{-\alpha}\left(\frac{1-\rho}{\mathsf{R}}\right)\\
&\qquad\qquad=\sum_{n=0}^{\infty}(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^nP_n^{(\alpha,\beta)}(x),
\end{aligned}
$$

Generalized expansions for Jacobi polynomials

Theorem

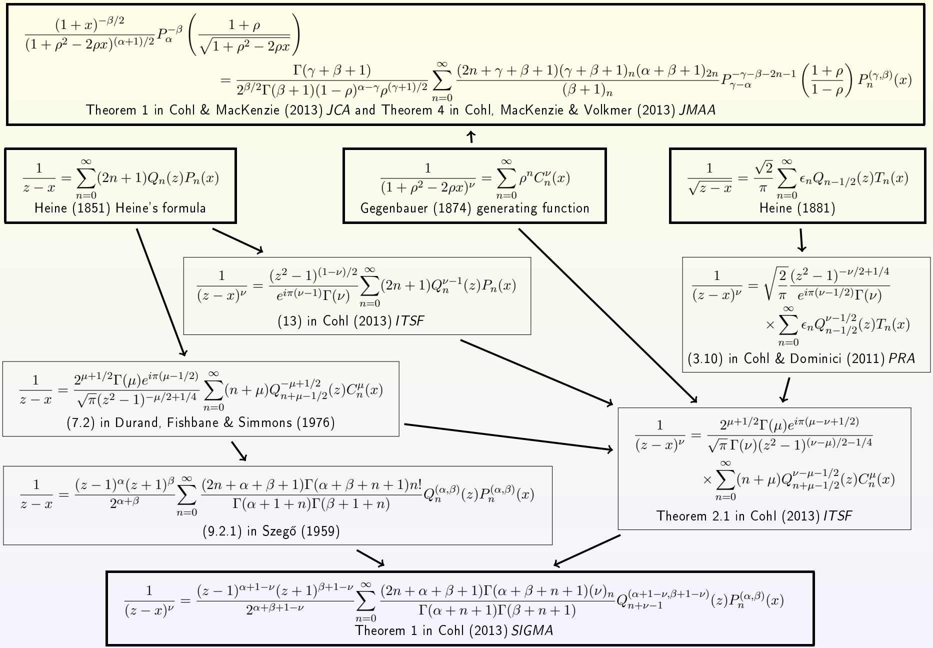
Let $\alpha \in \mathbf{C}$, γ, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1, 0]$, $x \in [-1, 1]$. *Then*

$$\frac{(1+x)^{-\beta/2}}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}} P_\alpha^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

$$= \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2}\Gamma(\beta+1)(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}}$$

$$\times \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k\,(\alpha+\beta+1)_{2k}}{(\beta+1)_k}$$

$$\times P_{\gamma-\alpha}^{-\gamma-\beta-2k-1}\left(\frac{1+\rho}{1-\rho}\right) P_k^{(\gamma,\beta)}(x).$$



$$\frac{(1+x)^{-\beta/2}}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}}P_\alpha^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right) = \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2}\Gamma(\beta+1)(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}}\sum_{n=0}^{\infty}\frac{(2n+\gamma+\beta+1)(\gamma+\beta+1)_n(\alpha+\beta+1)_{2n}}{(\beta+1)_n}P_{\gamma-\alpha}^{-\gamma-\beta-2n-1}\left(\frac{1+\rho}{1-\rho}\right)P_n^{(\gamma,\beta)}(x)$$

Theorem 1 in Cohl & MacKenzie (2013) *JCA* and Theorem 4 in Cohl, MacKenzie & Volkmer (2013) *JMAA*

$$\frac{1}{z-x} = \sum_{n=0}^{\infty}(2n+1)Q_n(z)P_n(x)$$

Heine (1851) Heine's formula

$$\frac{1}{(1+\rho^2-2\rho x)^{\nu}} = \sum_{n=0}^{\infty}\rho^n C_n^{\nu}(x)$$

Gegenbauer (1874) generating function

$$\frac{1}{\sqrt{z-x}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\epsilon_n Q_{n-1/2}(z)T_n(x)$$

Heine (1881)

$$\frac{1}{(z-x)^{\nu}} = \frac{(z^2-1)^{(1-\nu)/2}}{e^{i\pi(\nu-1)}\Gamma(\nu)}\sum_{n=0}^{\infty}(2n+1)Q_n^{\nu-1}(z)P_n(x)$$

(13) in Cohl (2013) *ITSF*

$$\frac{1}{(z-x)^{\nu}} = \sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{-\nu/2+1/4}}{e^{i\pi(\nu-1/2)}\Gamma(\nu)} \times\sum_{n=0}^{\infty}\epsilon_n Q_{n-1/2}^{\nu-1/2}(z)T_n(x)$$

(3.10) in Cohl & Dominici (2011) *PRA*

$$\frac{1}{z-x} = \frac{2^{\mu+1/2}\Gamma(\mu)e^{i\pi(\mu-1/2)}}{\sqrt{\pi}(z^2-1)^{-\mu/2+1/4}}\sum_{n=0}^{\infty}(n+\mu)Q_{n+\mu-1/2}^{-\mu+1/2}(z)C_n^{\mu}(x)$$

(7.2) in Durand, Fishbane & Simmons (1976)

$$\frac{1}{(z-x)^{\nu}} = \frac{2^{\mu+1/2}\Gamma(\mu)e^{i\pi(\mu-\nu+1/2)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{(\nu-\mu)/2-1/4}} \times\sum_{n=0}^{\infty}(n+\mu)Q_{n+\mu-1/2}^{\nu-\mu-1/2}(z)C_n^{\mu}(x)$$

Theorem 2.1 in Cohl (2013) *ITSF*

$$\frac{1}{z-x} = \frac{(z-1)^{\alpha}(z+1)^{\beta}}{2^{\alpha+\beta}}\sum_{n=0}^{\infty}\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}Q_n^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(x)$$

(9.2.1) in Szegő (1959)

$$\frac{1}{(z-x)^{\nu}} = \frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}}\sum_{n=0}^{\infty}\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z)P_n^{(\alpha,\beta)}(x)$$

Theorem 1 in Cohl (2013) *SIGMA*

Other Jacobi generating functions

Connection relation (1 free parameter)

$$P_n^{(\alpha,\beta)}(x) = \frac{(\beta+1)_n}{(\gamma+\beta+1)(\gamma+\beta+2)_n}$$

$$\times \sum_{k=0}^{n} \frac{(\gamma+\beta+2k+1)(\gamma+\beta+1)_k\,(n+\beta+\alpha+1)_k(\alpha-\gamma)_{n-k}}{(\beta+1)_k\,(n+\gamma+\beta+2)_k(n-k)!} P_k^{(\gamma,\beta)}(x).$$

Generating function for Jacobi polynomials: DLMF (18.12.1)

$$\frac{2^{\alpha+\beta}}{\mathsf{R}\,(1+\mathsf{R}-\rho)^{\alpha}\,(1+\mathsf{R}+\rho)^{\beta}} = \sum_{n=0}^{\infty} \rho^n P_n^{(\alpha,\beta)}(x),$$

where $\mathsf{R} := \sqrt{1+\rho^2-2\rho x}$.

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, γ, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1,1]$. *Then*

$$\frac{2^{\alpha+\beta}}{\mathsf{R}\,(1+\mathsf{R}-\rho)^{\alpha}\,(1+\mathsf{R}+\rho)^{\beta}}$$

$$= \frac{1}{\gamma+\beta+1} \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k \left(\frac{\alpha+\beta+1}{2}\right)_k \left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+\beta+1)_k \left(\frac{\gamma+\beta+2}{2}\right)_k \left(\frac{\gamma+\beta+3}{2}\right)_k}$$

$$\times {}_3F_2 \left(\begin{array}{c} \beta+k+1, \alpha+\beta+2k+1, \alpha-\gamma \\ \alpha+\beta+k+1, \gamma+\beta+2k+2 \end{array} ; \rho \right) \rho^k P_k^{(\gamma,\beta)}(x).$$

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, γ, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1,1]$. *Then*

$$\left(\frac{2}{(1-x)\rho}\right)^{\alpha/2}\left(\frac{2}{(1+x)\rho}\right)^{\beta/2} J_\alpha\left(\sqrt{2(1-x)\rho}\right) I_\beta\left(\sqrt{2(1+x)\rho}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\rho^n}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}P_n^{(\alpha,\beta)}(x)$$

$$=\frac{1}{(\gamma+\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)}$$

$$\times\sum_{k=0}^{\infty}\frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k\left(\frac{\alpha+\beta+1}{2}\right)_k\left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+1)_k\,(\beta+1)_k\,(\alpha+\beta+1)_k\left(\frac{\gamma+\beta+2}{2}\right)_k\left(\frac{\gamma+\beta+3}{2}\right)_k}$$

$$\times\,{}_2F_3\left(\begin{matrix}2k+\alpha+\beta+1,\alpha-\gamma\\ \alpha+\beta+k+1,\gamma+\beta+2k+2,\alpha+k+1\end{matrix};\rho\right)\rho^k P_k^{(\gamma,\beta)}(x)$$

Generalized expansions for Gegenbauer polynomials

Theorem

Let $\alpha, \mu \in \mathbf{C}$, $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. *Then*

$$(1-x^2)^{1/4-\mu/2}$$

$$\times P^{1/2-\mu}_{\mu-\alpha-1/2}\left(\sqrt{1+\rho^2-2\rho x}+\rho\right) \mathrm{P}^{1/2-\mu}_{\mu-\alpha-1/2}\left(\sqrt{1+\rho^2-2\rho x}-\rho\right)$$

$$= \frac{2^{1/2-\mu}}{\Gamma(\mu+1/2)} \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(2\mu-\alpha)_n\,(\mu)_n}{(2\mu)_n\,(\nu)_n\,\Gamma(\mu+n+1/2)} \rho^{\mu+n-1/2}$$

$$\times\, {}_6F_5\left(\begin{array}{c} \frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}, \frac{2\mu-\alpha+n}{2}, \frac{2\mu-\alpha+n+1}{2}, \mu-\nu, \mu+n \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+\frac{1}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}, \nu+1+n \end{array} ; \rho^2\right) C^{\nu}_{n}(x).$$

Wilson polynomials

$$W_n(x^2;a,b,c,d) := (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ;1\right).$$

Connection relation with **one free parameter** for the **Wilson polynomials**

$$W_n\left(x^2;a,b,c,d\right) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} W_k\left(x^2;a,b,c,h\right) \times \frac{(n+a+b+c+d-1)_k\,(d-h)_{n-k}\,(k+a+b)_{n-k}\,(k+a+c)_{n-k}}{(k+b+c)_{n-k}^{-1}(k+a+b+c+h-1)_k\,(2k+a+b+c+h)_{n-k}}.$$

Generalized generating function for Wilson polynomials

Theorem

Let $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in (0,\infty)$, $\Re a, \Re b, \Re c, \Re d, \Re h > 0$ *and non-real parameters* a, b, c, d, h *occurring in conjugate pairs. Then*

$$ {}_2F_1\left(\begin{array}{c} a+ix,\, b+ix \\ a+b \end{array} ; \rho\right) {}_2F_1\left(\begin{array}{c} c-ix,\, d-ix \\ c+d \end{array} ; \rho\right) $$

$$ = \sum_{k=0}^{\infty} \frac{(k+a+b+c+d-1)_k}{(k+a+b+c+h-1)_k\,(a+b)_k\,(c+d)_k\,k!} $$

$$ \times {}_4F_3\left(\begin{array}{c} d-h,\, 2k+a+b+c+d-1,\, k+a+c,\, k+b+c \\ k+a+b+c+d-1,\, 2k+a+b+c+h,\, k+c+d \end{array} ; \rho\right) $$

$$ \times \rho^k\, W_k\left(x^2; a, b, c, h\right). $$

Generalized generating function for Wilson polynomials

Theorem

Let $\rho \in \mathbf{C}$, $|\rho| < 1$, $x \in (0,\infty)$, *and* a,b,c,d,h *complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among* a,b,c,d *and* a,b,c,h. *Then*

$$(1-\rho)^{1-a-b-c-d}$$

$$\times {}_4F_3\left(\begin{matrix}\frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix\\ a+b, a+c, a+d\end{matrix}; -\frac{4\rho}{(1-\rho)^2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(k+a+b+c+d-1)_k(a+b+c+d-1)_k}{(k+a+b+c+h-1)_k(a+b)_k(a+c)_k(a+d)_k k!}\rho^k$$

$$\times {}_3F_2\left(\begin{matrix}2k+a+b+c+d-1, d-h, k+b+c\\ 2k+a+b+c+h, a+d+k\end{matrix}; \rho\right) W_k(x^2; a,b,c,h)$$

Continuous dual Hahn polynomials

$$S_n(x^2;a,b,c) := (a+b)_n(a+c)_n\, {}_3F_2\left(\begin{matrix}-n, a+ix, a-ix\\ a+b, a+c\end{matrix};1\right),$$

where $a,b,c>0$, except for possibly a pair of complex conjugates with positive real parts. **Connection coefficient** for the **continuous dual Hahn polynomials** with **two free parameters**.

Lemma

Let $x\in(0,\infty)$, and $a,b,c,f,g\in\mathbf{C}$ with positive real parts and non-real values appearing in conjugate pairs among a,b,c and a,f,g. Then

$$S_n(x^2;a,b,c)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(k+a+b)_{n-k}(k+a+c)_{n-k}S_k(x^2;a,f,g)$$

$$\times\, {}_3F_2\left(\begin{matrix}k-n, k+a+f, k+a+g\\ k+a+b, k+a+c\end{matrix};1\right).$$

Generalized gen. fn. for continuous dual-Hahn polynomials

Theorem

Let $\rho \in \mathbf{C}$ *with* $|\rho| < 1$, $x \in (0, \infty)$ *and* $a, b, c, d, f > 0$ *except for possibly pairs of complex conjugates with positive real parts among* a, b, c *and* a, d, f*. Then*

$$(1-\rho)^{-d+ix} {}_2F_1 \left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix} ; \rho \right) = \sum_{k=0}^{\infty} \frac{S_k(x^2; a, d, f)\rho^k}{(a+b)_k k!} {}_2F_1 \left(\begin{matrix} b-f, k+a+d \\ k+a+b \end{matrix} ; \rho \right).$$

Theorem

Let $\rho \in \mathbf{C}$, $x \in (0, \infty)$, *and* $a, b, c, d > 0$ *except for possibly a pair of complex conjugates with positive real parts among* a, b, c *and* a, b, d*. Then*

$$e^{\rho} {}_2F_2 \left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix} ; -\rho \right) = \sum_{k=0}^{\infty} \frac{\rho^k S_k(x^2; a, b, d)}{(a+b)_k (a+c)_k k!} {}_1F_1 \left(\begin{matrix} c-d \\ k+a+c \end{matrix} ; \rho \right).$$

Connection relation for Meixner-Pollaczek polynomials

If $\lambda > 0$ and $\phi \in (0, \pi)$, then the **Meixner-Pollaczek** polynomials are orthogonal

$$P_n^{(\lambda)}(x;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{matrix}-n, \lambda + ix\\ 2\lambda\end{matrix}; 1 - e^{-2i\phi}\right).$$

Connection relation with one free parameter:

Lemma

Let $\lambda > 0, \phi, \psi \in (0, \pi)$. *Then*

$$P_n^{(\lambda)}(x;\phi) = \frac{1}{\sin^n \psi} \sum_{k=0}^{n} \frac{(2\lambda + k)_{n-k}}{(n-k)!} \sin^k \phi \sin^{n-k}(\psi - \phi) P_k^{(\lambda)}(x;\psi).$$

Expansions for Meixner-Pollaczek polynomials

Theorem

Let $\lambda > 0$, $\psi, \phi \in (0, \pi)$, $x \in \mathbf{R}$, *and* $\rho \in \mathbf{C}$ *such that*

$$|\rho|(\sin\phi + |\sin(\psi-\phi)|) < \sin\psi.$$

Then

$$(1-e^{i\phi}\rho)^{-\lambda+ix}(1-e^{-i\phi}\rho)^{-\lambda-ix} = \left(1-\rho\frac{\sin(\psi-\phi)}{\sin\psi}\right)^{-2\lambda}\sum_{k=0}^{\infty} P_k^{(\lambda)}(x;\psi)\tilde{\rho}^k,$$

where

$$\tilde{\rho} = \frac{\rho\sin\phi}{\sin\psi - \rho\sin(\psi-\phi)}.$$

etc. for continuous Hahn polynomials

Some results for q-Pochhammer symbols

$$(a;q)_{n+k} = (a;q)_k(aq^k;q)_n,$$

$$(aq^n;q)_k = \frac{(a;q)_k}{(a;q)_n}(aq^k;q)_n,$$

$$(aq^{-n};q)_n = (-a)^n q^{-n-\binom{n}{2}}(q/a;q)_n,$$

$$(aq^{-n};q)_k = q^{-nk}\frac{(q/a;q)_n}{(q^{1-k}/a;q)_n}(a;q)_k,$$

$$(a^2;q^2)_n = (a,-a;q)_n.$$

$$(a;q)_{2n} = (a,aq;q^2)_n = (\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_n,$$

$$(aq^n;q)_n = \frac{(\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_n}{(a;q)_n},$$

$$(-a^2;q^2)_n = (ia,-ia;q)_n.$$

The q-ultraspherical/Rogers polynomials

Definition **(orthogonal)** for $|\beta| < 1$

$$C_n(x;\beta|q) := \frac{(\beta^2;q)_n\beta^{-n/2}}{(q;q)_n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n},\beta^2q^n,\beta^{1/2}e^{i\theta},\beta^{1/2}e^{-i\theta} \\ \beta q^{1/2},-\beta,-\beta q^{1/2}\end{array};q,q\right)$$

$$= \frac{(\beta^2;q)_n\beta^{-n}e^{-in\theta}}{(q;q)_n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n},\beta,\beta e^{2i\theta} \\ \beta^2,0\end{array};q,q\right)$$

$$= \frac{(\beta;q)_n e^{in\theta}}{(q;q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},\beta \\ \beta^{-1}q^{1-n}\end{array};q,q\beta^{-1}e^{-2i\theta}\right),$$

where $x = \cos\theta$.

Theorem

Let $x \in [-1,1]$, $\beta \in (-1,1) \setminus \{0\}$, $q \in (0,1)$. *Then*

$$
{}_2\phi_1\left(\begin{matrix} \beta^{\frac{1}{2}}e^{i\theta}, (\beta q)^{\frac{1}{2}}e^{i\theta} \\ \beta q^{\frac{1}{2}} \end{matrix} ; q, e^{-i\theta}t\right) {}_2\phi_1\left(\begin{matrix} -\beta^{\frac{1}{2}}e^{-i\theta}, -\beta^{\frac{1}{2}}q^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}} \end{matrix} ; q, e^{i\theta}t\right)
$$

$$
= \frac{1}{1-\gamma}\sum_{n=0}^{\infty} \frac{(\beta, -\beta, -\beta q^{\frac{1}{2}}; q)_n (1-\gamma q^n) t^n}{(\beta^2, \beta q^{\frac{1}{2}}, q\gamma; q)_n} C_n(x;\gamma|q)
$$

$$
\times\, {}_{10}\phi_9\left(\begin{matrix} \beta/\gamma, \beta q^n, i(\beta q^n)^{\frac{1}{2}}, -i(\beta q^n)^{\frac{1}{2}}, i(\beta q^{n+1})^{\frac{1}{2}}, -i(\beta q^{n+1})^{\frac{1}{2}}, \\ \gamma q^{n+1}, \beta q^{n/2}, -\beta q^{n/2}, \beta q^{(n+1)/2}, -\beta q^{(n+1)/2}, (\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \\ i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, -i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, i(\beta q^{n+3/2})^{\frac{1}{2}}, -i(\beta q^{n+3/2})^{\frac{1}{2}} \\ -(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, (\beta q^{n+3/2})^{\frac{1}{2}}, -(\beta q^{n+3/2})^{\frac{1}{2}} \end{matrix} ; q, \gamma t^2\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{(-\beta, -\beta q^{1/2}; q)_n}{(\beta^2, \beta q^{1/2}; q)_n} C_n(x;\beta|q) t^n.
$$

Theorem

Let $x \in [-1,1]$, $\beta \in (-1,1) \setminus \{0\}$, $q \in (0,1)$. *Then*

$$\frac{(\gamma e^{i\theta}t;q)_\infty}{(e^{i\theta}t;q)_\infty}\, {}_3\phi_2\left(\begin{array}{c}\gamma,\beta,\beta e^{2i\theta}\\ \beta^2,\gamma e^{i\theta}t\end{array};q,e^{-i\theta}t\right)$$

$$=\frac{1}{1-\gamma}\sum_{n=0}^{\infty}\frac{(\beta,\gamma;q)_n(1-\gamma q^n)t^n}{(\beta^2,q\gamma;q)_n}C_n(x;\gamma|q)$$

$$\times\, {}_6\phi_5\left(\begin{array}{c}\beta/\gamma,\beta q^n,(\gamma q^n)^{\frac12},-(\gamma q^n)^{\frac12},(\gamma q^{n+1})^{\frac12},-(\gamma q^{n+1})^{\frac12}\\ \beta q^{n/2},-\beta q^{n/2},\beta q^{(n+1)/2},-\beta q^{(n+1)/2},\gamma q^{n+1}\end{array};q,\gamma t^2\right)$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma;q)_n}{(\beta^2;q)_n}C_n(x;\beta|q)t^n,$$

etc. for q-ultraspherical/Rogers polynomials.

The q-Laguerre polynomials

Definition **(orthogonal)** $\alpha > -1$

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array};q,-q^{n+\alpha+1}x\right)$$

$$= \frac{1}{(q;q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},-x \\ 0 \end{array};q,q^{n+\alpha+1}\right).$$

Connection relation

$$L_n^{(\alpha)}(x;q)$$

$$= \frac{q^{n(\alpha-\beta)}}{(q;q)_n}\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}(q^{n-j+1};q)_j(q^{j-n+\beta-\alpha+1};q)_{n-j}\,L_j^{(\beta)}(x;q)$$

Generalized generating function for q-Laguerre

Theorem

Let $\alpha > -1$. *Then*

$$\frac{1}{(t;q)_\infty}\,{}_0\phi_1\left(\begin{array}{c}-\\ q^{\alpha+1}\end{array};q,-xtq^{\alpha+1}\right)$$

$$=\sum_{n=0}^\infty \frac{(q^{(\alpha-\beta)}t)^n L_n^{(\beta)}(x;q)}{(q^{\alpha+1};q)_n}\,{}_2\phi_1\left(\begin{array}{c}q^{\alpha-\beta},0\\ q^{\alpha+n+1}\end{array};q,t\right),$$

$$=\sum_{n=0}^\infty \frac{L_n^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_n}t^n.$$

Generalized generating function for q-Laguerre (cont.)

Theorem

Let $\alpha > -1$. *Then*

$$(t;q)_\infty\,{}_0\phi_2\left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix}\,;q,-q^{\alpha+1}xt\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-tq^{\alpha-\beta})^n q^{\binom{n}{2}} L_n^{(\beta)}(x;q)}{(q^{\alpha+1};q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix}\,;q,q^n t\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q^{\alpha+1};q)_n} L_n^{(\alpha)}(x;q) t^n.$$

etc. for q-Laguerre and etc. for little q-Laguerre.

Summary of generalized generating functions

OP	–	generating function	coef.
Askey-Wilson	p_n	${}_2\phi_1\left(\begin{matrix}ae^{i\theta},be^{i\theta}\\ab\end{matrix};q,te^{-i\theta}\right){}_2\phi_1\left(\begin{matrix}ce^{-i\theta},de^{-i\theta}\\cd\end{matrix};q,te^{i\theta}\right)$	${}_4\phi_3(\alpha t)$
Rogers	C_n	$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}$	${}_2\phi_1(\gamma t^2)$
”	”	$(te^{-i\theta};q)_\infty\,{}_2\phi_1\left(\begin{matrix}\beta,\beta e^{2i\theta}\\\beta^2\end{matrix};q,te^{-i\theta}\right)$	${}_2\phi_5(\gamma(\beta t)^2q^{2n+1})$
”	”	$\frac{1}{(te^{i\theta};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\beta,\beta e^{2i\theta}\\\beta^2\end{matrix};q,te^{-i\theta}\right)$	${}_6\phi_5(\gamma t^2)$
”	”	$\frac{(\gamma te^{i\theta};q)_\infty}{(te^{i\theta};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\gamma,\beta,\beta e^{2i\theta}\\\beta^2,\gamma te^{i\theta}\end{matrix};q,te^{-i\theta}\right)$	${}_6\phi_5(\alpha t^2)$
”	”	${}_2\phi_1\left(\begin{matrix}\frac{\sqrt{\beta}}{e^{-i\theta}},-\frac{\sqrt{\beta}}{e^{-i\theta}}\\-\beta\end{matrix};q,\frac{t}{e^{i\theta}}\right){}_2\phi_1\left(\begin{matrix}\frac{\sqrt{q\beta}}{e^{i\theta}},-\frac{\sqrt{q\beta}}{e^{i\theta}}\\-q\beta\end{matrix};q,\frac{t}{e^{-i\theta}}\right)$	${}_{10}\phi_9(\gamma t^2)$
”	”	${}_2\phi_1\left(\begin{matrix}-\beta,\sqrt{\beta q}\\-\beta\sqrt{q}\end{matrix};q,\frac{t}{e^{i\theta}}\right){}_2\phi_1\left(\begin{matrix}\frac{\sqrt{\beta q}}{e^{i\theta}},-\frac{\sqrt{\beta}}{e^{i\theta}}\\-\beta\sqrt{q}\end{matrix};q,\frac{t}{e^{-i\theta}}\right)$	${}_{10}\phi_9(\gamma t^2)$
”	”	${}_2\phi_1\left(\begin{matrix}\frac{\sqrt{\beta}}{e^{-i\theta}},\frac{\sqrt{q\beta}}{e^{-i\theta}}\\\beta\sqrt{q}\end{matrix};q,\frac{t}{e^{i\theta}}\right){}_2\phi_1\left(\begin{matrix}-\frac{\sqrt{\beta}}{e^{i\theta}},-\frac{\sqrt{q\beta}}{e^{i\theta}}\\\beta\sqrt{q}\end{matrix};q,\frac{t}{e^{-i\theta}}\right)$	${}_{10}\phi_9(\gamma t^2)$
”	”	${}_2\phi_1\left(\begin{matrix}\frac{\sqrt{\beta}}{e^{-i\theta}},-\frac{\sqrt{q\beta}}{e^{-i\theta}}\\-\beta\sqrt{q}\end{matrix};q,\frac{t}{e^{i\theta}}\right){}_2\phi_1\left(\begin{matrix}\frac{\sqrt{q\beta}}{e^{i\theta}},-\frac{\sqrt{\beta}}{e^{i\theta}}\\-\beta\sqrt{q}\end{matrix};q,\frac{t}{e^{-i\theta}}\right)$	${}_{10}\phi_9(\gamma t^2)$

Summary of generalized generating functions (cont.)

OP	–	generating function	coef.
little q-Laguerre	p_n	$\frac{(t;q)_\infty}{(xt;q)_\infty}\,{}_0\phi_1\left(\begin{matrix}-\\ aq\end{matrix};q,aqxt\right)$	${}_1\phi_1(btq^{n+1})$
q-Laguerre	$L_n^{(\alpha)}$	$\frac{1}{(t;q)_\infty}\,{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-xtq^{\alpha+1}\right)$	${}_2\phi_1(t)$
,,	,,	$(t;q)_\infty\,{}_0\phi_2\left(\begin{matrix}-\\ q^{\alpha+1},t\end{matrix};q,-xtq^{\alpha+1}\right)$	${}_1\phi_1(tq^n)$
,,	,,	$\frac{(\gamma t;q)_\infty}{(t;q)_\infty}\,{}_1\phi_2\left(\begin{matrix}\gamma\\ q^{\alpha+1},\gamma t\end{matrix};q,-xtq^{\alpha+1}\right)$	${}_2\phi_1(t)$

Continuous orthogonality for q-Laguerre polynomials

Corollary (Cohl & Koornwinder). The **continuous orthogonality** relation for q-Laguerre polynomials is given as follows. Let $\alpha \in (-1,\infty)$, $m,n \in \mathbf{N}_0$. Then

$$\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)\frac{x^\alpha}{(-x;q)_\infty}dx$$

$$=\frac{\delta_{m,n}}{q^n}\begin{cases}\dfrac{\pi(q^{-\alpha};q)_\infty(q^{\alpha+1};q)_n}{\sin(\pi\alpha)(q;q)_\infty(q;q)_n} & \text{if } \alpha\in(-1,\infty)\setminus\mathbf{N}_0,\\[2ex] \dfrac{(q^{n+1};q)_\alpha\log q}{q^{\alpha(\alpha+1)/2}} & \text{if } \alpha\in\mathbf{N}_0.\end{cases}$$

$$(\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)\frac{x^\alpha}{(-x;q)_\infty}dx$$

$$=\frac{(q^{-\alpha};q)_\infty(q^{\alpha+1};q)_n}{q^n(q;q)_\infty(q;q)_n}\Gamma(-\alpha)\Gamma(\alpha+1)\delta_{m,n}.)$$

Let $n \in \mathbf{N}_0$, $\alpha, \beta \in (-1, \infty)$, $q \in (0,1)$, $|t| < 1$. Then

$$\int_0^\infty {}_0\phi_1\left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix}; q, -xtq^{\alpha+1}\right) L_n^{(\beta)}(x;q) \frac{x^\beta}{(-x;q)_\infty} dx$$

$$= \frac{-\left(tq^{\alpha-\beta}\right)^n (t;q)_\infty}{q^n (q^{\alpha+1};q)_n} {}_2\phi_1\left(\begin{matrix} q^{\alpha-\beta}, 0 \\ q^{\alpha+n+1} \end{matrix}; q, t\right) \begin{cases} \dfrac{\pi(q^{-\beta};q)_\infty (q^{\beta+1};q)_n}{\sin(\pi\beta)(q;q)_\infty (q;q)_n} \\[2ex] \dfrac{(q^{n+1};q)_\beta \log q}{q^{\beta(\beta+1)/2}} \end{cases}$$

$$\int_0^\infty {}_0\phi_2\left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix}; q, -xtq^{\alpha+1}\right) L_n^{(\beta)}(x;q) \frac{x^\beta}{(-x;q)_\infty} dx$$

$$= \frac{-\left(-tq^{\alpha-\beta}\right)^n}{q^n (t;q)_\infty (q^{\alpha+1};q)_n} {}_1\phi_1\left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix}; q, tq^n\right) \begin{cases} \dfrac{\pi(q^{-\beta};q)_\infty (q^{\beta+1};q)_n}{\sin(\pi\beta)(q;q)_\infty (q;q)_n} & \text{if } \\[2ex] \dfrac{(q^{n+1};q)_\beta \log q}{q^{\beta(\beta+1)/2}} & \text{if } \end{cases}$$

Little q-Laguerre polynomials (discrete orthogonality)

Satisfy a **discrete orthogonality relation**, namely for $a \in (0, \frac{1}{q})$, we have

$$\sum_{k=0}^{\infty} p_m(q^k; a|q) p_n(q^k; a|q) \frac{(aq)^k}{(q;q)_k} = \frac{(aq)^n (q;q)_n}{(aq;q)_\infty (aq;q)_n} \delta_{m,n}.$$

This leads us to the following infinite series.

Let $n \in \mathbf{N}_0$, $q \in (0,1)$, $\alpha, \beta \in (0, \frac{1}{q})$, $|t| < 1$. Then

$$\sum_{k=0}^{\infty} \frac{(q\beta)^k}{(tq^k;q)_\infty (q;q)_k} {}_0\phi_1\left(\begin{matrix} - \\ q\alpha \end{matrix}; q, t\alpha q^{k+1}\right) p_n\left(q^k; \beta|q\right)$$

$$= \frac{q^{\binom{n}{2}} (-q\beta t)^n}{(t, q\beta; q)_\infty (q\alpha; q)_n} {}_1\phi_1\left(\begin{matrix} \alpha/\beta \\ \alpha q^{n+1} \end{matrix}; q, t\beta q^{n+1}\right)$$

q-Laguerre polynomials (Jackson q-integral orthogonality)

One type of orthogonality for the q-Laguerre polynomials is

$$\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)\frac{x^\alpha}{(-x;q)_\infty}d_qx = \frac{(1-q)(q,-q^{\alpha+1},-q^{-\alpha};q)_\infty(q^{\alpha+1};}{2q^n(q^{\alpha+1},-q,-q;q)_\infty(q;q)_n}$$

Using this orthogonality relation we can obtain new Jackson q-integrals using our generalized generating functions.

Let $n \in \mathbf{N}_0$, $q \in (0,1)$, $\alpha, \beta \in (-1,\infty)$, $|t|<1$. Then

$$\int_0^\infty {}_0\phi_1\left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix};q,-xtq^{\alpha+1}\right)L_n^{(\beta)}(x;q)\frac{x^\beta}{(-x;q)_\infty}d_qx$$

$$= \frac{(1-q)\left(tq^{\alpha-\beta}\right)^n(t,q,-q^{\beta+1},-q^{-\beta};q)_\infty(q^{\beta+1};q)_n}{2q^n(q^{\beta+1},-q,-q;q)_\infty(q,q^{\alpha+1};q)_n}\,{}_2\phi_1$$

q-Laguerre polynomials (bilateral discrete orthogonality)

The q-Laguerre polynomials also satisfy a bilateral discrete orthogonality $\alpha\in(-1,\infty)$, $c>0$,

$$\sum_{k=-\infty}^{\infty} L_m^{(\alpha)}(cq^k;q)L_n^{(\alpha)}(cq^k;q)\frac{q^{(\alpha+1)k}}{(-cq^k;q)_\infty}$$
$$=\frac{(q,-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty(q^{\alpha+1};q)_n}{q^n(q^{\alpha+1},-c,-q/c;q)_\infty(q;q)_n}\delta_{m,n}.$$

Let $n\in\mathbf{N}_0$, $q\in(0,1)$, $\alpha,\beta\in(-1,\infty)$, $|t|<1$, $c>0$. Then

$$\sum_{k=-\infty}^{\infty} {}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-ctq^{\alpha+k+1}\right)L_n^{(\beta)}(cq^k;q)\frac{q^{(\beta+1)k}}{(-cq^k;q)_\infty}$$
$$=\frac{\left(tq^{\alpha-\beta}\right)^n(t,q,-cq^{\beta+1},-q^{-\beta}/c;q)_\infty(q^{\beta+1};q)_n}{q^n(q^{\beta+1},-c,-q/c;q)_\infty(q,q^{\alpha+1};q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{\alpha-\beta},0\\ q^{\alpha+n+1}\end{matrix};q,t\right).$$

Ongoing work

- **orthogonality** etc. for **Rogers** and **Askey-Wilson / justification**
- **Big/little q-Jacobi**, **continuous q-Jacobi**
- *Mourad*'s **connection coefficient** trick:
  - Meixner polynomials (3 gf's, 2 fp's)
  - Krawtchouk polynomials (3 gf's, 2 fp's)
  - Bessel polynomials (1 gf, 1 fp)
  - Charlier polynomials (1 gf, 1 fp)
  - Al-Salam-Chihara polynomials (4 gf's, 2 fp's)
  - q-Meixner-Pollaczek polynomials (2 gf's, 1 fp)
  - dual q-Krawtchouk polynomials (1 gf's, 2 fp's)
  - continuous big q-Hermite polynomials (3 gf's, 1 fp)
  - continuous q-Laguerre polynomials (4 gf's, 1 fp)
  - Al-Salam-Carlitz {I,II} polynomials {(1 gf, 1 fp),(2 gf's, 1 fp)}
  - continuous q-Hermite polynomials (3 gf's, **0** fp's)
  - discrete q-Hermite {I,II} polynomials {(3 gf's, **0** fp's),(2 gf's, **0** fp's)}

Publications

- "On a generalization of the generating function for Gegenbauer polynomials," H. S. Cohl, 2013, *Integral Transforms and Special Functions*, **24**, 10, 807–816, 10 pp.
- "Generalizations and specializations of generating functions for Jacobi, Gegenbauer, Chebyshev and Legendre polynomials with definite integrals," H. S. Cohl and Connor MacKenzie, 2013, *Journal of Classical Analysis*, **3**, 1, 17-33, 17 pp.
- "Fourier, Gegenbauer and Jacobi expansions for a power-law fundamental solution of the polyharmonic equation and polyspherical addition theorems," H. S. Cohl, 2013, *Symmetry, Integrability and Geometry: Methods and Applications*, **9**, 042, 26 pp.
- "Generalizations of generating functions for hypergeometric orthogonal polynomials with definite integrals," H. S. Cohl, Connor MacKenzie, and H. Volkmer, 2013, *Journal of Mathematical Analysis and Applications*, **407**, 2, 211-225, 15 pp.
- "Generalized generating functions for higher continuous orthogonal polynomials in the Askey scheme," M. A. Baeder, H. S. Cohl, H. Volkmer, 2014 (submitted).
- "Generalizations of generating functions for basic hypergeometric orthogonal polynomials in the q-Askey scheme," H. S. Cohl, R. S. Costas-Santos, P. R. Hwang, 2014 (in preparation).