AREA LAW DOES NOT GENERALIZE TO CUT-LAW

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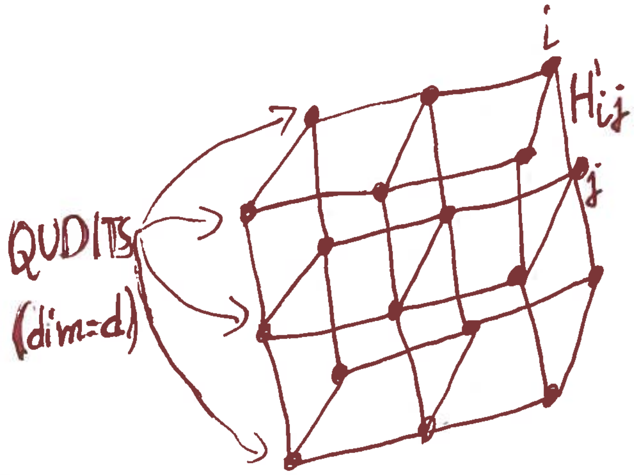
2-LOCAL HAMILTONIANS



\# OF NODES $= n$

\# OF EDGES $= m$

$$\dim(\mathcal{H}) = d^n$$

INTERACTION GRAPH G

$$\|H_{ij}\|_{op} = \text{STRENGTH OF INTERACTION}$$

$$H = \sum_{(i,j)\in G} H_{ij}$$

ASSUMPTION: EACH H_{ij} IS POSITIVE SEMIDEFINITE

ENERGY OF φ:

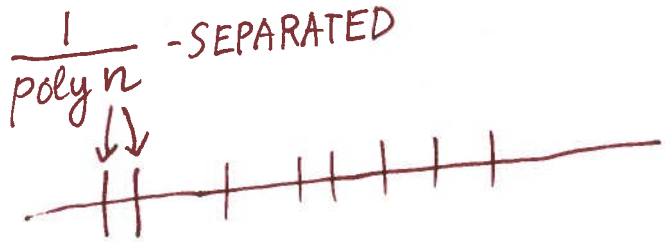
$$\sum_{(i,j)\in G} \underbrace{\langle \varphi | H_{ij} | \varphi \rangle}_{\text{NON NEGATIVE}}$$

GAPPED HAMILTONIANS

SPECTRUM

UNIQUE GROUND STATE:

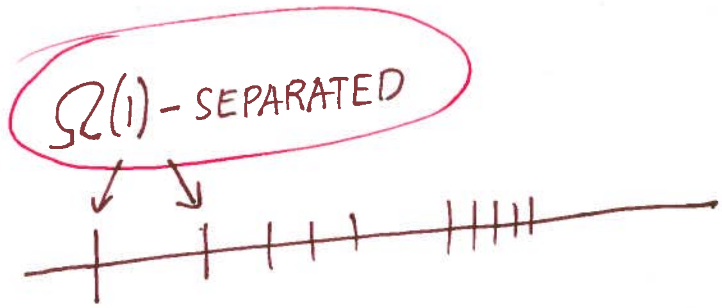
MORE UNIQUE GROUND STATE:



VERY UNIQUE GROUND STATE:

|| BY DEFINITION

<u>GAPPED HAMILTONIAN</u>



(NORMALIZATION: ALL INTERACTION STRENGTHS $\leq O(1)$)

n = # OF SITES = # OF NODES IN THE INTERACTION GRAPH G

$H = \sum_{ij} H_{ij}$; $\|H_{ij}\|_{op}$ = INTERACTION STRENGTH BTWN i AND j

ENTANGLEMENT ENTROPY

LET $\varphi \in \mathcal{A} \otimes \mathcal{B}$

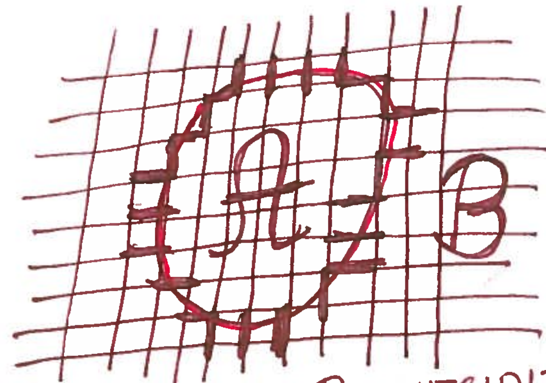
$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi) = \text{ENTROPY}\left(tr_{\mathcal{A}}\,\varphi\right) =$$

$$\text{ENTROPY}\left(tr_{\mathcal{B}}\,\varphi\right) =$$

$$-\sum \lambda_i \log \lambda_i$$

SCHMIDT COEFFICIENTS

AREA LAW CONJECTURE



$\mathcal{A}$ = INSIDE $\mathcal{B}$ = OUTSIDE

- H IS GAPPED, GROUND STATE = φ_0
- INTERACTION GRAPH IS k-DIMENSIONAL GRID

$$\Downarrow$$

$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi_0) \lesssim_k |\mathrm{CUT}(\mathcal{A},\mathcal{B})|$$

CUT($\mathcal{A},\mathcal{B}$) = {LOCAL OPERATORS CONNECTING $\mathcal{A}$ WITH $\mathcal{B}$}

RESULTS

① SEVERAL PHYSICISTS NOTICING THE AREA LAW FOR SPEC. SYSTEMS

② HASTINGS: MATHEMATICAL PROOF OF AREA LAW FOR 1-D SYSTEMS 

③ D. AHARONOV, I. ARAD, Z. LANDAU, U. VAZIRANI:
COMBINATORIAL APPROACH TO 1-D

④ I. ARAD ~~AHARONOV~~, A. KITAEV, Z. LANDAU, U. VAZIRANI:
SUB-EXPONENTIAL ALGORITHM FOR 1-D SYSTEMS
EXPONENTIAL IMPROVEMENT ($\exp(\frac{\log d}{\varepsilon}) \rightarrow \frac{\log^3 d}{\varepsilon}$) OVER HASTINGS

TECHNIQUES IMPLY

⑤ Z. LANDAU, U. VAZIRANI, T. VIDICK:
POLINOMIAL TIME ALGORITHM FOR 1-D SYSTEMS

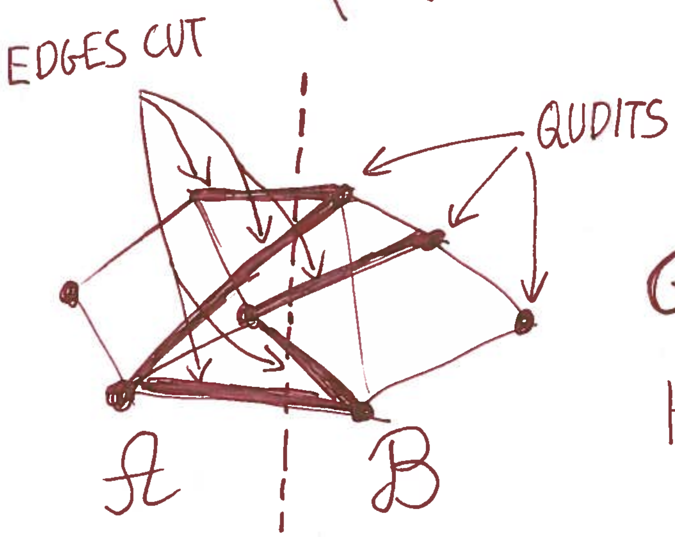
REVIEW ARTICLE BY J. EISERT, M CRAMER, M.B. PLENIO

OTHER RELEVANT LITERATURE:
SANDY IRANI
GOTTESMAN–HASTINGS

} TRADEOFF BTWN GAP & AMOUNT OF ENTANGLEMENT

WHEN THE LOCAL TERMS COMMUTE

$$\left(H_{ij} H_{jk} = H_{jk} H_{ij} \right)$$



$G = \underline{\underline{\text{ANY}}}$ INTERACTION GRAPH

$H = \sum_{(i,j)\in G} H_{ij}$ IS GAPPED

$$\text{SCHMIDT RANK}_{\mathcal{A},\mathcal{B}}(\varphi_0) \leq d^{|\text{CUT}(\mathcal{A},\mathcal{B})|}$$

$$\downarrow$$

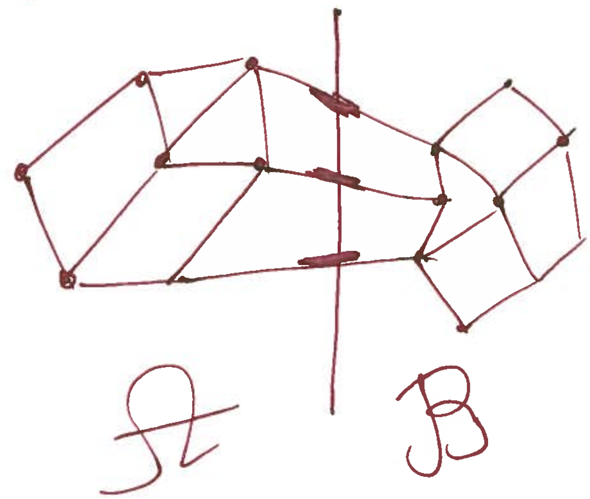
$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi_0) \lesssim_d |\text{CUT}(\mathcal{A},\mathcal{B})|$$

AREA LAW HOLDS, AND EVEN STRONGER

[SCHMIDT RANK = # OF NON-ZERO λ_iS IN $\varphi = \sum \lambda_i \underset{\mathcal{A}}{|a_i\rangle} \underset{\mathcal{B}}{|b_i\rangle}$]

GRAPH-AREA LAW CONJECTURE (FALSE)

$\forall$ LOCALITY STRUCTURE G

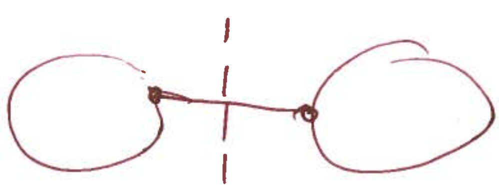


$H = \sum_{(i,j)\in G} H_{ij}$ GAPPED

$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi_0) \lesssim |\mathrm{CUT}(\mathcal{A},\mathcal{B})|$$

GROUND STATE OF H

OUR RESULT: FALSE IN GENERAL

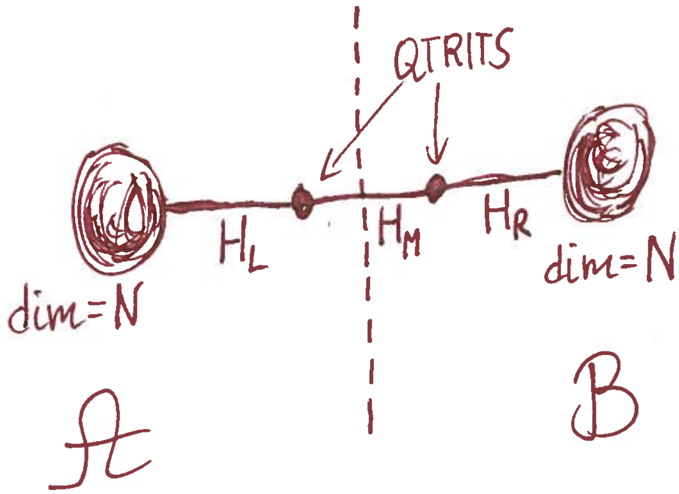


$|\mathrm{CUT}|=1$

$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi_0) \gg O(1)$$

PROOF OUTLINE:

FIRST STEP: CONSTRUCT

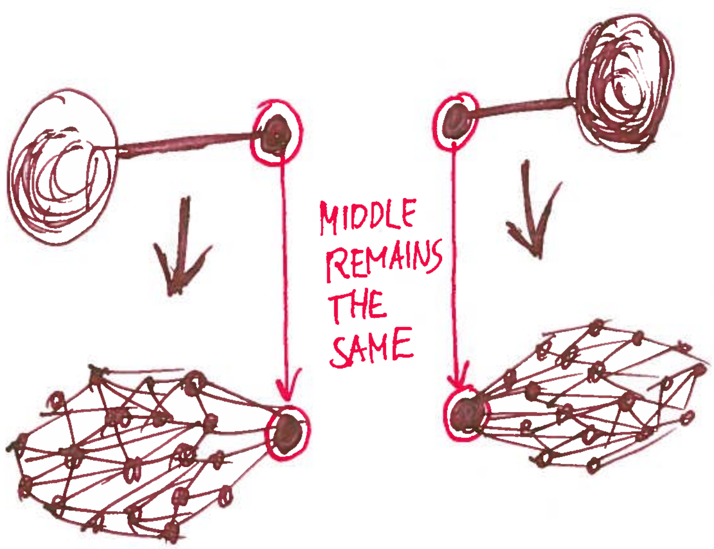


$$H = H_L + H_M + H_R$$

GAPPED

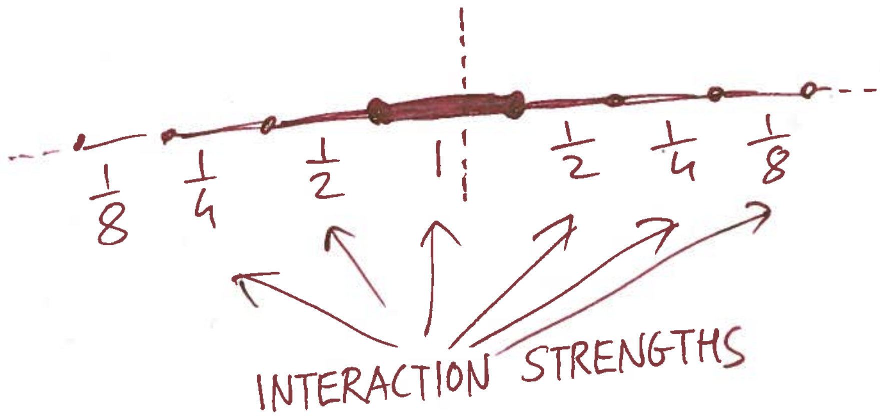
$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi_0) \approx \log N$$

SECOND STEP: REPLACE



(DETOUR)

FRANK'S EXAMPLE:



LARGE ENTANGLEMENT ACROSS MIDDLE LINE, BUT NO GAP. STRONG MIDDLE INTERACTION CREATES THE ENTANGLEMENT

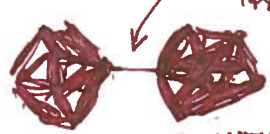
OUR EXAMPLE: MIDDLE IS (RELATIVELY) WEAK

PROPOSITION: IF $H_L + H_M + H_R$ (AS IN OUR CASE) IS AN UNFRUSTRATED EXAMPLE TO THE GRAPH AREA LAW NOT HOLDING THEN SO IS

$$\lambda H_L + H_M + \lambda' H_R$$

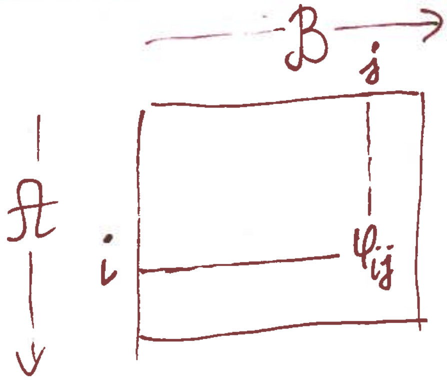
FOR ANY $\lambda, \lambda' \geq 1$.

MIDDLE WEAK INTERACTION CREATES A LOT OF ENTANGLEME



SIDE INTERACTIONS CAN BE MADE ARBITRARILY STRONG

RECTANGLE REPRESENTATION OF STATES IN $\mathcal{A}\otimes\mathcal{B}$



REPRESENTS $\varphi = \sum_{i,j} \varphi_{ij} \underset{\in \mathcal{A}}{|i\rangle} \underset{\in \mathcal{B}}{|j\rangle} \in \mathcal{A}\otimes\mathcal{B}$

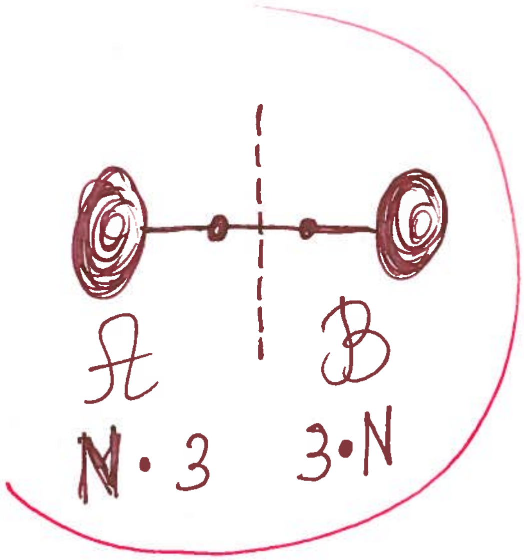
$$\sum |\varphi_{ij}|^2 = 1$$

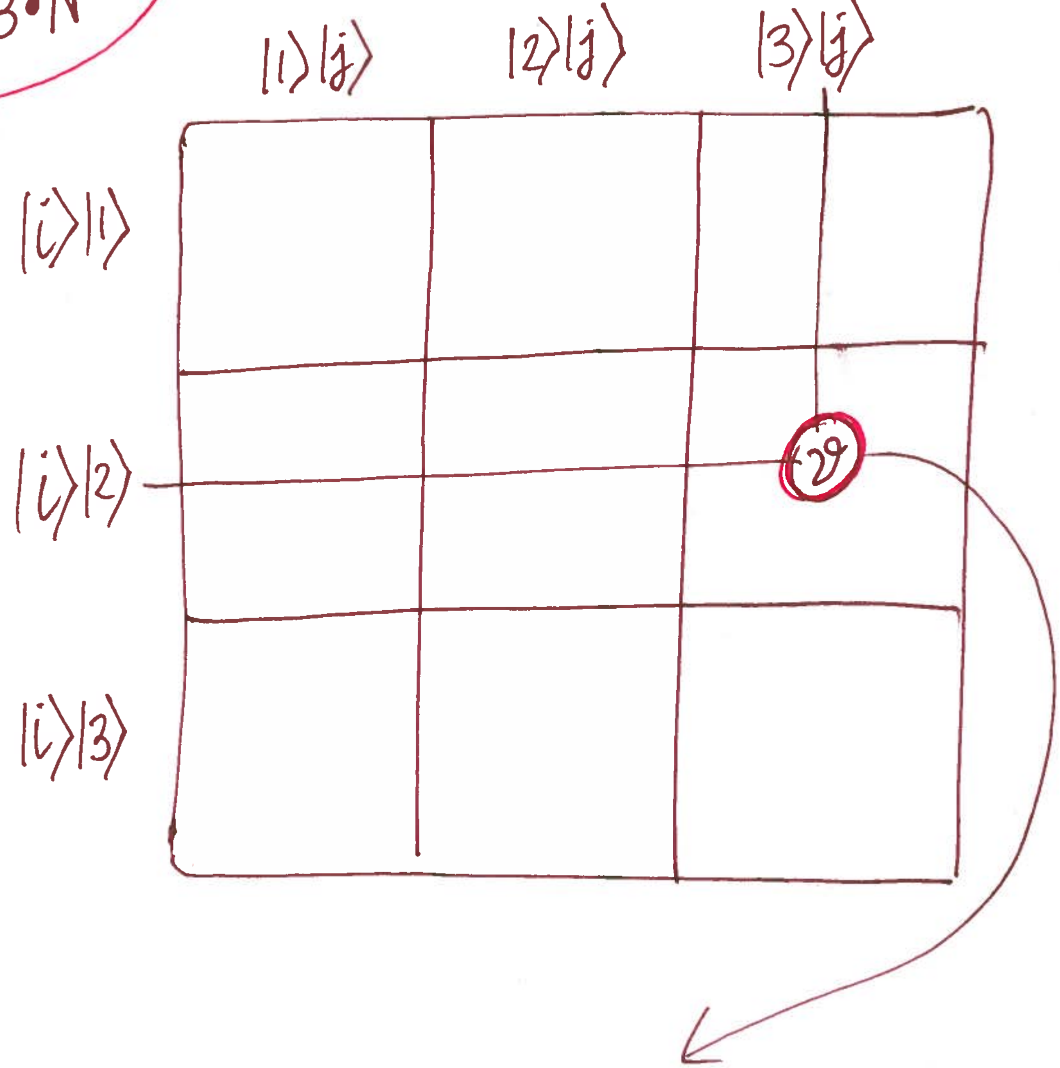
NOT TO BE CONFUSED WITH THE DENSITY MATRIX FOR φ!

FACT: $\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi) = -\sum \lambda_i \log \lambda_i$, WHERE

$\lambda_1, \dots, \lambda_m$ ARE THE SINGULAR VALUES OF $[\varphi_{ij}]_{ij}$

($[\varphi_{ij}]_{ij} = M \longrightarrow MM^*$ AND M^*M ARE THE REDUCED DENSITY MATRICES)





$$\varphi = \ldots + \underline{\underline{\vartheta}}\,|i\rangle|2\rangle|3\rangle|j\rangle + \cdots$$

$$N\times N \text{ MATRICES}$$

$$\mathbb{C}^{N\cdot 3} \geqslant \mathcal{S}_L = \left\{ (\underbrace{x}_{N}, \underbrace{Ax}_{+N}, \underbrace{Bx}_{+N}) \;\middle|\; x \in \mathbb{C}^N \right\}$$

$$\mathbb{C}^{3\cdot N} \geqslant \mathcal{S}_R = \left\{ (\underbrace{y^*}_{N}, \underbrace{y^*A}_{+N}, \underbrace{y^*B}_{+N}) \;\middle|\; y \in \mathbb{C}^N \right\}$$

$$\mathbb{C}^{3\cdot 3} \geqslant \mathcal{S}_M = \left\{ \begin{aligned} &a|1\rangle|1\rangle + \overbrace{b|1\rangle|2\rangle + b|2\rangle|1\rangle}^{\text{SAME}} + \overbrace{c|1\rangle|3\rangle + c|3\rangle|1\rangle}^{\text{SAME}} + \\ &d|2\rangle|2\rangle + e|2\rangle|3\rangle + f|3\rangle|2\rangle + g|3\rangle|3\rangle \end{aligned} \right\}$$

$$\left.\begin{aligned} H'_L &= P_{\mathcal{S}_L} \\ H'_M &= P_{\mathcal{S}_M} \\ H'_R &= P_{\mathcal{S}_R} \end{aligned}\right\} \text{PROJECTORS}$$

$$H_L = I_{\mathcal{B}} \otimes H'_L, \quad \text{e.t.c.}$$

GROUND STATE φ_0 OF $H_L + H_M + H_R$

$$\in \mathcal{S}_L \otimes \mathcal{B} \;\cap\; \mathcal{S}_R \otimes \mathcal{A} \;\cap\; \mathcal{S}_M \otimes \mathcal{C}$$

$$\varphi_0 = \begin{array}{|c|c|c|} \hline X & XA & XB \\ \hline AX & AXA & AXB \\ \hline BX & BXA & BXB \\ \hline \end{array}$$

MUST BE OF THIS FORM

$$\text{WITH} \quad \begin{cases} AX = XA \\ BX = XB \end{cases}$$

AND THIS MUST HOLD

GAPPED! ✓

$$\varphi_0 = \boxed{\begin{matrix} I & A & B \\ A & A^2 & AB \\ B & BA & B^2 \end{matrix}}$$

IS A GROUND STATE (WHEN NORMALIZED)

IF (A,B) IS A QUANTUM EXPANDER, I.E.

i.) A, B ARE UNITARY

ii.) $\forall\, X$ WITH $tr(X) = 0$, $\|X\|_{FROB} = 1$:

$$|X + AXA^{-1} + BXB^{-1}| < 3 - \varepsilon$$

THEN EVERY ψ WHICH IS ORTHOGONAL TO φ_0 WE HAVE:

$$\langle \psi | H_L | \psi \rangle + \langle \psi | H_M | \psi \rangle + \langle \psi | H_R | \psi \rangle \geq \frac{\varepsilon}{10000}$$

HUGE PENALTY FOR ANY ψ $(\langle \psi | \varphi_0 \rangle = 0)$

OF THE FORM $\begin{pmatrix} Z & ZA & ZB \\ AZ & AZA & AZB \\ BZ & BZA & BZB \end{pmatrix}$ BY EXPANDER PROPERTY

ENTANGLED

$$\varphi_0 = \begin{pmatrix} I & A & B \\ A & A^2 & AB \\ B & BA & B^2 \end{pmatrix} \cdot \frac{1}{3\sqrt{N}}$$

$$\mathcal{E}_{\mathcal{A},\mathcal{B}}(\varphi_0) \simeq \log N$$

SECOND STEP: REPLACE

LOCALIZING PROJECTIONS

- $\mathcal{H} = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$

  SITES

- $\mathcal{S} \leq \mathcal{H}$

- WANT TO CREATE A LOCAL GAPPED** HAMILTONIAN WITH GROUND SUBSPACE $\mathcal{S}$. MUST BE FRUSTRATION-FREE

** $\varphi \in \mathcal{S}^{\perp}$ MUST HAVE $\Omega(1)$ ENERGY

CAVEATS:

① NEED TO ADD AN ANCILLA SPACE

② NEED TO ADD ERROR

WHAT WE GET IS AN H
WITH GROUND SUBSPACE
SPANNED BY

$$\sqrt{\varepsilon}\,\phi_i' + \sqrt{1-\varepsilon}\,\boxed{\phi_i} \otimes \psi$$

ARBITRARILY
SMALL POSITIVE CONST.

ERROR VECTOR
(UNIT)

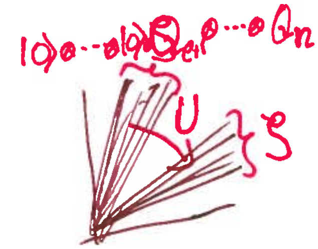
ORTHOGONAL
UNIT VECTORS
THAT SPAN S

FIXED UNIT VECTOR
IN THE AUXILIARY
SPACE

CONSTRUCTION OF H

ASSUME THAT $\dim(\mathcal{S}) = \dim(Q_{\ell+1}) \cdot \ldots \cdot \dim(Q_n)$

LET U BE A UNITARY S.T.



$$U \underbrace{|0\rangle|0\rangle\cdots|0\rangle}_{\ell} \underbrace{Q_{\ell+1}Q_{\ell+2}\cdots Q_n}_{\dim_1} = \underbrace{\mathcal{S}}_{\dim_2}$$

$$\dim_1 = \dim_2$$

- BUILD A QUANTUM ~~COMPUTER~~ CIRCUIT THAT COMPUTES U
- CONVERT THE CIRCUIT INTO A LOCAL HAMILTONIAN AS IN KITAEV
- CAVEAT: MAKE SURE THAT AFTER THE ACTUAL COMPUTATION TERMINATES AFTER τ STEPS, THERE ARE ADDITIONAL $T-\tau$ STEPS WHEN THE CIRCUIT REMAINS IN THE SAME STATE

- MULTIPLY ALL INTERACTIONS BY A LARGE ENOUGH NUMBER α TO ACHIEVE CONSTANT GAP $(\alpha = \alpha(N, \varepsilon))$

- APPLY THE NAGAJ-CAO GADGET TO REDUCE EACH LOCAL INTERACTION STRENGTH TO CONSTANT (THIS WILL MAKE THE DEGREE OF THE INTERACTION GRAPH UNBOUNDED)

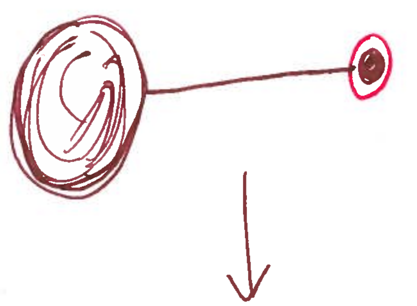
GROUND STATES (WITHOUT THE NAGAJ-CAO GADGET):

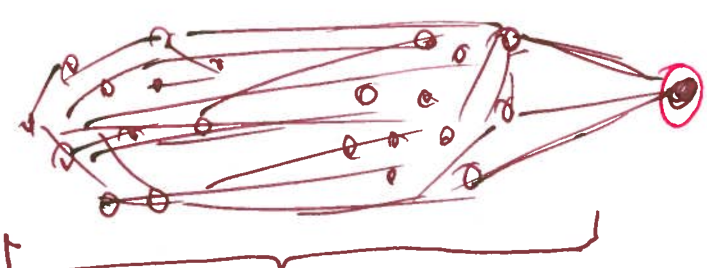
$x \in Q_{\ell+1} \otimes \cdots \otimes Q_n \qquad \phi_x = U \underbrace{|0\rangle \cdots |0\rangle}_{\ell} |x\rangle$

$$\underbrace{\frac{1}{\sqrt{T}} \sum_{t=1}^{\tau} |\varphi_{x,t}\rangle}_{\text{ERROR TERM}} + \underbrace{|\phi_x\rangle}_{\in \mathcal{S}} \otimes \underbrace{\underbrace{\frac{1}{\sqrt{T}} \sum_{t=\tau+1}^{T} |t\rangle}_{\text{CLOCK}} \otimes \underbrace{|0 \cdots 0\rangle}_{\text{ANCILLA}}}_{\psi \ \text{INDEPENDENT OF } x}$$

APLICATION TO H_L

(H_R IS SIMILAR)





OPEN

① OUR LOCAL HAMILTONIAN HAS LARGE INTERACTION STRENGTH OR THE INTERACTION GRAPH HAS UNBOUNDED DEGREE.
MAKE BOTH BOUNDED

② PROVE OR DISPROVE THAT THE MIDDLE ~~HAMILTONIAN~~ INTERACTION SHOULD BE BETWEEN QUDITS WITH $d \geq 3$