Space-Time Circuit-to-Hamiltonian construction and its applications

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Circuit-to-Hamiltonian (Feynman-Kitaev)

Mapping from time-dependent circuit to the ground-state of a Hamiltonian H.

Circuit has n qubits and gates U_1, ….U_L. Introduce a clock register |t>: |t=0>….|t=L> and let

$$H_{circuit} = \sum_{t=1}^{L} \left(-U_t \otimes |t\rangle\langle t-1| + h.c. + |t-1\rangle\langle t-1| + |t\rangle\langle t|\right)$$

Ground-state of H_{circ} is history state of the circuit (for any ξ)

$$|\varphi_{his}\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} U_t \ldots . U_1 \, |\xi\rangle \otimes |t\rangle$$

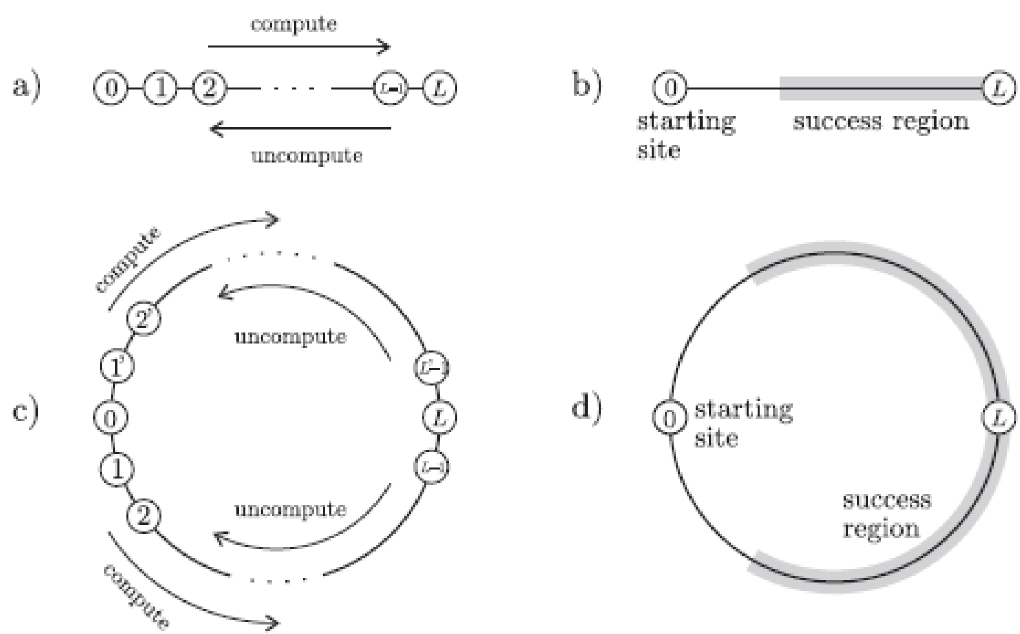
Features of Circuit-to-Hamiltonian Mapping

- View as particle hopping on a time-line, gate U_t is executed when particle hops from position t-1 to t.
- Realize clock register using a domain wall clock (e.g. $|111100000\rangle$) or particle clock (e.g. $|0000100000\rangle$ so that $|t-1\rangle\langle t|, |t\rangle\langle t|$ acts on O(1) (3 resp. 2) clock qubits.
- Read out answer of computation from history state
- Spectrum of H, independent of gates: $E_k \propto 1 - \cos(\frac{\pi k}{L+1})$. Gap $\Delta \geq \Theta({}^{1}/_{L^2})$

$$|\varphi_{his}\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=0}^{L} U_t \ldots . U_1 \, |\xi\rangle \otimes |t\rangle$$

Use of Circuit-to-Hamiltonian Mapping I: universal continuous quantum walk

Quantum Walk (e.g. Nagaj) on a line or circle:
Start walk in t=0 state, evolve with $H_{circuit}$ for random time s$\sim L^2$ (L gates in original quantum circuit) such that one approximately samples a time from the (uniform) distribution.
If we finds a time in the output region, output of entire computation is available.



Use of Circuit-to-Hamiltonian Mapping II

Proof that quantum adiabatic computation is equivalent to quantum computation with a circuit model.

$$H_{circuit} = \sum_{t=1}^{L} (-U_t \otimes |t\rangle\langle t-1| + h.c. + |t-1\rangle\langle t-1| + |t\rangle\langle t|)$$

Quantum Adiabatic Computation with $H_{circuit}(t)$ such that at t=0, $H_{circuit}(t=0) = H_{circuit}(U_1 = I, \dots, U_L = I)$ and $H_{circuit}(t=T) = H_{circuit}(U_1, \dots, U_L)$.

Adiabatic theorem applies as Gap $\Delta(t) \geq \Theta({}^1/_{L^2})$

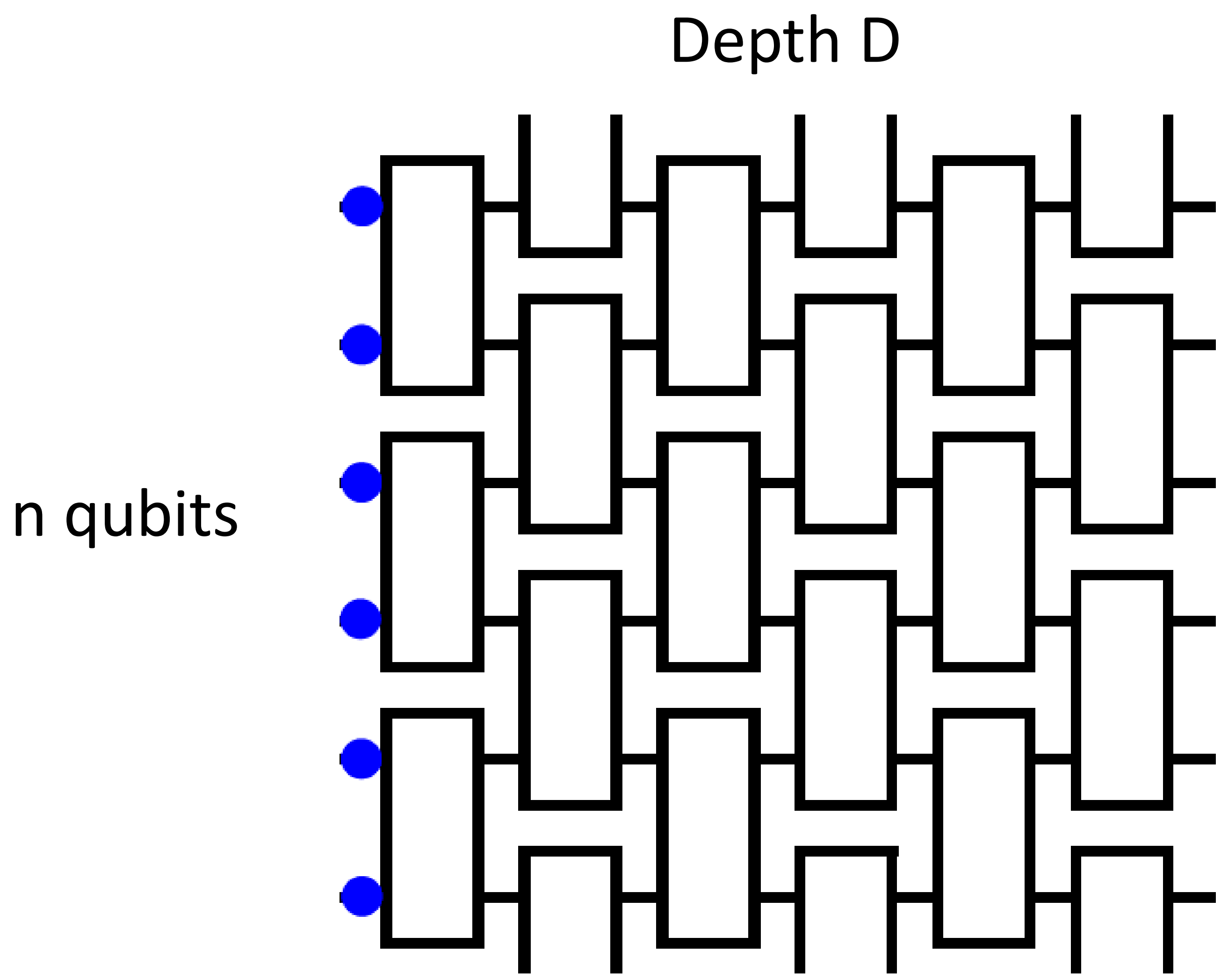
(Original construction uses linear interpolation: $H(t) = tH_{circuit} + (1-t)H_{init}$)

Use of Circuit-to-Hamiltonian Mapping III

Quantum Cook-Levin Theorem (Kitaev): proof that determining the lowest energy of a n-qubit Hamiltonian with 1/poly(n) accuracy is QMA-complete, i.e. hard for quantum computers.

Idea: any problem in QMA (quantum NP) has a verification circuit which (approximately) outputs 0 or 1 depending on input being a valid proof.

Construct $H = H_{circuit} + H_{input} + H_{output}$ which has low energy state iff there exists an input (proof) such that circuit outputs 1 and only high-energy states when circuit outputs 0.



For simplicity, we assume we have a 1D quantum circuit with nearest-neighbor interactions (and periodic boundary conditions). Such circuit is universal for computation if D=poly(n).

A different construction?

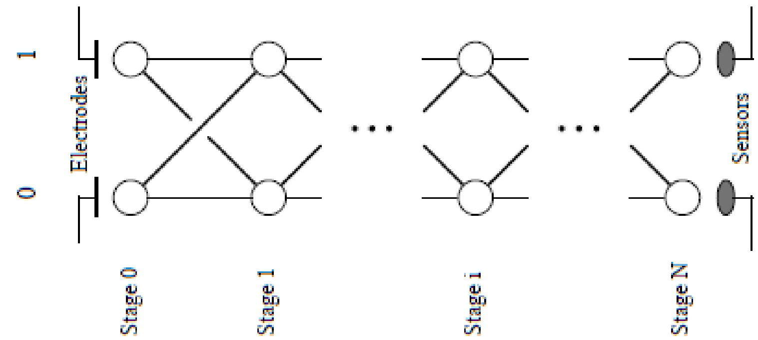
Mizel et al. ‘Ground State Quantum Computation’ in 1999 & PRL 99, 070502 (2007)) consider a fermionic Hamiltonian (for adiabatic QC) with the following features:

A qubit q in a quantum circuit of depth D is represented by 2(D+1) fermionic modes, $a_i(q), b_i(q), i = 0 \dots D$

For example: electron in left/right quantum dot or electron spin.

Particles (fermions) can hop on this track of length D.

Track of length D

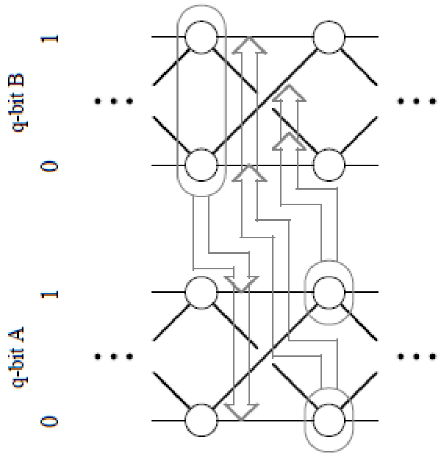


Construct $H_{circuit}$ such that

- Single qubit gate U is represented by a single particle hopping on such track (and changing its internal state).

Mizel et al. construction

- Two-qubit gate C(U), e.g. CNOT is represented as pairs of particles hopping together. They can only move forward or backward when they are both 'at the gate'.



New circuit-to-Hamiltonian construction

If quantum circuit is 1D, then Mizel et al. Hamiltonian $H_{circuit}$ is an interacting fermion 2D Hamiltonian (or 2D qubit Hamiltonian with 4 qubit interactions).

Properties of this construction are not well understood.

We show that their construction is an example of a general space-time circuit-to-Hamiltonian construction through which we can get new QMA (quantum adiabatic QC & quantum walk) results.

Space-Time Circuit-to-Hamiltonian

Define a clock for each qubit q: $|t_q = 0, \dots, D\rangle$.

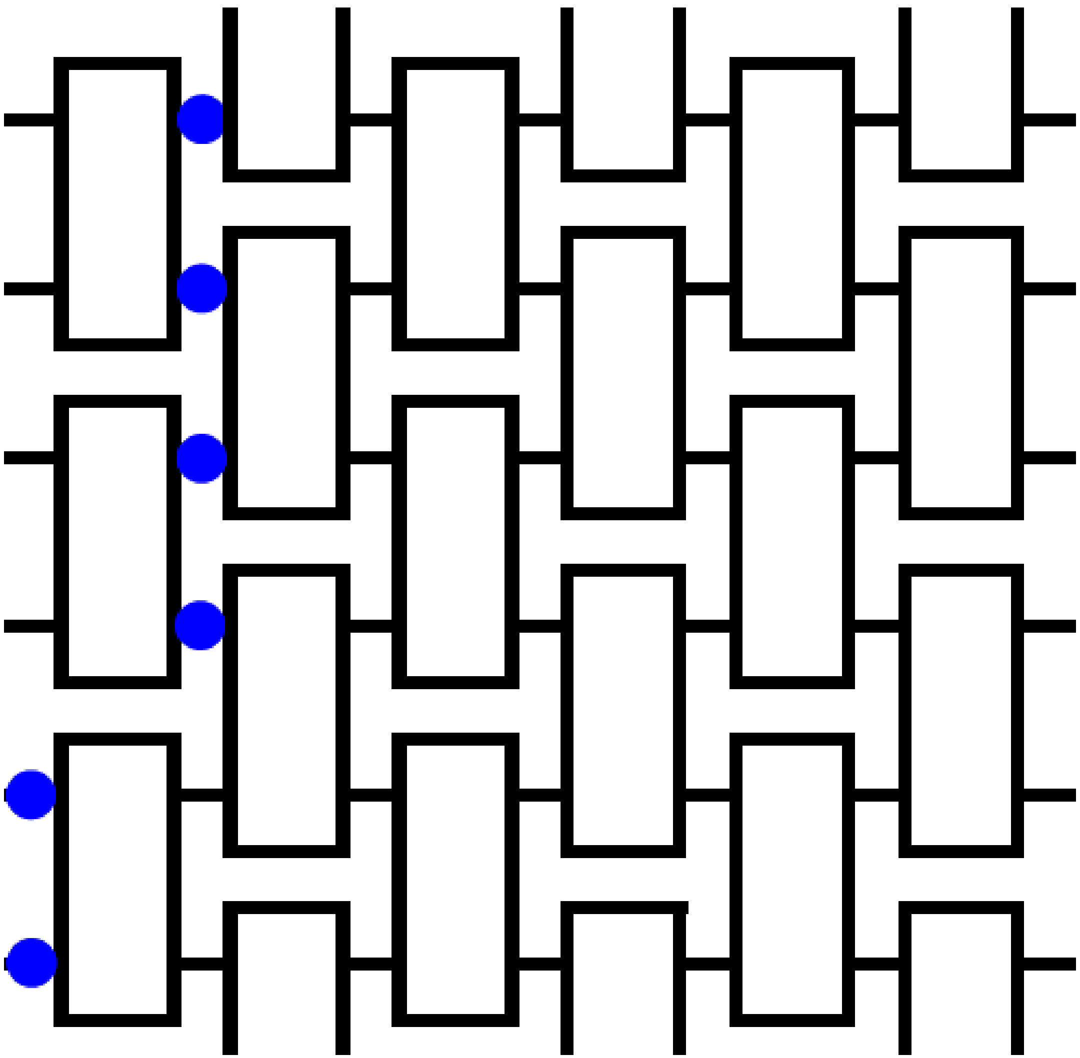
Time configuration $\boldsymbol{t} = (t_1, t_2 \dots . t_n)$

Term in $H_{circuit}$ for a two-qubit gate U on *qubit q, p* at time s+1

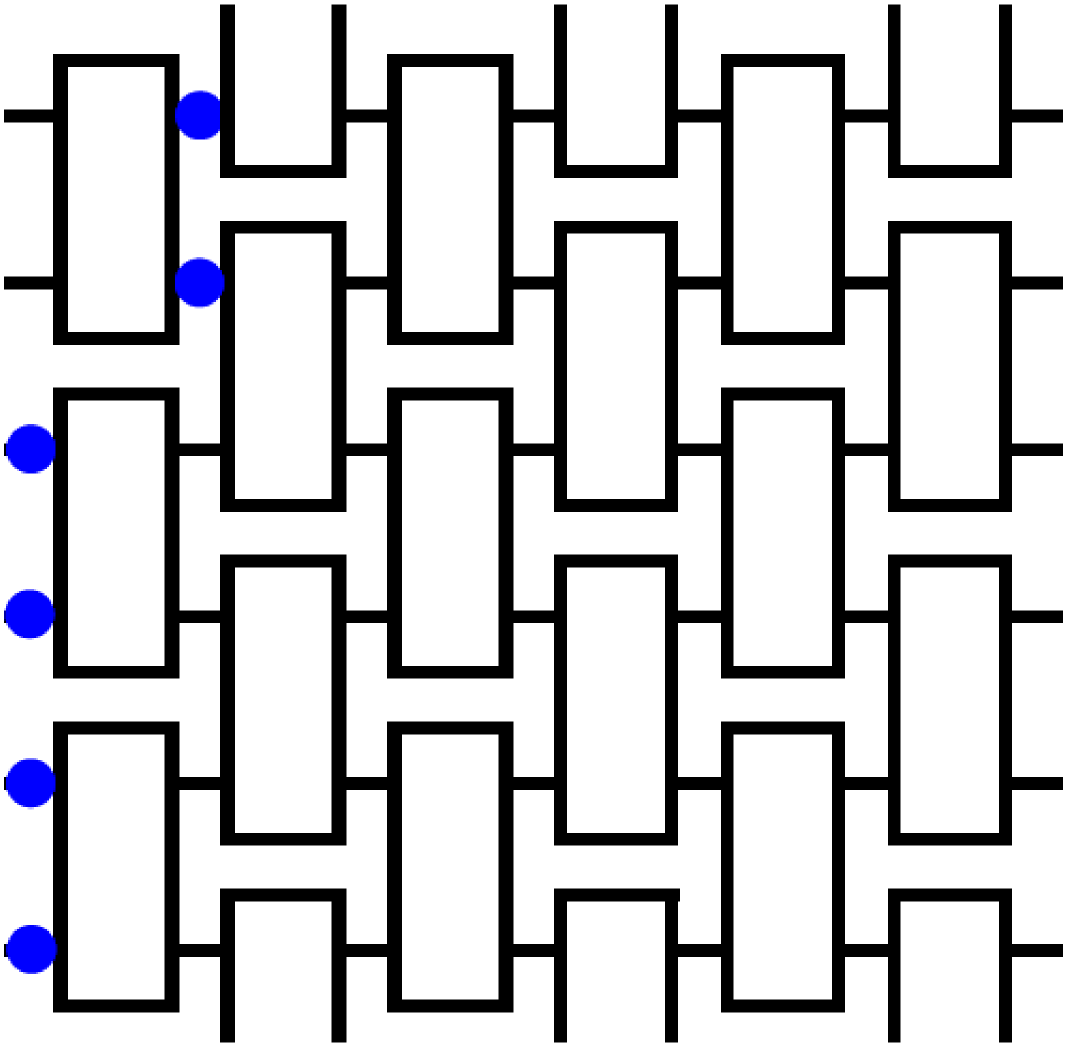
$$-U \otimes \left|t_p = s+1, t_q = s+1\right\rangle\!\left\langle t_c = s, t_q = s\right| + h.c.$$
$$+|t_p = s, t_q = s\rangle\langle t_p = s, t_q = s|$$
$$+|t_p = s+1, t_q = s+1\rangle\langle t_p = s+1, t_q = s+1|$$

"Times of interacting qubits are moved ahead/backward if they are synchronized".

The previous fermionic model effectively corresponds to a certain clock realization (is thus unitarily equivalent).

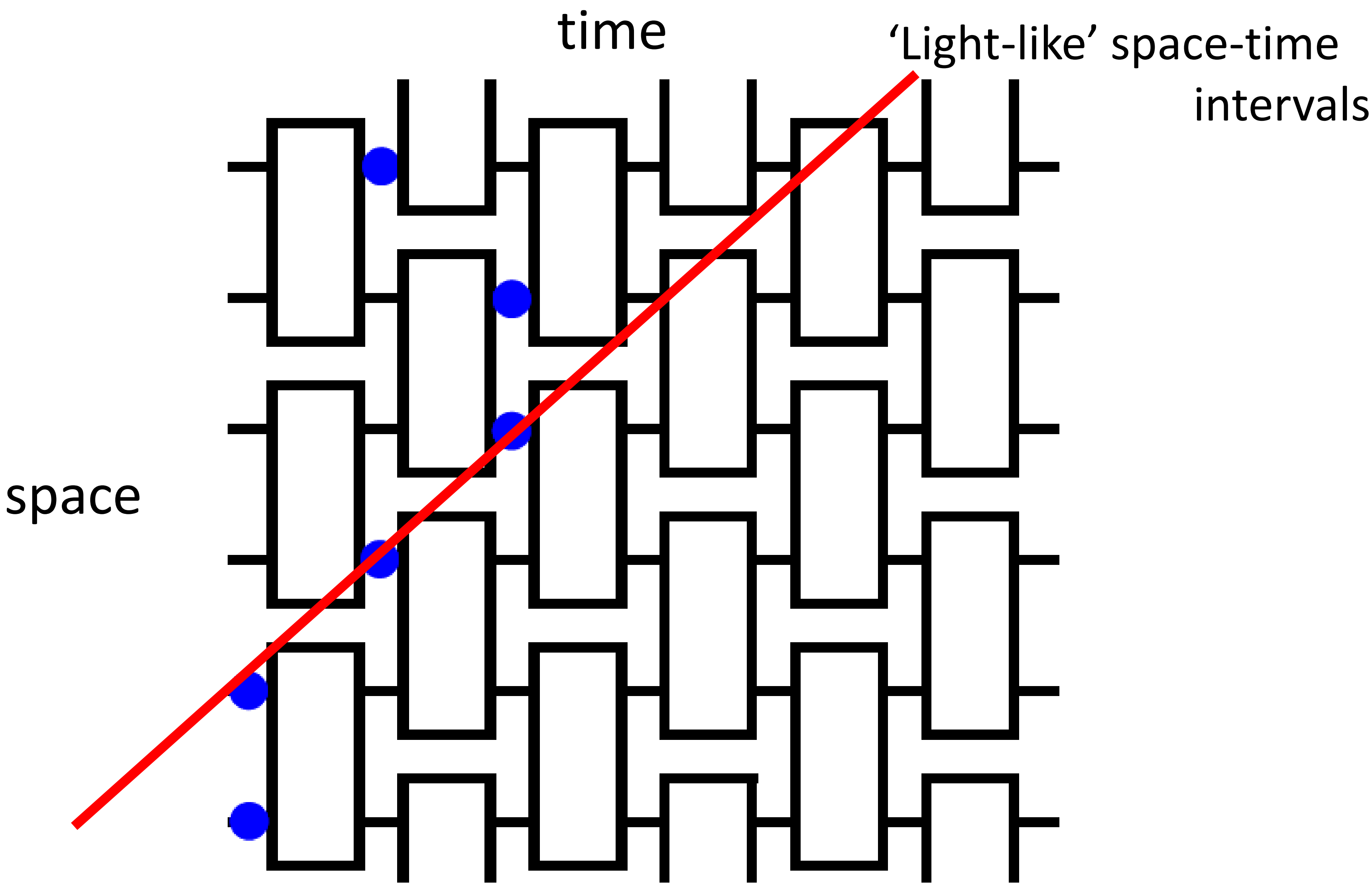


Possible time-configuration

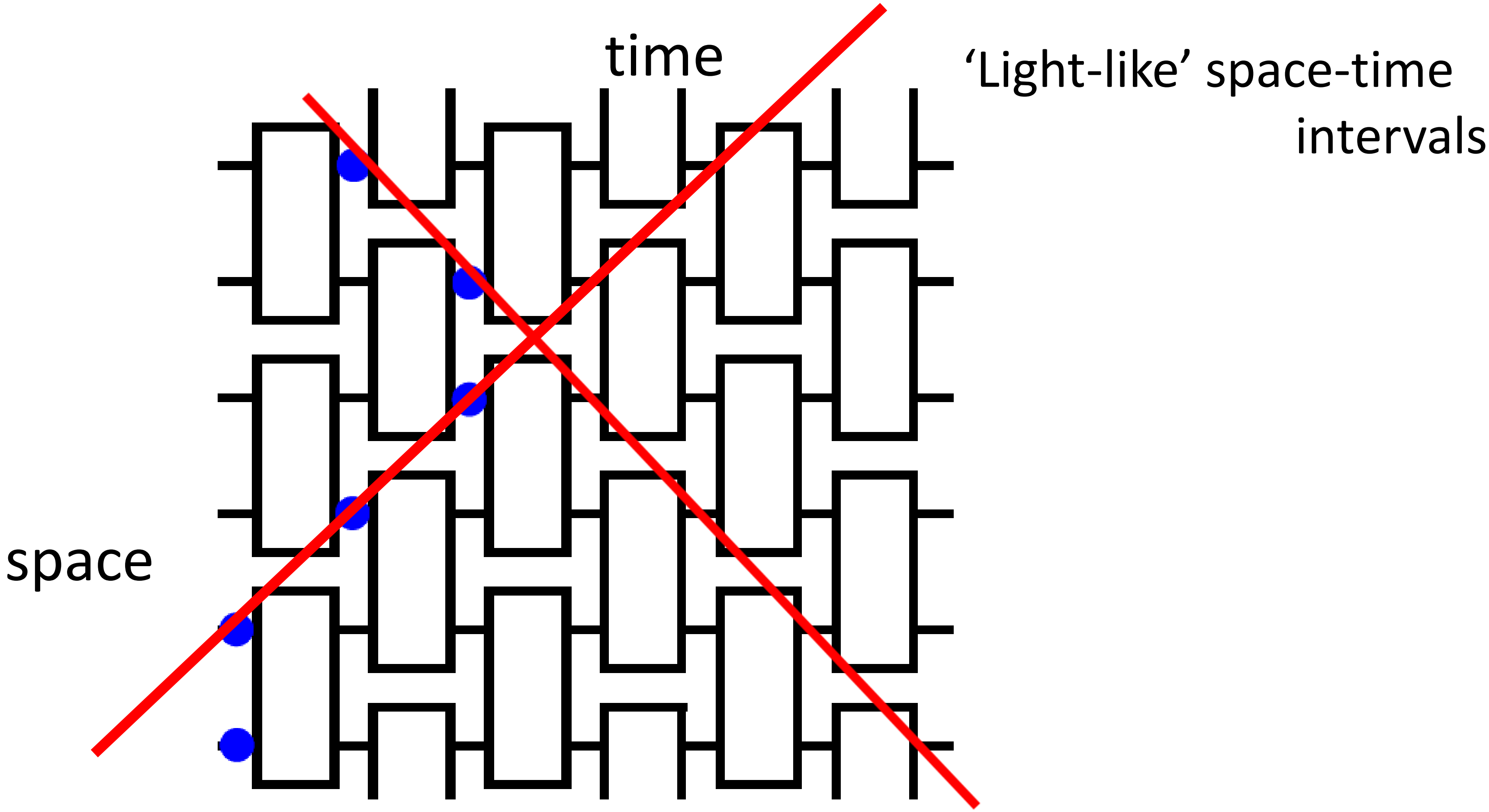


Definition valid time-configuration ***t*** (informally): for no pair of qubits interacting at time t in the circuit is the clock of one qubit past t and the clock of the other qubit before t.

$H_{circuit}$ preserves the subspace of valid time-configurations



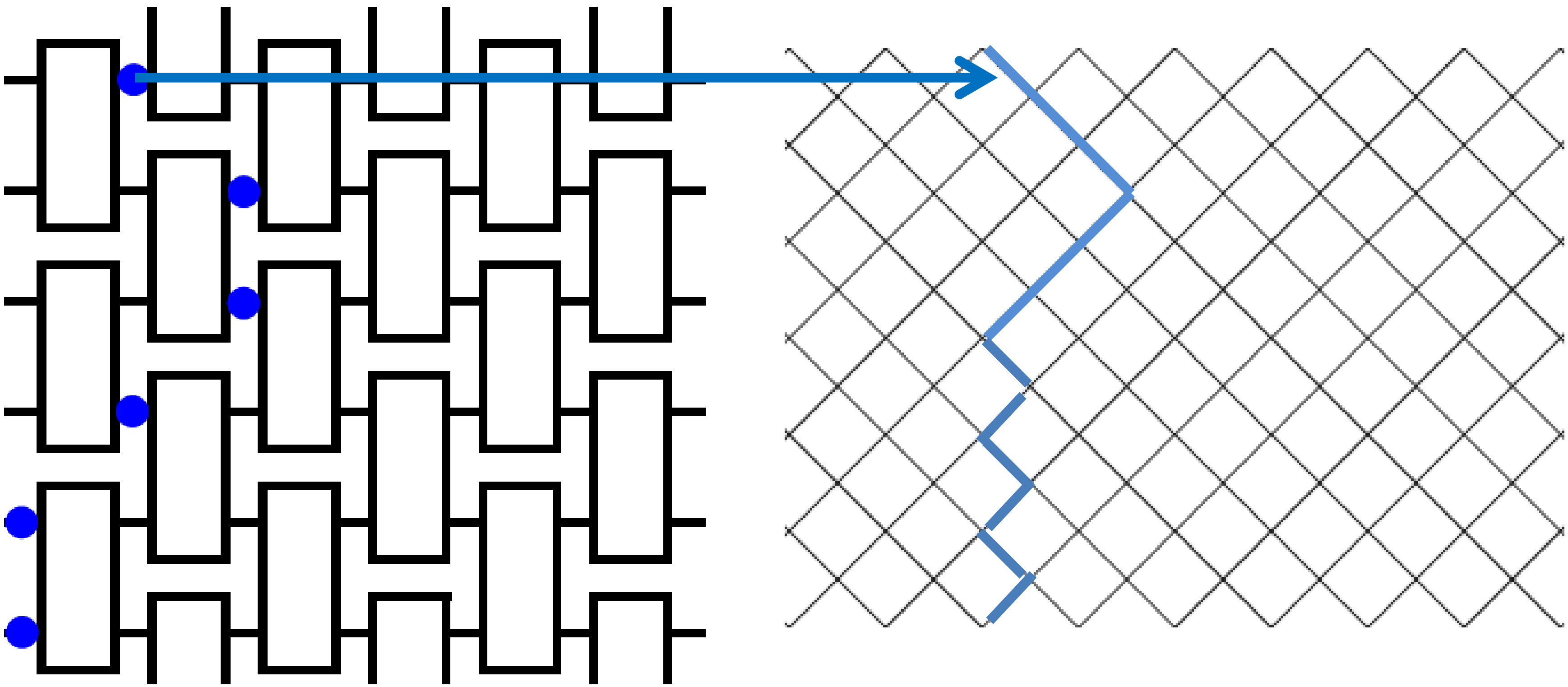
A valid time-configuration has only space-like or light-like space-time intervals while for an invalid time-configuration some intervals are time-like (with respect to a 2D Minkowski metric $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$).



Zero-energy pure light-like **t** can be avoided. The ground-state of the circuit Hamiltonian equals history state
$\sum_{proper\ \boldsymbol{t}} V(\boldsymbol{t} \leftarrow \boldsymbol{0})|\xi\rangle|\boldsymbol{t} = t_1 \ldots t_n\rangle$.
with $V(\boldsymbol{t} \leftarrow \boldsymbol{0})$ those unitaries which are applied to go from **0** (all clocks reading t=0) to time-string **t**.

A very useful representation

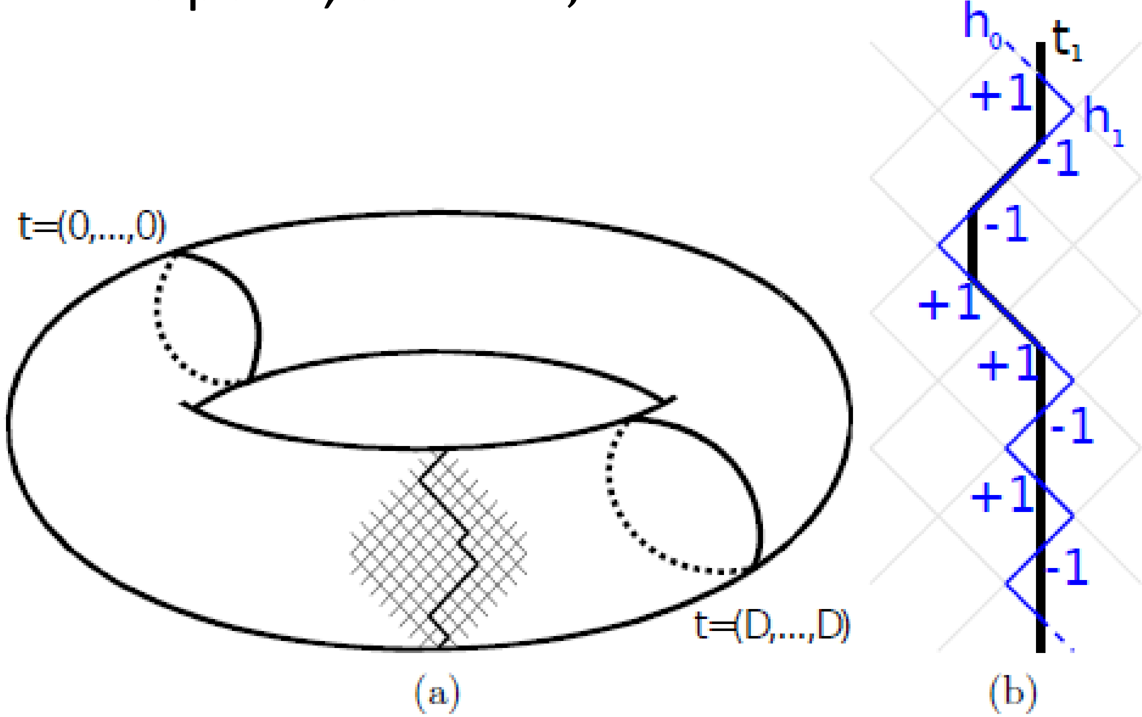
Valid time-configurations are closed strings on a cylinder if we have a one-dimensional circuit on a circle.



Picture on the right: each gate is represented as square and blue string is vertex in the graph. Points in time are on the edges. Strings (vertices) are connected by transitions  $\leftrightarrow$  .

Diffusion of a String on a Torus

We will consider a circuit Hamiltonian with periodic boundary in time (original circuit has beginning and an end) for technical reasons. String $\boldsymbol{t}$ is equivalently described by boundary point τ and $x_i = \pm 1$ deviations from this point, so $\boldsymbol{t} \equiv \tau, x$



Interesting model: 2D growth model/diffusion of domain wall of Ising ferromagnet at T=0.

Results

The circuit Hamiltonian can be represented in the relabeled basis $|\tau, x>$ and we can define plane-wave eigenstates

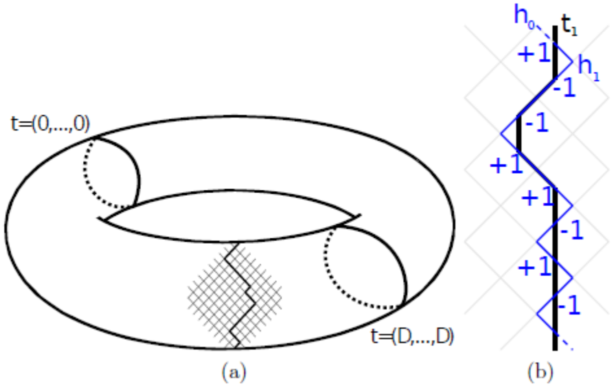
$$|\psi_k\rangle = \frac{1}{\sqrt{D}} \sum_{\tau \in Z_D} e^{2\pi i\ k\tau/D} |\tau\rangle, k = 0..D-1$$

such that $H_{circuit}|\psi_k\rangle|\xi\rangle = |\psi_k\rangle H(k)|\xi\rangle$, where *H(k)* is a 1D spin-1/2 Heisenberg chain with a twisted (k-dependent) boundary (and $\sum_i Z_i = 0$).

Theorem: $\lambda_1(H_{circuit}) \geq \frac{\pi^2}{4D^2(n-1)n} + O\left(\frac{1}{n^4 D^2}\right)$.

(For 1D circuits of the form given and periodic boundaries in time)

Scaling as $1/S^2$ with S the size of circuit.



Application for QMA

Using this lowerbound on the gap, we can prove that (informally)

“determining the lowest eigenvalue of a two-dimensional interacting fermion model (periodic boundary conditions in both directions) in the sector where there is one fermion per line is QMA-complete.”

To prove this one needs a.o. to add spatially-local terms to Hamiltonian in order to penalize improper time-configuration (realized by blue quartic operators in the picture)

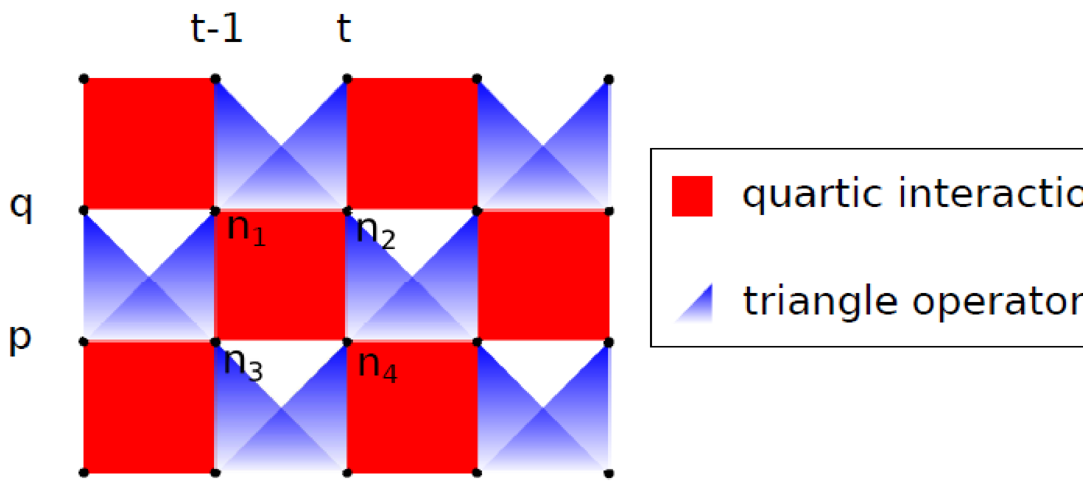


Figure 5: The black dots are fermionic sites, each with two modes (an ↑ or ↓ s say). The (red) squares represent the quartic gate interactions and the (blue) tria operators penalize improper fermionic configurations (improper time-configuratio A (blue) triangle operator with top corner a and bottom corners b and c eq $n_a(1-n_b-n_c)$. The lattice has periodic boundary conditions in both directions.

Some Open Questions

- Complexity Perspective: how does gap depend on geometric structure of the quantum circuit (i.e. D-dimensional circuits, expander circuits, MERA circuits).

- Can this model be useful for a 2D fermionic universal quantum walk? Yes, potentially but it depends on some more properties of the spectrum (ongoing work).