

Generalizations of generating functions for hypergeometric and q -hypergeometric orthogonal polynomials

Howard S. Cohl*

*Applied and Computational Mathematics Division,
National Institute of Standards and Technology, Gaithersburg, Maryland

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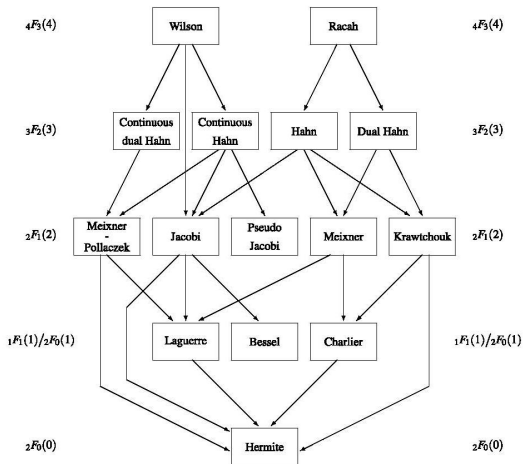
Generalized hypergeometric orthogonal polynomials

- **Hans Volkmer**, University of Wisconsin-Milwaukee
- **Michael A. Baeder**, student Harvey Mudd College

Basic hypergeometric orthogonal polynomials

- **Roberto Costas-Santos**, Universidad de Alcalá, Spain
- **Philbert Hwang**, student at Poolesville High School

ASKEY SCHEME
OF
HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS



calculus

Pochhammer symbol

$$(a)_n := (a)(a+1)\dots(a+n-1), \quad (a)_0 := 1$$

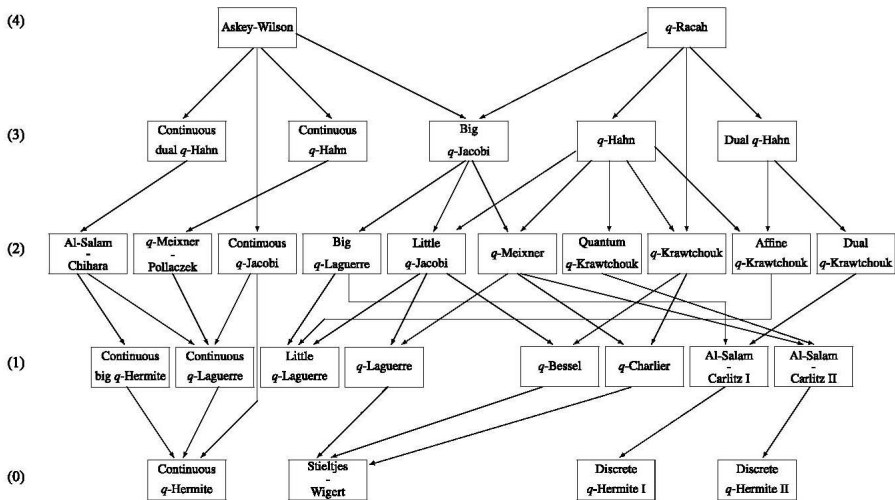
Notation

$$(a_1, \dots, a_r)_n := (a_1)_n \dots (a_r)_n$$

Generalized hypergeometric series

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n z^n}{(b_1, \dots, b_s)_n n!}$$

SCHEME OF BASIC HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS



q-calculus

q-Pochhammer symbol

$$(a; q)_n := (1 - a)(1 - aq) \dots (1 - aq^{n-1}), \quad (a; q)_0 := 1$$

Notation

$$(a_1, \dots, a_r; q)_n := (a_1; q)_n \dots (a_r; q)_n$$

Basic hypergeometric series

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} z^n$$

$${}_{s+1}\phi_s \left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{s+1}; q)_n}{(b_1, \dots, b_s; q)_n} \frac{z^n}{(q; q)_n}$$

Generalizations of generating function families

Generalized hypergeometric orthogonal polynomials

- Laguerre polynomials
- Jacobi, Gegenbauer, Chebyshev and Legendre polynomials
- Wilson polynomials
- Continuous Hahn polynomials
- Continuous dual Hahn polynomials
- Meixner-Pollaczek polynomials

Basic hypergeometric orthogonal polynomials

- | | |
|--|--|
| ■ q -ultraspherical
/Rogers polynomials | ■ Al-Salam-Carlitz II |
| ■ q -Laguerre polynomials | ■ big/little q -Jacobi polynomials (ongoing) |
| ■ big q -Laguerre | ■ Stieltjes-Wigert polynomials (ongoing) |
| | ■ Askey-Wilson (ongoing) |

Example: Al-Salam-Carlitz II generating function

Definition (orthogonal, monic):

$$V_n^{(\alpha)}(x; q) = (-\alpha)^n q^{-\binom{n}{2}} {}_2\phi_0 \left(\begin{matrix} q^{-n}, x \\ - \end{matrix}; q, \frac{q^n}{\alpha} \right), \quad \alpha < 0$$

Connection relation (1 free parameter)

$$V_n^{(\alpha)}(x; q) = \sum_{k=0}^n c_{n,k}(\alpha; \beta) V_k^{(\beta)}(x),$$

where

$$c_{n,k}(\alpha; \beta) := \frac{\beta^{n-k} q^{k(k-n)} (q^{k-n+1} \alpha / \beta; q)_{n-k} (q^{n-k+1}; q)_k}{(q; q)_k}$$

Generating function:

$$(\alpha\rho; q)_\infty {}_1\phi_1 \left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho \right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} \rho^n}{(q; q)_n} V_n^{(\alpha)}(x; q)$$

Example: Al-Salam-Carlitz II (cont.)

Therefore:

$$\begin{aligned}
 (\alpha\rho; q)_\infty {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)}\rho^n}{(q; q)_n} \\
 &\times \sum_{k=0}^n \frac{\beta^{n-k}q^{k(k-n)}(q^{k-n+1}\alpha/\beta; q)_{n-k}(q^{n-k+1}; q)_k}{(q; q)_k} V_k^{(\beta)}(x)
 \end{aligned}$$

Reverse the **order of summation** and let $n \mapsto n + k$

$$\begin{aligned}
 (\alpha\rho; q)_\infty {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) &= \sum_{k=0}^{\infty} \frac{\rho^k V_k^{(\beta)}(x, q)}{(q; q)_k} \\
 &\times \sum_{n=0}^{\infty} \frac{\beta^n q^{-nk}(\alpha q^{1-n}/\beta; q)_n (q^{n+1}; q)_k q^{(n+k)(n+k-1)} \rho^n}{(q; q)_{n+k}}
 \end{aligned}$$

Example: Al-Salam-Carlitz II (cont.)

Using the **properties** of q -**Pochhammer** symbols:

$$(a; q)_{n+k} = (a; q)_n (aq^n, q)_k,$$

$$(aq^{-n}; q)_n = (q/a; q)_n (-a)^n q^{-n - \binom{n}{2}}, \quad a \neq 0,$$

one obtains after **cancellation**

$$(\alpha\rho; q)_\infty {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{k=0}^{\infty} \frac{\rho^k q^{k(k-1)} V_k^{(\beta)}(x, q)}{(q; q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\frac{\beta}{\alpha}; q)_n (-\alpha\rho q^k)^n}{(q; q)_n}$$

therefore with $\alpha, \beta < 0$, one has

$$(\alpha\rho; q)_\infty {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{n=0}^{\infty} \frac{\rho^n q^{n(n-1)} V_n^{(\beta)}(x, q)}{(q; q)_n} {}_1\phi_1\left(\begin{matrix} \beta/\alpha \\ 0 \end{matrix}; q, \alpha\rho q^n\right)$$

Classical expansions for orthogonal polynomials

Gegenbauer generating function (1874)

$$\frac{1}{(1 + \rho^2 - 2\rho x)^\nu} = \sum_{n=0}^{\infty} \rho^n C_n^\nu(x)$$

Legendre generating function (1783)

$$\frac{1}{\sqrt{1 + \rho^2 - 2\rho x}} = \sum_{n=0}^{\infty} \rho^n P_n(x)$$

Heine's reciprocal square root identity (1881)

$$\frac{1}{\sqrt{z - x}} = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2}(z) T_n(x), \quad \epsilon_n := \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, 2, 3, \dots \end{cases}$$

$$\frac{1}{\sqrt{1 + \rho^2 - 2\rho x}} = \frac{1}{\pi\sqrt{\rho}} \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2}\left(\frac{1 + \rho^2}{2\rho}\right) T_n(x)$$

Obtaining generalizations: Gegenbauer generating function

Gegenbauer generating function:

$$\frac{1}{(1 + \rho^2 - 2\rho x)^\nu} = \sum_{n=0}^{\infty} \rho^n C_n^\nu(x)$$

Connection relation:

$$C_n^\nu(x) = \frac{1}{\mu} \sum_{k=0}^{\lfloor n/2 \rfloor} (\mu + n - 2k) \frac{(\nu - \mu)_k (\nu)_{n-k}}{k! (\mu + 1)_{n-k}} C_{n-2k}^\mu(x).$$

Gegenbauer expansion:

$$\frac{1}{(z - x)^\nu} = \frac{2^{\mu + \frac{1}{2}} \Gamma(\mu) e^{i\pi(\mu - \nu + \frac{1}{2})}}{\sqrt{\pi} \Gamma(\nu) (z^2 - 1)^{\frac{\nu - \mu}{2} - \frac{1}{4}}} \sum_{n=0}^{\infty} (n + \mu) Q_{n + \mu - \frac{1}{2}}^{\nu - \mu - \frac{1}{2}}(z) C_n^\mu(x) \quad (1)$$

Implies **Chebyshev 1st kind expansion** (see Cohl & Dominici (2011)):

$$\frac{1}{(z - x)^\nu} = \sqrt{\frac{2}{\pi}} \frac{e^{i\pi(\frac{1}{2} - \nu)}}{\Gamma(\nu) (z^2 - 1)^{\frac{\nu}{2} - \frac{1}{4}}} \sum_{n=0}^{\infty} \epsilon_n Q_{n - \frac{1}{2}}^{\nu - \frac{1}{2}}(z) T_n(x),$$

Jacobi generalizations of Gegenbauer generating function

Gegenbauer \rightarrow Jacobi generating function:

$$\frac{1}{(1 + \rho^2 - 2\rho x)^\nu} = \sum_{n=0}^{\infty} \rho^n \frac{(2\nu)_n}{(\nu + \frac{1}{2})_n} P_n^{(\nu - \frac{1}{2}, \nu - \frac{1}{2})}(x)$$

Jacobi connection relation (2 free parameters) Ismail (2005)

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n c_{n,k}(\alpha, \beta; \gamma, \delta) P_k^{(\gamma, \delta)}(x),$$

$$c_{n,k}(\alpha, \beta; \gamma, \delta) := \frac{(\alpha + k + 1)_{n-k} (n + \alpha + \beta + 1)_k \Gamma(\gamma + \delta + k + 1)}{(n - k)! \Gamma(\gamma + \delta + 2k + 1)} \\ \times {}_3F_2 \left(\begin{matrix} -n + k, n + k + \alpha + \beta + 1, \gamma + k + 1 \\ \alpha + k + 1, \gamma + \delta + 2k + 2 \end{matrix} ; 1 \right)$$

Jacobi generalization of Gegenbauer generating function

Theorem

Let $\alpha, \beta > -1$, $z \in \mathbf{C} \setminus (-\infty, 1] \rightarrow \mathbf{C}$, on any ellipse with foci at ± 1 and x in the interior of that ellipse. Then

$$\frac{1}{(z-x)^\nu} = \frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}}$$

$$\times \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} Q_{n+\nu-1}^{(\alpha+1-\nu, \beta+1-\nu)}(z) P_n^{(\alpha, \beta)}(x)$$

Jacobi polynomials: $P_n^{(\alpha, \beta)} : \mathbb{C} \rightarrow \mathbb{C}$ DLMF (18.12.3) **generating function**

$$\begin{aligned} & \frac{1}{(1+\rho)^{\alpha+\beta+1}} {}_2F_1 \left(\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2} \\ \beta+1 \end{matrix}; \frac{2\rho(1+x)}{(1+\rho)^2} \right) \\ &= \left(\frac{2}{\rho(1+x)} \right)^{\beta/2} \frac{\Gamma(\beta+1)}{R^{\alpha+1}} P_\alpha^{-\beta} \left(\frac{1+\rho}{R} \right) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} \rho^n P_n^{(\alpha, \beta)}(x), \end{aligned}$$

Ismail (2005) (4.3.2) **generating function**

$$\begin{aligned} & \frac{(\alpha+\beta+1)(1+\rho)}{(1-\rho)^{\alpha+\beta+2}} {}_2F_1 \left(\begin{matrix} \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \\ \alpha+1 \end{matrix}; \frac{-2\rho(1-x)}{(1-\rho)^2} \right) \\ &= \left(\frac{2}{\rho(1-x)} \right)^{\alpha/2} \frac{(\alpha+\beta+1)(1+\rho)\Gamma(\alpha+1)}{R^{\beta+2}} P_{\beta+1}^{-\alpha} \left(\frac{1-\rho}{R} \right) \\ &= \sum_{n=0}^{\infty} (2n+\alpha+\beta+1) \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} \rho^n P_n^{(\alpha, \beta)}(x), \end{aligned}$$

Generalized expansions for Jacobi polynomials

Theorem

Let $m \in \mathbf{N}_0$, $\alpha, \beta > -1$, $x \in [-1, 1]$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1, 0]$. Then

$$\begin{aligned} & \frac{(1+x)^{-\beta/2}}{R^{\alpha+m+1}} P_{\alpha+m}^{-\beta} \left(\frac{1+\rho}{R} \right) \\ &= \frac{\rho^{-(\alpha+1)/2}}{2^{\beta/2}(1-\rho)^m} \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{\Gamma(\beta+n+1)} \\ & \quad \times P_{-m}^{-\alpha-\beta-2n-1} \left(\frac{1+\rho}{1-\rho} \right) P_n^{(\alpha,\beta)}(x). \end{aligned}$$

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, $\gamma, \beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1, 0]$, $x \in [-1, 1]$. Then

$$\begin{aligned} & \frac{(1+x)^{-\beta/2}}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}} P_\alpha^{-\beta} \left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}} \right) \\ &= \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2} \Gamma(\beta+1) (1-\rho)^{\alpha-\gamma} \rho^{(\gamma+1)/2}} \\ & \quad \times \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k (\alpha+\beta+1)_{2k}}{(\beta+1)_k} \\ & \quad \times P_{\gamma-\alpha}^{-\gamma-\beta-2k-1} \left(\frac{1+\rho}{1-\rho} \right) P_k^{(\gamma,\beta)}(x). \end{aligned}$$

$$\frac{(1+x)^{-\beta/2}}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}} P_\alpha^{-\beta} \left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}} \right) \\ = \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2}\Gamma(\beta+1)(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}} \sum_{n=0}^{\infty} \frac{(2n+\gamma+\beta+1)(\gamma+\beta+1)_n(\alpha+\beta+1)_{2n}}{(\beta+1)_n} P_n^{-\gamma-\beta-2n-1} \left(\frac{1+\rho}{1-\rho} \right) P_n^{(\gamma,\beta)}(x)$$

Theorem 1 in Cohl & MacKenzie (2013) *JCA* and Theorem 4 in Cohl, MacKenzie & Volkmer (2013) *JMAA*

$$\frac{1}{z-x} = \sum_{n=0}^{\infty} (2n+1)Q_n(z)P_n(x)$$

Heine (1851) Heine's formula

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \sum_{n=0}^{\infty} \rho^n C_n^\nu(x)$$

Gegenbauer (1874) generating function

$$\frac{1}{\sqrt{z-x}} = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2}(z) T_n(x)$$

Heine (1881)

$$\frac{1}{(z-x)^\nu} = \frac{(z^2-1)^{(1-\nu)/2}}{e^{i\pi(\nu-1)}\Gamma(\nu)} \sum_{n=0}^{\infty} (2n+1)Q_n^{\nu-1}(z)P_n(x)$$

(13) in Cohl (2013) *ITSF*

$$\frac{1}{(z-x)^\nu} = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{-\nu/2+1/4}}{e^{i\pi(\nu-1/2)}\Gamma(\nu)} \\ \times \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2}^{\nu-1/2}(z) T_n(x)$$

(3.10) in Cohl & Dominci (2011) *PRA*

$$\frac{1}{z-x} = \frac{2^{\mu+1/2}\Gamma(\mu)e^{i\pi(\mu-1/2)}}{\sqrt{\pi}(z^2-1)^{-\mu/2+1/4}} \sum_{n=0}^{\infty} (n+\mu)Q_{n+\mu-1/2}^{-\mu+1/2}(z)C_n^\mu(x)$$

(7.2) in Durand, Fishbane & Simmons (1976)

$$\frac{1}{z-x} = \frac{(z-1)^\alpha(z+1)^\beta}{2^{\alpha+\beta}} \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} Q_n^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(x)$$

(9.2.1) in Szegő (1959)

$$\frac{1}{(z-x)^\nu} = \frac{2^{\mu+1/2}\Gamma(\mu)e^{i\pi(\mu-1/2)}}{\sqrt{\pi}\Gamma(\nu)(z^2-1)^{(\nu-\mu)/2-1/4}} \\ \times \sum_{n=0}^{\infty} (n+\mu)Q_{n+\mu-1/2}^{\nu-\mu-1/2}(z)C_n^\mu(x)$$

Theorem 2.1 in Cohl (2013) *ITSF*

$$\frac{1}{(z-x)^\nu} = \frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}} \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} Q_{n+\nu-1}^{\alpha+1-\nu,\beta+1-\nu}(z)P_n^{(\alpha,\beta)}(x)$$

Theorem 1 in Cohl (2013) *SIGMA*

Other Jacobi generating functions

Connection relation (1 free parameter)

$$P_n^{(\alpha, \beta)}(x) = \frac{(\beta + 1)_n}{(\gamma + \beta + 1)(\gamma + \beta + 2)_n} \\ \times \sum_{k=0}^n \frac{(\gamma + \beta + 2k + 1)(\gamma + \beta + 1)_k (n + \beta + \alpha + 1)_k (\alpha - \gamma)_{n-k}}{(\beta + 1)_k (n + \gamma + \beta + 2)_k (n - k)!} P_k^{(\gamma, \beta)}(x).$$

Generating function for Jacobi polynomials: DLMF (18.12.1)

$$\frac{2^{\alpha+\beta}}{R(1+R-\rho)^\alpha(1+R+\rho)^\beta} = \sum_{n=0}^{\infty} \rho^n P_n^{(\alpha, \beta)}(x),$$

where $R := \sqrt{1 + \rho^2 - 2\rho x}$.

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, $\gamma, \beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. Then

$$\begin{aligned} & \frac{2^{\alpha+\beta}}{R(1+R-\rho)^\alpha(1+R+\rho)^\beta} \\ &= \frac{1}{\gamma+\beta+1} \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k \left(\frac{\alpha+\beta+1}{2}\right)_k \left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+\beta+1)_k \left(\frac{\gamma+\beta+2}{2}\right)_k \left(\frac{\gamma+\beta+3}{2}\right)_k} \\ & \quad \times {}_3F_2 \left(\begin{matrix} \beta+k+1, \alpha+\beta+2k+1, \alpha-\gamma \\ \alpha+\beta+k+1, \gamma+\beta+2k+2 \end{matrix} ; \rho \right) \rho^k P_k^{(\gamma,\beta)}(x). \end{aligned}$$

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, $\gamma, \beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. Then

$$\begin{aligned} & \left(\frac{2}{(1-x)\rho} \right)^{\alpha/2} \left(\frac{2}{(1+x)\rho} \right)^{\beta/2} J_{\alpha} \left(\sqrt{2(1-x)\rho} \right) I_{\beta} \left(\sqrt{2(1+x)\rho} \right) \\ &= \frac{1}{(\gamma + \beta + 1)\Gamma(\alpha + 1)\Gamma(\beta + 1)} \\ & \times \sum_{k=0}^{\infty} \frac{(2k + \gamma + \beta + 1)(\gamma + \beta + 1)_k \left(\frac{\alpha + \beta + 1}{2} \right)_k \left(\frac{\alpha + \beta + 2}{2} \right)_k}{(\alpha + 1)_k (\beta + 1)_k (\alpha + \beta + 1)_k \left(\frac{\gamma + \beta + 2}{2} \right)_k \left(\frac{\gamma + \beta + 3}{2} \right)_k} \\ & \quad \times {}_2F_3 \left(\begin{matrix} 2k + \alpha + \beta + 1, \alpha - \gamma \\ \alpha + \beta + k + 1, \gamma + \beta + 2k + 2, \alpha + k + 1 \end{matrix} ; \rho \right) \rho^k P_k^{(\gamma, \beta)}(x) \\ &= \sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(\alpha + 1 + n)\Gamma(\beta + 1 + n)} P_n^{(\alpha, \beta)}(x). \end{aligned}$$

Gegenbauer polynomials: $C_n^\mu : \mathbb{C} \rightarrow \mathbb{C}$

Generating function from (9.8.32) of Koekoek, Lesky & Swarttouw (2010)

$$\begin{aligned}
 & {}_2F_1 \left(\begin{matrix} \lambda, 2\alpha - \lambda \\ \alpha + \frac{1}{2} \end{matrix}; \frac{1 - R - \rho}{2} \right) {}_2F_1 \left(\begin{matrix} \lambda, 2\alpha - \lambda \\ \alpha + \frac{1}{2} \end{matrix}; \frac{1 - R + \rho}{2} \right) \\
 &= \frac{\Gamma^2(\alpha + \frac{1}{2})}{2^{1/2-\alpha}} (1 - x^2)^{1/4-\alpha/2} \rho^{1/2-\alpha} \\
 &\quad \times P_{\alpha-\lambda-1/2}^{1/2-\alpha} \left(\sqrt{1 + \rho^2 - 2\rho x} + \rho \right) P_{\alpha-\lambda-1/2}^{1/2-\alpha} \left(\sqrt{1 + \rho^2 - 2\rho x} - \rho \right) \\
 &= \sum_{n=0}^{\infty} \frac{(\lambda)_n (2\alpha - \lambda)_n}{(2\alpha)_n (\alpha + \frac{1}{2})_n} \rho^n C_n^\alpha(x).
 \end{aligned}$$

Generalized expansions for Gegenbauer polynomials

Theorem

Let $\alpha, \mu \in \mathbf{C}$, $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$.

Then

$$\begin{aligned}
 & (1-x^2)^{1/4-\mu/2} \\
 & \times P_{\mu-\alpha-1/2}^{1/2-\mu} \left(\sqrt{1+\rho^2-2\rho x} + \rho \right) P_{\mu-\alpha-1/2}^{1/2-\mu} \left(\sqrt{1+\rho^2-2\rho x} - \rho \right) \\
 & = \frac{2^{1/2-\mu}}{\Gamma(\mu+1/2)} \sum_{n=0}^{\infty} \frac{(\alpha)_n (2\mu-\alpha)_n (\mu)_n}{(2\mu)_n (\nu)_n \Gamma(\mu+n+1/2)} \rho^{\mu+n-1/2} \\
 & \times {}_6F_5 \left(\begin{matrix} \frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}, \frac{2\mu-\alpha+n}{2}, \frac{2\mu-\alpha+n+1}{2}, \mu-\nu, \mu+n \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+1/2}{2}, \frac{\mu+n+3/2}{2}, \nu+1+n \end{matrix} ; \rho^2 \right) C_n^\nu(x).
 \end{aligned}$$

Associated Legendre function, Ferrers functions, and Gegenbauer polynomials

If $n \in \mathbf{N}_0$, then through DLMF (14.3.22)

$$P_{n+\mu-1/2}^{1/2-\mu}(z) = \frac{2^{\mu-1/2}\Gamma(\mu)n!}{\sqrt{\pi}\Gamma(2\mu+n)}(z^2-1)^{\mu/2-1/4}C_n^\mu(z),$$

and from DLMF (14.3.21), one has

$$P_{n+\mu-1/2}^{1/2-\mu}(x) = \frac{2^{\mu-1/2}\Gamma(\mu)n!}{\sqrt{\pi}\Gamma(2\mu+n)}(1-x^2)^{\mu/2-1/4}C_n^\mu(x).$$

Gegenbauer expansions

For specific values can be this **nice expansion**

$$C_m^\mu(R + \rho)C_m^\mu(R - \rho) = \frac{(2\mu)_m^2}{(m!)^2} \sum_{n=0}^m \frac{(-m)_n(2\mu + m)_n}{(2\mu)_n(\mu + \frac{1}{2})_n} \rho^n C_n^\mu(x),$$

and from the above **generalized result** we have

$$C_m^\mu(R + \rho)C_m^\mu(R - \rho) = \frac{(2\mu)_m^2}{\nu(m!)^2} \sum_{n=0}^m \frac{(\nu + n)(-m)_n(2\mu + m)_n(\mu)_n}{(2\mu)_n(\mu + 1/2)_n(\nu + 1)_n} \\ \times {}_6F_5 \left(\begin{matrix} \frac{-m+n}{2}, \frac{-m+n+1}{2}, \frac{2\mu+m+n}{2}, \frac{2\mu+m+n+1}{2}, \mu - \nu, \mu + n \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+\frac{1}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}, \nu + 1 + n \end{matrix} ; \rho^2 \right) \rho^n C_n^\nu(x),$$

which reduces to **above formula** when $\nu = \mu$.

Wilson polynomials

$$W_n(x^2; a, b, c, d) := (a+b)_n (a+c)_n (a+d)_n \\ \times {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; 1 \right).$$

Connection relation with one free parameter for the Wilson polynomials

$$W_n(x^2; a, b, c, d) = \sum_{k=0}^n \frac{n!}{k!(n-k)!} W_k(x^2; a, b, c, h) \\ \times \frac{(n+a+b+c+d-1)_k (d-h)_{n-k} (k+a+b)_{n-k} (k+a+c)_{n-k}}{(k+b+c)_{n-k}^{-1} (k+a+b+c+h-1)_k (2k+a+b+c+h)_{n-k}}.$$

Generalized generating function for Wilson polynomials

Theorem

Let $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in (0, \infty)$, $\Re a, \Re b, \Re c, \Re d, \Re h > 0$ and non-real parameters a, b, c, d, h occurring in conjugate pairs. Then

$$\begin{aligned}
 & {}_2F_1 \left(\begin{matrix} a + ix, b + ix \\ a + b \end{matrix} ; \rho \right) {}_2F_1 \left(\begin{matrix} c - ix, d - ix \\ c + d \end{matrix} ; \rho \right) \\
 &= \sum_{k=0}^{\infty} \frac{(k + a + b + c + d - 1)_k}{(k + a + b + c + h - 1)_k (a + b)_k (c + d)_k k!} \\
 & \quad \times {}_4F_3 \left(\begin{matrix} d - h, 2k + a + b + c + d - 1, k + a + c, k + b + c \\ k + a + b + c + d - 1, 2k + a + b + c + h, k + c + d \end{matrix} ; \rho \right) \\
 & \quad \times \rho^k W_k(x^2; a, b, c, h).
 \end{aligned}$$

Generalized generating function for Wilson polynomials

Theorem

Let $\rho \in \mathbf{C}$, $|\rho| < 1$, $x \in (0, \infty)$, and a, b, c, d, h complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among a, b, c, d and a, b, c, h . Then

$$\begin{aligned}
 & (1 - \rho)^{1-a-b-c-d} \\
 & \times {}_4F_3 \left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; -\frac{4\rho}{(1-\rho)^2} \right) \\
 & = \sum_{k=0}^{\infty} \frac{(k+a+b+c+d-1)_k (a+b+c+d-1)_k}{(k+a+b+c+h-1)_k (a+b)_k (a+c)_k (a+d)_k k!} \rho^k \\
 & \times {}_3F_2 \left(\begin{matrix} 2k+a+b+c+d-1, d-h, k+b+c \\ 2k+a+b+c+h, a+d+k \end{matrix} ; \rho \right) W_k(x^2; a, b, c, h)
 \end{aligned}$$

Continuous dual Hahn polynomials

$$S_n(x^2; a, b, c) := (a+b)_n (a+c)_n {}_3F_2 \left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1 \right),$$

where $a, b, c > 0$, except for possibly a pair of complex conjugates with positive real parts. **Connection coefficient** for the **continuous dual Hahn polynomials** with **two free parameters**.

Lemma

Let $x \in (0, \infty)$, and $a, b, c, f, g \in \mathbf{C}$ with positive real parts and non-real values appearing in conjugate pairs among a, b, c and a, f, g . Then

$$S_n(x^2; a, b, c) = \sum_{k=0}^n \frac{n!}{k!(n-k)!} (k+a+b)_{n-k} (k+a+c)_{n-k} S_k(x^2; a, f, g) \\ \times {}_3F_2 \left(\begin{matrix} k-n, k+a+f, k+a+g \\ k+a+b, k+a+c \end{matrix}; 1 \right).$$

Generalized gen. fn. for continuous dual-Hahn polynomials

Theorem

Let $\rho \in \mathbf{C}$ with $|\rho| < 1$, $x \in (0, \infty)$ and $a, b, c, d, f > 0$ except for possibly pairs of complex conjugates with positive real parts among a, b, c and a, d, f . Then

$$(1 - \rho)^{-d+ix} {}_2F_1 \left(\begin{matrix} a + ix, b + ix \\ a + b \end{matrix} ; \rho \right) = \sum_{k=0}^{\infty} \frac{S_k(x^2; a, d, f) \rho^k}{(a + b)_k k!} {}_2F_1 \left(\begin{matrix} b - f, k + a + d \\ k + a + b \end{matrix} ; \rho \right).$$

Theorem

Let $\rho \in \mathbf{C}$, $x \in (0, \infty)$, and $a, b, c, d > 0$ except for possibly a pair of complex conjugates with positive real parts among a, b, c and a, b, d . Then

$$e^{\rho} {}_2F_2 \left(\begin{matrix} a + ix, a - ix \\ a + b, a + c \end{matrix} ; -\rho \right) = \sum_{k=0}^{\infty} \frac{\rho^k S_k(x^2; a, b, d)}{(a + b)_k (a + c)_k k!} {}_1F_1 \left(\begin{matrix} c - d \\ k + a + c \end{matrix} ; \rho \right).$$

Connection relation for Meixner-Pollaczek polynomials

If $\lambda > 0$ and $\phi \in (0, \pi)$, then the **Meixner-Pollaczek** polynomials are orthogonal

$$P_n^{(\lambda)}(x; \phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix}; 1 - e^{-2i\phi} \right).$$

Connection relation with one free parameter:

Lemma

Let $\lambda > 0, \phi, \psi \in (0, \pi)$. Then

$$P_n^{(\lambda)}(x; \phi) = \frac{1}{\sin^n \psi} \sum_{k=0}^n \frac{(2\lambda + k)_{n-k}}{(n-k)!} \sin^k \phi \sin^{n-k}(\psi - \phi) P_k^{(\lambda)}(x; \psi).$$

Expansions for Meixner-Pollaczek polynomials

Theorem

Let $\lambda > 0$, $\psi, \phi \in (0, \pi)$, $x \in \mathbf{R}$, and $\rho \in \mathbf{C}$ such that

$$|\rho|(\sin \phi + |\sin(\psi - \phi)|) < \sin \psi.$$

Then

$$\begin{aligned} (1 - e^{i\phi}\rho)^{-\lambda+ix}(1 - e^{-i\phi}\rho)^{-\lambda-ix} \\ = \left(1 - \rho \frac{\sin(\psi - \phi)}{\sin \psi}\right)^{-2\lambda} \sum_{k=0}^{\infty} P_k^{(\lambda)}(x; \psi) \tilde{\rho}^k, \end{aligned}$$

where

$$\tilde{\rho} = \frac{\rho \sin \phi}{\sin \psi - \rho \sin(\psi - \phi)}.$$

etc. for continuous Hahn polynomials

Some results for q -Pochhammer symbols

$$(a; q)_{n+k} = (a; q)_n (aq^n; q)_k,$$

$$(a; q)_{2n} = (a, aq; q^2)_n,$$

$$(aq^{-n}; q)_n = (a^{-1}q; q)_n (-a)^n q^{-n - \binom{n}{2}}, \quad a \neq 0,$$

$$(aq^n; q)_k = \frac{(a; q)_k}{(a; q)_n} (aq^k; q)_n,$$

$$(a^2; q^2)_n = (a, -a; q)_n,$$

$$(-a^2; q^2)_n = (ia, -ia; q)_n.$$

The q -ultraspherical/Rogers polynomials

Definition (**orthogonal**) for $|\beta| < 1$

$$\begin{aligned}
 C_n(x; \beta|q) &:= \frac{(\beta^2; q)_n \beta^{-n/2}}{(q; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \beta^2 q^n, \beta^{1/2} e^{i\theta}, \beta^{1/2} e^{-i\theta} \\ \beta q^{1/2}, -\beta, -\beta q^{1/2} \end{matrix} ; q, q \right) \\
 &= \frac{(\beta^2; q)_n \beta^{-n} e^{-in\theta}}{(q; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, \beta, \beta e^{2i\theta} \\ \beta^2, 0 \end{matrix} ; q, q \right) \\
 &= \frac{(\beta; q)_n e^{in\theta}}{(q; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, \beta \\ \beta^{-1} q^{1-n} \end{matrix} ; q, q\beta^{-1} e^{-2i\theta} \right),
 \end{aligned}$$

where $x = \cos \theta$.

Theorem

Let $x \in [-1, 1]$, $\beta \in (-1, 1) \setminus \{0\}$, $q \in (0, 1)$. Then

$$\begin{aligned}
 & {}_2\phi_1 \left(\begin{matrix} \beta^{\frac{1}{2}} e^{i\theta}, (\beta q)^{\frac{1}{2}} e^{i\theta} \\ \beta q^{\frac{1}{2}} \end{matrix}; q, e^{-i\theta} t \right) {}_2\phi_1 \left(\begin{matrix} -\beta^{\frac{1}{2}} e^{-i\theta}, -\beta^{\frac{1}{2}} q^{\frac{1}{2}} e^{-i\theta} \\ \beta q^{\frac{1}{2}} \end{matrix}; q, e^{i\theta} t \right) \\
 &= \frac{1}{1-\gamma} \sum_{n=0}^{\infty} \frac{(\beta, -\beta, -\beta q^{\frac{1}{2}}; q)_n (1-\gamma q^n) t^n}{(\beta^2, \beta q^{\frac{1}{2}}, q\gamma; q)_n} C_n(x; \gamma|q) \\
 &\times {}_{10}\phi_9 \left(\begin{matrix} \beta/\gamma, \beta q^n, i(\beta q^n)^{\frac{1}{2}}, -i(\beta q^n)^{\frac{1}{2}}, i(\beta q^{n+1})^{\frac{1}{2}}, -i(\beta q^{n+1})^{\frac{1}{2}}, \\ \gamma q^{n+1}, \beta q^{n/2}, -\beta q^{n/2}, \beta q^{(n+1)/2}, -\beta q^{(n+1)/2}, (\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, \\ i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, -i(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, i(\beta q^{n+3/2})^{\frac{1}{2}}, -i(\beta q^{n+3/2})^{\frac{1}{2}}, \\ -(\beta q^{n+\frac{1}{2}})^{\frac{1}{2}}, (\beta q^{n+3/2})^{\frac{1}{2}}, -(\beta q^{n+3/2})^{\frac{1}{2}} \end{matrix}; q, \gamma t^2 \right) \\
 &= \sum_{n=0}^{\infty} \frac{(-\beta, -\beta q^{1/2}; q)_n}{(\beta^2, \beta q^{1/2}; q)_n} C_n(x; \beta|q) t^n.
 \end{aligned}$$

Theorem

Let $x \in [-1, 1]$, $\beta \in (-1, 1) \setminus \{0\}$, $q \in (0, 1)$. Then

$$\begin{aligned} & \frac{(\gamma e^{i\theta} t; q)_\infty}{(e^{i\theta} t; q)_\infty} {}_3\phi_2 \left(\begin{matrix} \gamma, \beta, \beta e^{2i\theta} \\ \beta^2, \gamma e^{i\theta} t \end{matrix}; q, e^{-i\theta} t \right) \\ &= \frac{1}{1-\gamma} \sum_{n=0}^{\infty} \frac{(\beta, \gamma; q)_n (1-\gamma q^n) t^n}{(\beta^2, q\gamma; q)_n} C_n(x; \gamma|q) \\ & \quad \times {}_6\phi_5 \left(\begin{matrix} \beta/\gamma, \beta q^n, (\gamma q^n)^{\frac{1}{2}}, -(\gamma q^n)^{\frac{1}{2}}, (\gamma q^{n+1})^{\frac{1}{2}}, -(\gamma q^{n+1})^{\frac{1}{2}} \\ \beta q^{n/2}, -\beta q^{n/2}, \beta q^{(n+1)/2}, -\beta q^{(n+1)/2}, \gamma q^{n+1} \end{matrix}; q, \gamma t^2 \right) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma; q)_n}{(\beta^2; q)_n} C_n(x; \beta|q) t^n, \end{aligned}$$

etc. for q -ultraspherical/Rogers polynomials.

The q -Laguerre polynomials

Definition (**orthogonal**) $\alpha > -1$

$$\begin{aligned} L_n^{(\alpha)}(x; q) &= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}; q, -q^{n+\alpha+1}x \right) \\ &= \frac{1}{(q; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix}; q, q^{n+\alpha+1} \right). \end{aligned}$$

Connection relation

$$\begin{aligned} L_n^{(\alpha)}(x; q) &= \frac{q^{n(\alpha-\beta)}}{(q; q)_n} \sum_{j=0}^n (-1)^{n-j} q^{\binom{n-j}{2}} (q^{n-j+1}; q)_j (q^{j-n+\beta-\alpha+1}; q)_{n-j} L_j^{(\beta)}(x; q) \end{aligned}$$

Generalized generating function for q -Laguerre

Theorem

Let $\alpha > -1$. Then

$$\begin{aligned} \frac{1}{(t; q)_{\infty}} {}_0\phi_1 \left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix} ; q, -xtq^{\alpha+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{(q^{(\alpha-\beta)}t)^n L_n^{(\beta)}(x; q)}{(q^{\alpha+1}; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{\alpha-\beta}, 0 \\ q^{\alpha+n+1} \end{matrix} ; q, t \right), \\ &= \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x; q)}{(q^{\alpha+1}; q)_n} t^n. \end{aligned}$$

Generalized generating function for q -Laguerre (cont.)

Theorem

Let $\alpha > -1$. Then

$$\begin{aligned}
 (t; q)_{\infty} {}_0\phi_2 \left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix}; q, -q^{\alpha+1}xt \right) \\
 &= \sum_{n=0}^{\infty} \frac{(-tq^{\alpha-\beta})^n q^{\binom{n}{2}} L_n^{(\beta)}(x; q)}{(q^{\alpha+1}; q)_n} {}_1\phi_1 \left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix}; q, q^n t \right) \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} L_n^{(\alpha)}(x; q) t^n}{(q^{\alpha+1}; q)_n}.
 \end{aligned}$$

etc. for q -Laguerre and etc. for little q -Laguerre.

Final Remarks & see publications

- **Definite integrals from orthogonality for the basic case**
- **Discrete hypergeometric orthogonal polynomials**
- **Big/little q -Jacobi, continuous q -Jacobi, Askey-Wilson**
- “On a generalization of the generating function for Gegenbauer polynomials,” H. S. Cohl, 2013, *Integral Transforms and Special Functions*, **24**, 10, 807–816, 10 pp.
- “Generalizations and specializations of generating functions for Jacobi, Gegenbauer, Chebyshev and Legendre polynomials with definite integrals,” H. S. Cohl and Connor MacKenzie, 2013, *Journal of Classical Analysis*, **3**, 1, 17-33, 17 pp.
- “Fourier, Gegenbauer and Jacobi expansions for a power-law fundamental solution of the polyharmonic equation and polyspherical addition theorems,” H. S. Cohl, 2013, *Symmetry, Integrability and Geometry: Methods and Applications*, **9**, 042, 26 pp.
- “Generalizations of generating functions for hypergeometric orthogonal polynomials with definite integrals,” H. S. Cohl, Connor MacKenzie, and H. Volkmer, 2013, *Journal of Mathematical Analysis and Applications*, **407**, 2, 211-225, 15 pp.
- “Generalized generating functions for higher continuous orthogonal polynomials in the Askey scheme,” M. A. Baeder, H. S. Cohl, H. Volkmer (submitted).