

CLASSIFICATION OF SYMMETRIC WAVELET TRANSFORMS

Christopher M. Brislawn
Group C-3, Computer Research
Los Alamos National Laboratory
Mail Stop B-265
Los Alamos, NM 87545
office: (505) 665-1165, e-mail: brislawn@LANL.GOV

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ABSTRACT

This paper describes and classifies a family of invertible digital signal transforms, referred to here as symmetric wavelet transforms (SWT's), for finite-length signals. SWT's are algorithms for applying multirate filter banks to symmetric extensions of finite-length input signals, thereby avoiding the sort of boundary artifacts introduced by simple periodization. This approach does for wavelet transforms what the discrete cosine transform does for the Fourier transform. A key point addressed here is when such symmetric decompositions can be formed with no increase in data storage requirements ("nonexpansive decompositions"). Transforms based on three types of symmetric signal extension and four classes of generalized linear phase filters are analyzed in terms of their memory requirements for general M -channel perfect-reconstruction filter banks. The classification is shown to be complete in the sense that it contains all possible nonexpansive SWT's. All such transforms are described explicitly in the case of two-channel systems, for both even- and odd-length signals. The individual channels in general M -channel systems are classified according to their individual memory requirements. Time-domain formulas are given for direct-form implementation of SWT analysis and synthesis filter banks. The paper classifies the particular SWT algorithms incorporated in the Federal Bureau of Investigation's digital fingerprint image coding standard.

CONTENTS

I.	INTRODUCTION.	1
I-A.	Symmetric Signal Extensions and Expansiveness.	2
I-B.	Prior Results.	4
I-C.	Organization and Scope of the Paper.	5
II.	PERIODIC MULTIRATE FILTER BANKS FOR SYMMETRIC SIGNALS.	6
II-A.	Signal Symmetries and Extension Operators.	6
	1. Six types of signal symmetry.	6
	2. Four types of linear phase filters.	8
	3. Symmetric extension operators.	9
II-B.	Convolution and Decimation.	11
	1. Downsampling WS-type channels.	14
	2. Downsampling HS-type channels.	15
II-C.	Subband Projection and Reconstruction.	16
	1. The extended subband coder.	17
II-D.	Generic Analysis/Synthesis Computations.	18
	1. Analysis.	18
	2. Synthesis.	18
	3. Causality.	19
II-E.	SWT Expansiveness.	20
	1. Nonuniformly downsampled filter banks.	21
	2. Completeness of the classification.	22
III.	TWO-CHANNEL SYMMETRIC SUBBAND CODERS.	23
III-A.	Two Types of Symmetric Extension.	23
	1. The extension $y = E_s^{(1,1)}x$	23
	2. The extension $y = E_s^{(2,2)}x$	24
III-B.	Two-Channel Discrete Wavelet Transforms.	25
	1. Symmetry/phase possibilities: nontrivial filter banks.	26
III-C.	Classification of Two-Channel Symmetric Wavelet Transforms.	27
	1. (1,1)-Symmetric wavelet transforms.	27
	2. (2,2)-Symmetric wavelet transforms.	28
	3. The FBI fingerprint coding standard.	28

IV.	M-CHANNEL SYMMETRIC SUBBAND CODERS.	29
IV-A.	Divisibility Constraints on Signal Length.	31
IV-B.	M Even.	32
	1. The extension $y = E_s^{(1,1)}x$	32
	2. The extension $y = E_s^{(2,2)}x$	33
IV-C.	M Odd.	34
	1. The extension $y = E_s^{(1,1)}x$	35
	2. The extension $y = E_s^{(1,2)}x$	36
	3. The extension $y = E_s^{(2,2)}x$	36
V.	CONCLUSIONS.	37
V-A.	Acknowledgements.	38
	REFERENCES	38

LIST OF FIGURES

1.	M -Channel Subband Coder.	1
2.	Extended M -Channel SWT Subband Coder.	3
3.	Whole- and Half-Sample Symmetry.	7
4.	Symmetric Extensions, $y = E_s^{(i,j)}x$, for Signal Analysis.	10
5.	The Symmetric Extension $b = E_s^{(2,1)}a$ for Subband Synthesis.	11
6.	Antisymmetric Extensions, $b = E_a^{(i,j)}a$, for Subband Synthesis.	12
7.	Analysis/Synthesis Channels for Extended Subband Coder.	18
8.	The Symmetric Signal Extension $y = E_s^{(1,3)}x$	22

LIST OF TABLES

I.	Symmetry Properties of N -Periodic Signals.	8
II.	Symmetric Extension Operators for Signal Analysis.	10
III.	Extension Operators for Subband Synthesis.	11
IV.	Symmetry and Center of the Convolution Product $u = y * h$	13
V.	Symmetry Properties of Down-Sampled Signals.	16
VI.	Symmetric Subband Reconstruction.	19
VII.	(1,1)-Symmetric Wavelet Transform Channels, $M = 2$	24
VIII.	(2,2)-Symmetric Wavelet Transform Channels, $M = 2$	25
IX.	Divisibility Constraints on Signal Length, N_0	32

X.	(1, 1)-Symmetric Wavelet Transform Channels, M Even.	33
XI.	(2, 2)-Symmetric Wavelet Transform Channels, M Even.	34
XII.	(1, 1)-Symmetric Wavelet Transform Channels, M Odd.	35
XIII.	(1, 2)-Symmetric Wavelet Transform Channels, M Odd.	36
XIV.	(2, 2)-Symmetric Wavelet Transform Channels, M Odd.	36

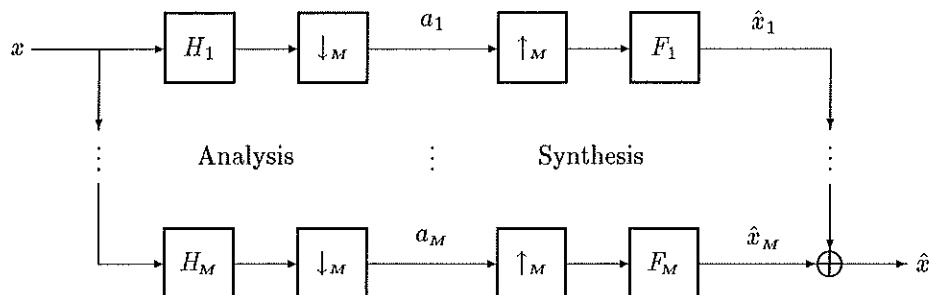


Figure 1: M -Channel Subband Coder.

I. INTRODUCTION.

A critically downsampled M -channel subband coder is a digital filter bank of the type shown in Figure 1 [1, 2, 3, 4]. Such filter banks have received a great deal of attention lately as a result of the recent development of wavelet transforms, which are special instances of this type of subband coder having a connection, in the infinite sampling-rate limit, with continuum approximation theory [5, 6, 7, 8]. The filter bank in Figure 1 is called a *perfect-reconstruction quadrature mirror filter* (PR QMF) bank if it has a linear time-invariant system function with, at worst, constant-amplitude or linear-phase distortion:

$$T(z) \equiv \frac{\hat{X}(z)}{X(z)} = Az^{-D} . \quad (\text{PR})$$

When condition (PR) is satisfied, we will refer to the linear transformation

$$x \mapsto \{a_1, \dots, a_M\} \quad (\text{DWT})$$

as a *discrete wavelet transform* (DWT), without insisting upon any particular degree of regularity for the infinitely iterated analysis bank, as is customarily done in the construction of “regular” continuous-time wavelets. The problem of specifying adequate filter-bank regularity properties is highly dependent on the application of interest and will not be dealt with in this paper. Nonetheless, every PR QMF bank corresponds, under very mild additional assumptions, to a wavelet frame for $L^2(\mathcal{R})$ [9], so we feel justified in referring to the transform (DWT) given by an arbitrary PR QMF bank as a discrete *wavelet* transform.

The first construction of nontrivial finite impulse response (FIR) filter banks satisfying condition (PR) was given by Smith and Barnwell [10] for two-channel systems ($M = 2$), under the assumption of power-complementarity,

$$|H_1(e^{j\omega})|^2 + |H_2(e^{j\omega})|^2 = 1 . \quad (\text{PC})$$

Such systems generate orthogonal DWT's. Shortly thereafter, the construction of compactly supported regular orthogonal wavelets by iteration of PR QMF banks satisfying condition (PC) was demonstrated by Daubechies [11]. Both of these constructions resulted in filters with asymmetric impulse responses since, as was already known [12], the only FIR solutions to (PC) with linear phase are trivial. Subsequent constructions of two-channel FIR PR QMF banks [13] and wavelets [14] with linear phase were given by relaxing condition (PC). Linear phase two-channel PR QMF banks satisfying condition (PC) are possible using infinite impulse response (IIR) filters, and symmetric extension methods using IIR filters have been studied in [15], but the present paper will only consider FIR filter banks. The design of M -channel FIR PR QMF banks has been studied by, e.g., [16, 17, 18, 19, 20]. Of particular interest for this paper, M -channel linear phase FIR PR QMF banks with $M > 2$ have been constructed recently in [21, 22, 23].

I-A. Symmetric Signal Extensions and Expansiveness.

A problem arises when we apply a PR QMF subband coder like the one in Figure 1 to finite-length signals, namely, the problem of deciding how to handle the boundary conditions at the ends of the signal. Let $x(n); n = 0, \dots, N_0 - 1$, be a discrete-time signal of length N_0 . When the filters in Figure 1 are all FIR filters, the coder can be applied to the periodized signal $\tilde{x}(n)$ of period N_0 . If the filters all have length less than or equal to N_0 and if M divides N_0 (indicated as " $M|N_0$ "), then this periodized filter bank defines a finite-length perfect-reconstruction DWT given by circular convolution and circular $M : 1$ downsampling. Since the output channels $\tilde{a}_1, \dots, \tilde{a}_M$ have period N_0/M , this transform is *nonexpansive*; i.e., it conserves storage costs in the sense of transforming an N_0 -point sequence into $M \times (N_0/M)$ transformed data points. This feature is clearly desirable for applications in which memory requirements are an issue, such as data storage or transmission. The periodized DWT suffers, however, from the defect of introducing a jump discontinuity in the data at the point where the ends of the signal are matched up. This boundary artifact usually results in added variance in the high-frequency subband(s), a phenomenon that often degrades the performance of systems acting on the output, such as data quantizers.

In image coding, a popular solution to this problem is to quantize the output of the (two-dimensional) discrete cosine transform (DCT) [24, 25]. The DCT can be identified with a phase shift of the first half of the discrete Fourier transform (DFT) expansion of the even signal, y , defined by the symmetric extension

$$y(n) = \begin{cases} x(n) ; & n = 0, \dots, N_0 - 1 ; \\ x(2N_0 - 1 - n) ; & n = N_0, \dots, 2N_0 - 1 . \end{cases} \quad (1)$$

While the $2N_0$ -point DFT is given mathematically by circular correlation with the Fourier kernel, $e^{j2\pi nk/2N_0}$, the periodization artifacts in the DCT are greatly reduced by the fact that the extension, y , is an even signal.

We can employ a similar strategy for transforming finite-length signals using a PR QMF bank. Given x of length N_0 , form a symmetric extension, y , of length $N \approx 2N_0$ by a procedure like (1). The extension, y ,

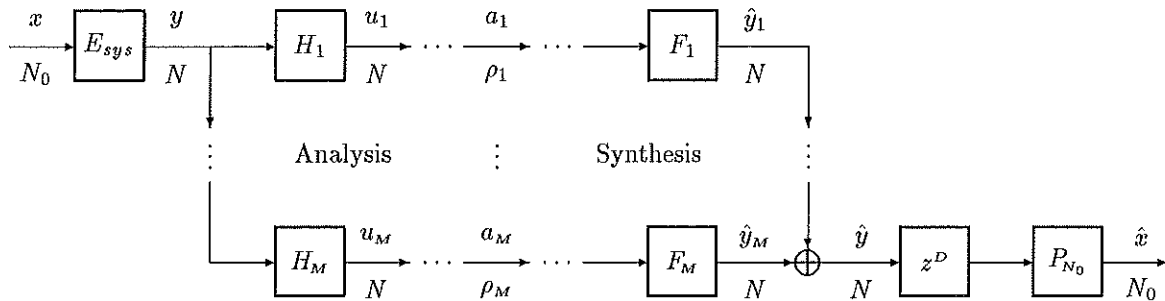


Figure 2: Extended M -Channel SWT Subband Coder.

can then be transformed by N -periodic circular convolution, eliminating the jump discontinuity that would have resulted from simple periodization of x . If the system transfer function is given by (PR), we can recover x by projecting the output of the synthesis bank onto the first N_0 coordinates:

$$\hat{x}(n) = \hat{y}(n + D) = Ax(n) ; n = 0, \dots, N_0 - 1 .$$

See Figure 2. Beneath each signal component we have indicated the length (or period) of that component. Note that a_1, \dots, a_M have been given (possibly) different lengths, ρ_1, \dots, ρ_M . We call the mapping

$$x \mapsto \{a_1, \dots, a_M\}$$

a *symmetric wavelet transform (SWT)* of dimension $N_1 = \rho_1 + \dots + \rho_M$. We insist that this mapping preserve the perfect reconstruction property of the filter bank; i.e., an SWT must be an *invertible* transformation. Clearly, we do not want an SWT for which $\rho_i = N/M$ since it would then have dimension $N_1 = N \approx 2N_0$ and the price we would pay for the symmetric extension would be a doubling of storage costs. For real signals, the DCT avoids this problem by using an even extension, so that only half of the DFT output (i.e., only the cosine modes) needs to be saved. In the SWT's we will construct, the output channels a_i will consist of one-half of a symmetric or antisymmetric subband, b_i . An SWT will be called *nonexpansive* if $N_1 = N_0$. Deriving the details of such procedures, which for clarity at this stage have been omitted from Figure 2, is the objective of this paper. The need to understand such details thoroughly to develop SWT implementations for digital image coding applications is what motivated the author to write this paper.

Image coding appears to be a lucrative source of SWT applications, since coding images (or even higher-dimensional digital data) involves transforming large numbers of relatively short row or column vectors. If the filter bank is implemented with several levels of cascade, boundary effects can propagate across a significant fraction of the transformed signal, so it is imperative to handle boundary conditions in a manner

that preserves perfect reconstructability and minimizes spectral coloration. Symmetric extrapolation meets these needs and furnishes several additional benefits. Since the period of the extended input is $\approx 2N_0$, a signal of length N_0 can be filtered with filters of length up to $\approx 2N_0$; simple periodization constrains filter lengths to at most N_0 . For cascaded two-channel filter banks, this allows computation of an additional level of cascade for given signal and filter lengths. The fact that the period of y is even means that a two-channel SWT forms nonexpansive decompositions for both even- and odd-length inputs; DWT's based on simple periodization are applicable only to even-length inputs. Examples of data coding applications utilizing these advantages of SWT's can be found in [26, 27, 28, 29].

I-B. Prior Results.

Symmetric signal extension has been considered by a number of other researchers in the context of subband coding. Karlsson and Vetterli [30] proposed several methods of continuous signal extension followed by linear convolution as alternatives to circular convolution for a two-channel subband coder. Also in the two-channel case, Smith and Eddins [31, 15, 32] observed that if the $2N_0$ -periodic extension, y , defined by (1) is filtered by circular convolution with even-length linear phase filters and 2:1 circular downsampling, the resulting subbands are symmetric or antisymmetric and can be reconstructed losslessly by saving only $N_0/2$ samples of each. (In the present notation, $\rho_1 = \rho_2 = N_0/2$.) This appears to be the first example of a nonexpansive SWT. A second example for even-length signals was published by Martucci [33, 34] using a $(2N_0 - 2)$ -periodic extension with odd-length linear phase filters. (Both of these transforms were developed independently by the author; the latter appeared in [26].)

More recently, there have been a few papers on the problem of defining symmetric extension algorithms for general M -channel PR QMF banks [35, 36, 37, 38]. The papers [37, 38] consider M -channel nonuniformly downsampled filter banks; constraints are derived on allowable combinations of signal extension method and filter symmetries. The authors also continue the analysis of nonexpansive symmetric transform schemes based on linear phase IIR filters begun in [15]. Some of the computational formulas derived in Section II of this paper appear, in slightly different forms, in the papers [36, 37, 38]; readers interested in a more concise overview of the M -channel case are directed to those papers. The author's goal in writing the present paper is to draw together a number of partial results on symmetric extension methods scattered throughout the literature and include them in a unified manner with a thorough analysis of *all* possible symmetric extension and filter bank combinations and the resulting storage requirements. Accordingly, he has devoted much effort to making the classification as straightforward and applications-oriented as he can.

It is possible to define smoother extrapolations than symmetric extensions; some such extrapolations appear in [30]. Transforms based on general linear extrapolation are considered in [39, 40, 41]. Methods based on computing nonlinear signal extensions have also appeared in the literature, e.g., [42, 43]. In a somewhat different vein, a method described recently by Wickerhauser [44] involves deforming the input signal by

smoothing it to match up the endpoints, followed by periodization of the smoothed signal. This requires additional computations near the signal boundaries during both analysis and synthesis. The symmetric extension method proposed here is thus a compromise between the algorithmic simplicity of methods like simple periodization or zero-padding [41] and the maximum possible smoothness obtained by nonlinear extrapolation methods. One advantage is that symmetric extension can be accomplished entirely by data addressing; no additional computations are required to form the signal extrapolation.

I-C. Organization and Scope of the Paper.

Section II contains preliminaries regarding the definitions of symmetric and antisymmetric signal extensions. We consider signal decompositions based on three types of symmetric extension. Since the goal of forming such extensions is to avoid introducing singularities and spurious high-band energy in the signal being transformed, we do not understand the motivation for the transforms based on antisymmetric extensions that were proposed in [36]. Thus, this paper only considers transforms based on symmetric extensions; antisymmetric extensions will be used exclusively for reconstructing antisymmetric subbands during synthesis.

A new result in Section II is a rigorous derivation of the ways in which a symmetric signal can remain symmetric after $M:1$ downsampling; this result is crucial to the classification of symmetric transform algorithms. Projection/extension and shift operations are developed that fill in the missing details in Figure 2, starting with a given linear phase PR QMF bank. This approach is complementary to the one taken in [37, 38], in which the authors start with a given number of channels and downsampling rates and a given symmetric extension and then derive constraints that a filter bank must be designed to satisfy in order to produce a (perfect reconstruction) SWT. The approach taken in the present paper is to assume that a PR QMF bank has already been designed, based solely on consideration of its filter characteristics, and then to classify the possible ways in which the given filter bank can be *implemented* in a system utilizing symmetric extension for finite-duration inputs. This decouples the filter bank design problem from the implementation issue.

Block diagrams and generic formulas are derived for direct-form implementations of SWT analysis and synthesis filter banks. It is beyond the scope of this paper to describe lattice implementations of SWT algorithms, nor does this paper consider the actual construction of linear phase PR QMF banks. These topics are addressed in the references cited above. As a result of these restrictions, the author found it unnecessary to use polyphase filter bank realizations in this paper. After describing direct-form implementations, the notion of causality is discussed for SWT systems; we argue that transforms based on symmetric extrapolation are inherently non-causal in nature. To conclude Section II, we introduce a general notion of subband expansiveness and explain how the classification given in this paper contains *all* nonexpansive SWT's.

Section III classifies all nonexpansive two-channel SWT algorithms in terms of signal length N_0 , filter symmetry, and filter phase. The necessary parameters are given for implementing the operations designed in Section II. The analysis is based on a new characterization of the group delays in a linear phase PR QMF

bank. The particular SWT algorithms incorporated by the author into the FBI digital fingerprint image coding standard [29] are also described; this paper forms the principal technical reference on these matters.

Section IV contains a complete but less specific classification of M -channel SWT's. The possible filter symmetry/phase combinations in M -channel linear phase PR QMF banks are not completely understood at present, so we enumerate all possible individual channels in M -channel SWT's and classify them according to their individual channel expansiveness. We comment on the design of M -channel SWT's and illustrate with some examples for the cases $M=3, 4$. It is shown that nonexpansive SWT's exist for input signals of arbitrary length when $M = 3$; the corresponding question for $M=4$ is open. Some interesting corollaries of our analysis of the expansiveness of M -channel SWT's are several new non-existence results for certain combinations of filter symmetries in FIR PR QMF banks.

Finally, Section V contains concluding remarks and acknowledgments.

II. PERIODIC MULTIRATE FILTER BANKS FOR SYMMETRIC SIGNALS.

In this section we analyze the effects of some standard multirate filter bank operations on symmetric extensions of input signals. The specific results needed to classify SWT's will be tabulated for later reference; most derivations are elementary and will be left to the reader, who is referred to [1, 4] for much more extensive treatments of multirate signal processing. The notation will generally follow the conventions in [45, 4], although we shall not use different notation to distinguish between a signal of length N and its periodic extension of period N . We *will* distinguish carefully between a finite-length signal and its various symmetric and antisymmetric extensions. We also restrict attention to real-valued signals and filters. This will simplify the analysis while still encompassing most conceivable applications. Moreover, since all operators involved are linear, the extension for complex signals is trivial.

II-A. Signal Symmetries and Extension Operators.

1. *Six types of signal symmetry.* There are two ways in which a discrete-time signal can be symmetric: it can be symmetric about one of its samples or about a point midway between two samples (see Figure 3). These cases are referred to as *whole-sample symmetry* (WSS) and *half-sample symmetry* (HSS) in [33, 34], and we retain the same terminology in this paper because of its mnemonic value. We call the axis of symmetry (e.g., $n = c$ or $n = c - 1/2$) a *center of symmetry* for y and require the entire signal to be symmetric about a given center; nonetheless, a signal can still have multiple centers. There are obvious antisymmetric analogues in both the whole- and half-sample cases; these are designated *whole-sample antisymmetry* (WSA) and *half-sample antisymmetry* (HSA). We only declare y to be whole-sample antisymmetric about c if $y(c) = 0$.

The above categories are not mutually exclusive since a signal may have different types of symmetry

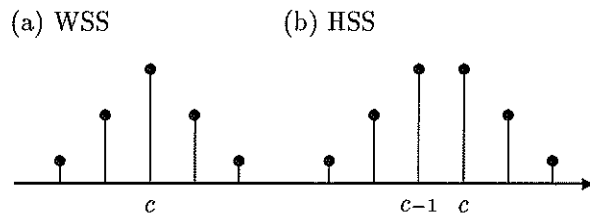


Figure 3: (a) Whole-Sample Symmetry About c .
 (b) Half-Sample Symmetry About $c - 1/2$.

about different centers; e.g., an alternating sequence like $(-1)^n$ is both WSS and HSA. To be precise, we must specify both a symmetry property and a center. Constant sequences show that there is no upper bound on the number of centers a signal may have, but for symmetric or antisymmetric periodic signals there is a lower bound of at least two. More specifically, let y be N -periodic with a center at c :

$$y(c + n) = \pm y(c - n) .$$

Then $c + N/2$ is also a center for y ; an analogous statement holds for a signal with a center at $c - 1/2$. If N is even then y is WSS (or WSA) about both c and $c + N/2$ in the former case, and HSS (or HSA) about both $c - 1/2$ and $c + N/2 - 1/2$ in the latter. If N is odd, however, y will be WSS (or WSA) at c and HSS (or HSA) at $c + N/2$. We refer to such signals as *odd-period symmetric* (OPS) or *odd-period antisymmetric* (OPA) signals. For illustrations of these six types of symmetry, the reader is referred ahead to Figures 4, 5, and 6, where we present signal extensions with all these various symmetries. Formal definitions of the symmetry properties are given in Table I in terms of time-domain criteria and equivalent frequency-domain characterizations. Since OPS and OPA signals possess both whole- and half-sample symmetries, they satisfy both whole- and half-sample frequency domain characterizations; one can verify that the DFT characterizations of whole- and half-sample symmetry are mathematically equivalent when N is odd.

To explain the last column in Table I, we need to define the *dimension* of a class of periodic signals sharing some symmetry property. The dimension, ρ , of such a class is the least value for which the class can be embedded losslessly in \mathbf{R}^ρ . This is the least upper bound on the number of nonredundant samples in an arbitrary signal from the class and thus depends on both the period, N , and the particular symmetry property. The dimension tells us how many memory registers are required to store an arbitrary signal from the class.

Consider, for instance, the class of all HSS signals of even period, N . This class will contain some signals possessing other symmetry properties, too, but *all* members will satisfy a condition of the form

$$y(c + n) = y(c - 1 - n) ,$$

Table I: Symmetry Properties of N -Periodic Signals.

Symmetry Property	Center	Symmetry Characterization:		Dimension
		Time Domain	DFT Domain	
WSS	c	$y(c+n) = y(c-n)$	$Y^*(k) = e^{j2\pi(2c)k/N}Y(k)$	$N/2 + 1$
WSA	c	$y(c+n) = -y(c-n)$	$Y^*(k) = -e^{j2\pi(2c)k/N}Y(k)$	$N/2 - 1$
HSS	$c - 1/2$	$y(c+n) = y(c-1-n)$	$Y^*(k) = e^{j2\pi(2c-1)k/N}Y(k)$	$N/2$
HSA	$c - 1/2$	$y(c+n) = -y(c-1-n)$	$Y^*(k) = -e^{j2\pi(2c-1)k/N}Y(k)$	$N/2$
OPS (N Odd)		WSS at c and HSS at $c + N/2$, or HSS at $c - 1/2$ and WSS at $c + (N - 1)/2$		$(N + 1)/2$ (N Odd)
OPA (N Odd)		WSA at c and HSA at $c + N/2$, or HSA at $c - 1/2$ and WSA at $c + (N - 1)/2$		$(N - 1)/2$ (N Odd)

so, in a single period $c - N/2 \leq n \leq c + N/2 - 1$, the samples in the “second half” of the period,

$$y(c+n) ; n = 0, \dots, N/2 - 1 ,$$

are redundant. An HSS signal therefore contains at most $N/2$ nonredundant samples, and since this upper bound is attained by symmetric ramps of the kind shown in Figure 4(d), the dimension of the HSS class is $N/2$ for even N . Similar arguments yield the figures in the last column of Table I. Also note that we do *not* count the end-samples that are necessarily zero when computing the dimension of the WSA and OPA classes. We will use the dimension of transform subbands to evaluate the storage requirements—the “expansiveness”—of the various transforms we construct.

2. Four types of linear phase filters. We use the four principal types of linear phase FIR filters corresponding to the whole- and half-sample symmetry and antisymmetry properties defined above. These four classes also appear in [45] under the headings “Type I–IV.” For the purpose of analyzing their effect on symmetric signals, it is most convenient to refer to linear phase filters in terms of their whole- or half-sample symmetry properties. Since the transforms defined later are based on circular convolution, for a given period, N , we consider only FIR filters with at most N real-valued impulse response taps.

At times, we shall make use of the fact that the center of symmetry for a linear phase FIR filter is equal to the filter’s group delay [45]. For this reason, we use γ to denote the center of a linear phase FIR filter. Note that γ is an integer for WS-type filters and an odd multiple of $1/2$ for HS-type filters. It follows that symmetric filters, of either type, satisfy the symmetry characterization

$$H^*(k) = e^{j2\pi(2\gamma)k/N}H(k) , \text{ or } H(z^{-1}) = z^{2\gamma}H(z) , \quad (2)$$

while antisymmetric filters satisfy

$$H^*(k) = -e^{j2\pi(2\gamma)k/N} H(k) \quad , \quad \text{or} \quad H(z^{-1}) = -z^{2\gamma} H(z) \quad . \quad (3)$$

3. Symmetric extension operators. We formalize the process of making a symmetric extension of a signal, x , of length N_0 by defining *extension operators* as linear transformations,

$$E_{sys} : \mathbf{R}^{N_0} \rightarrow \mathbf{R}^N \quad ,$$

of rank N_0 . An extension operator simply replicates values of x according to some symmetry relation to construct an extension, $y = Ex$, with one of the six symmetry properties described above. As mentioned in the Introduction, we will only consider signal analyses based on symmetric extensions, but we will need antisymmetric extension operators in SWT synthesis banks.

Aside from whether an extension is symmetric or antisymmetric, the main feature distinguishing different extensions is the number of times the first and last samples are repeated in the extension. A pair of digits, (i, j) , indicates the number of times the first and last samples occur in a single period of the extended signal. For instance, in the extension $E_s^{(1,2)}x$ shown in Figure 4(c), the sample $x(0)$ occurs once in each period while $x(N_0 - 1)$ occurs twice. (The open dots represent a second full period of the extension.) The three extensions shown in Figure 4 are the only ones that will be considered for extending a signal prior to coding. The reason for this particular choice is that these are the only extensions for which the SWT approach produces nonexpansive transforms; details will be given in Section II-E.

The three analysis extensions are defined formally in Table II. Each extension is initialized by setting

$$y(n) = x(n) \quad ; \quad n = 0, \dots, N_0 - 1 \quad ,$$

then extending from $n = N_0$ to $N - 1$ using the given symmetry relation. Note that since these extensions coincide with $x(n)$ for $0 \leq n \leq N_0 - 1$, the repeated value of $x(0)$ in $E_s^{(2,2)}x$ occurs at the end of the period; i.e., $y(2N_0 - 1) = x(0)$. All of the extensions we define will be centered at either 0 or $-1/2$.

This last fact makes it particularly simple to write down left inverses for the extension operators. Let

$$P_L : \mathbf{R}^N \rightarrow \mathbf{R}^L \quad ; \quad (x_0, \dots, x_{N-1}) \mapsto (x_0, \dots, x_{L-1})$$

be the projection onto the first L coordinates; i.e., a rectangular window of length L . Then

$$P_{N_0} E_{sys} x = x \quad .$$

These projection operators will find additional uses in Section II-C.

Finally, we have five extension operators that will be used exclusively for reconstructing signal subbands, $b_i = E_i a_i$, in the synthesis bank. The symmetric extension $E_s^{(2,1)}a$ pictured in Figure 5 is equivalent to a shift of the extension $E_s^{(1,2)}a$, so it will not be necessary to consider its use as an analysis extension. Four

Table II: Symmetric Extension Operators for Signal Analysis.

Operator E	Extension $y = Ex$:		Symmetry Characterization:	
	Period N	Sym., Center	Time Domain	DFT Domain
$E_s^{(1,1)}$	$2N_0 - 2$	WSS, 0	$y(n) = y(N - n)$	$Y^*(k) = Y(k)$
$E_s^{(1,2)}$	$2N_0 - 1$	OPS, 0	$y(n) = y(N - n)$	$Y^*(k) = Y(k)$
$E_s^{(2,2)}$	$2N_0$	HSS, $-1/2$	$y(n) = y(N - 1 - n)$	$Y^*(k) = e^{-j2\pi k/N} Y(k)$

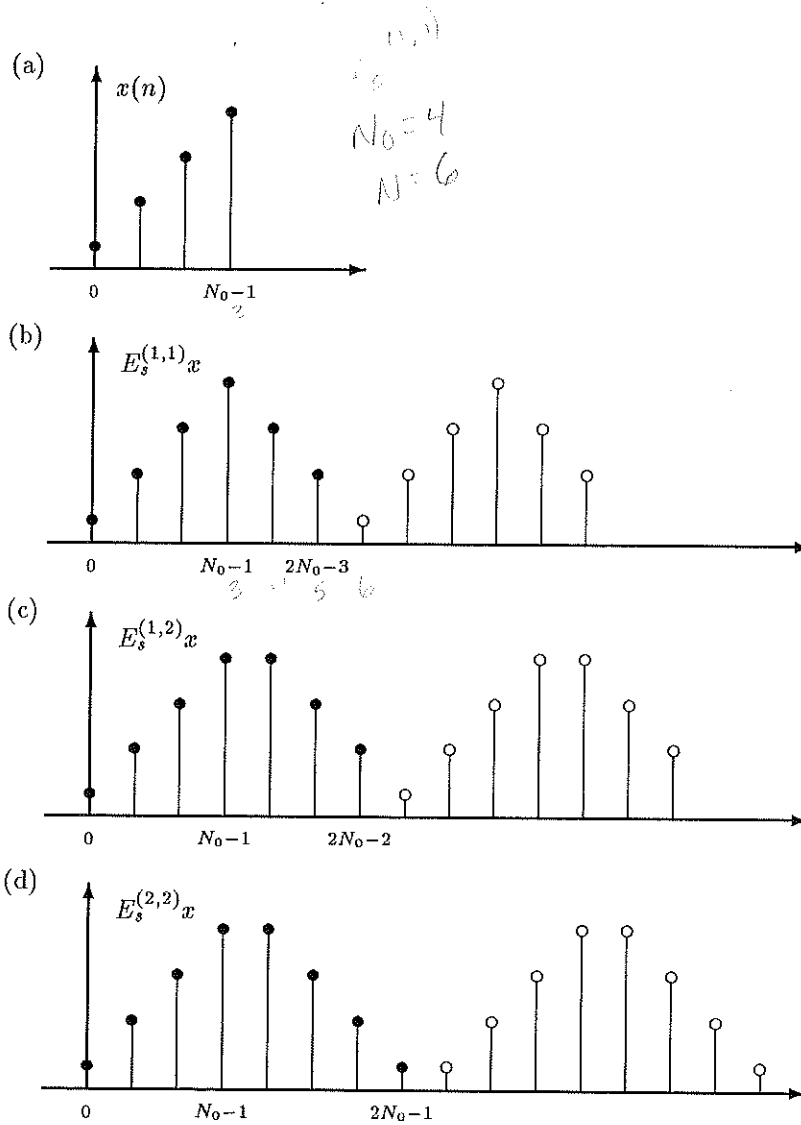


Figure 4: Symmetric Extensions, $y = E_s^{(i,j)}x$, for Signal Analysis.

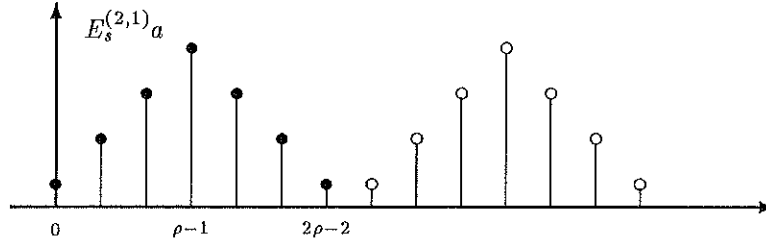


Figure 5: The Symmetric Extension $b = E_s^{(2,1)} a$ for Subband Synthesis.

Table III: Extension Operators for Subband Synthesis.

Operator	Extension $b = Ea$:			Symmetry Relation: Time Domain
	Period K	Sym., Center	Shift η	
$E_s^{(2,1)}$	$2\rho - 1$	OPS, $-1/2$	0	$b(n) = b(K - 1 - n)$
$E_a^{(1,1)}$	$2\rho + 2$	WSA, 0	1	$b(n) = -b(K - n);$ $b(0) = 0 = b(\rho + 1)$
$E_a^{(1,2)}$	$2\rho + 1$	OPA, 0	1	$b(n) = -b(K - n);$ $b(0) = 0$
$E_a^{(2,1)}$	$2\rho + 1$	OPA, $-1/2$	0	$b(n) = -b(K - 1 - n);$ $b(\rho) = 0$
$E_a^{(2,2)}$	2ρ	HSA, $-1/2$	0	$b(n) = -b(K - 1 - n)$

antisymmetric extensions are shown in Figure 6. Note that ρ is the length of a and that operators with an “ a ” subscript are antisymmetric extensions. Formal definitions are given in Table III; the presence of antisymmetric extensions complicates the initialization procedure slightly, however. Because we want WS-type extensions to have centers at 0, the signals being extended with whole-sample antisymmetry at 0 need to be delayed by one sample so we can insert the necessary value $b(0) = 0$. We introduce a parameter, η , to accommodate this delay and initialize the subband extensions

$$b(k) = a(k - \eta) ; k = \eta, \dots, \eta + \rho - 1 .$$

Table III shows that $\eta = 0$ for all extensions except $E_a^{(1,1)}$ and $E_a^{(1,2)}$, which receive a one-sample delay.

II-B. Convolution and Decimation.

The symmetry properties that result from applying linear phase filters to symmetric signal extensions by N -point circular convolution, $u = y * h$, are readily obtained from frequency-domain characterizations. For

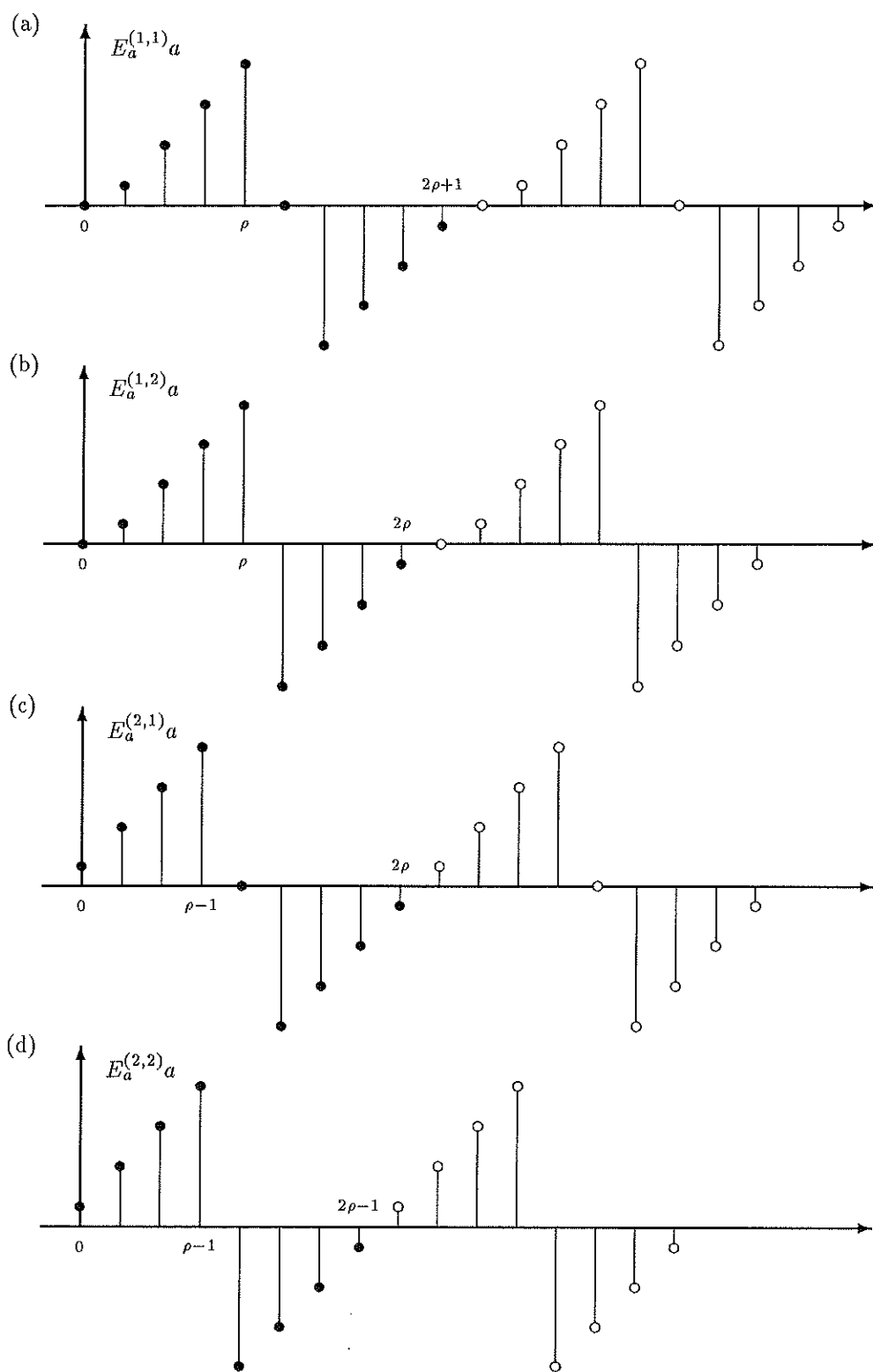


Figure 6: Antisymmetric Extensions, $b = E_a^{(i,j)} a$, for Subband Synthesis.

Table IV: Symmetry and Center of the Convolution Product $u = y * h$.

		Symmetry, Center of Filter h :			
		WSS, γ	WSA, γ	HSS, γ	HSA, γ
Signal y :	WSS, 0	WSS, γ	WSA, γ	HSS, γ	HSA, γ
	HSS, $-1/2$	HSS, $\gamma - 1/2$	HSA, $\gamma - 1/2$	WSS, $\gamma - 1/2$	WSA, $\gamma - 1/2$

instance, if $y = E_s^{(1,1)}x$ and h is HSS then, using (2) and Table II,

$$\begin{aligned}
 U^*(k) &= Y^*(k)H^*(k) \\
 &= Y(k)H(k)e^{j2\pi(2\gamma)k/N} \\
 &= e^{j2\pi(2\gamma)k/N}U(k) .
 \end{aligned}$$

Since 2γ is odd, Table I shows that u is HSS, centered at γ . The other possible combinations of signal and filter symmetry are listed in Table IV; note that we do not distinguish even from odd periods.

A more complicated problem is the matter of determining when $M : 1$ decimation of a symmetric signal, u , results in a symmetric subband, b . While the answer appears to have been intuited correctly in a couple recent papers, nowhere in the literature is there a precise statement, let alone a rigorous mathematical derivation, of this fundamental result. Accordingly, we provide proofs here and record the possibilities in Table V for later use.

To answer the decimation question, we first observe the following rule for interchanging decimators and shifts: if $\delta_M(n)$ is a unit impulse at $n = M$ (an M -sample delay) and $b = \downarrow_M u$, i.e., $b(n) = u(Mn)$, then

$$\downarrow_M (\delta_M * u) = \delta_1 * (\downarrow_M u) = \delta_1 * b . \quad (4)$$

When $M|N$, (4) also holds for $M : 1$ circular downsampling and circular shifts. We shall always insist that the signal length, the analysis extension and decimation factor be constrained so that $M|N$; this ensures that circular operations on length N signals coincide with linear operations on N -periodic signals. Without the restriction $M|N$, $M:1$ decimation of an N -periodic signal doesn't result in an $M : 1$ reduction in the amount of information in the signal, which is the goal of performing decimation in the first place.

The downsampling operation we are using, \downarrow_M , preserves samples occurring at integer multiples of M ,

$$(\downarrow_M u)(n) = u(Mn) . \quad (5)$$

The paper [36] considers two different types of decimation, termed ‘‘Decimation I,’’ which corresponds to (5), and ‘‘Decimation II,’’ which amounts to (5) preceded by a phase shift of $M/2$ samples. Since any such phase shift can be incorporated into the phase of the analysis filter, h , this paper will only make use of

the downsampler defined by (5) and will classify symmetric transforms in terms of the group delays of the analysis filters. Similarly, we shall always define the upsampling (or interpolation) operator to be

$$(\uparrow_M a)(n) = \begin{cases} a(n/M) & \text{if } M|n \text{ ,} \\ 0 & \text{otherwise .} \end{cases} \quad (6)$$

The analogue of (4) for upsampling is

$$\uparrow_M (\delta_1 * u) = \delta_M * (\uparrow_M u) . \quad (7)$$

1. Downsampling WS-type channels. Let u be WSS with center at 0; we want to know all values of c for which $b = \downarrow_M (\delta_c * u)$ will be either WSS or HSS. The phase, c , has been temporarily separated from the symmetry property, now embodied in u , to clarify the following arguments. Because of (4), we only need to consider $0 \leq c \leq M - 1$. Note that c is independent of u ; i.e., we are determining the phases, c , for which $M : 1$ decimation of an arbitrary WSS signal with center c will result in another symmetric signal. The result, b , will be WSS with center n_0 if and only if $b(n_0 + n) = b(n_0 - n)$; i.e.,

$$u(M(n_0 + n) - c) = u(M(n_0 - n) - c) . \quad (8)$$

By considering simple examples of WSS signals like $u = \delta_m + \delta_{-m}$, it is easy to show that (8) holds for *all* WSS signals, u , only if

$$M(n_0 + n) - c = -(M(n_0 - n) - c) .$$

This says $Mn_0 = c$, so for $0 \leq c \leq M - 1$, the only solution is

$$c = n_0 = 0 .$$

The same result holds when u and b are WSA.

It is also possible to get an HSS signal by decimating a WSS signal, u . We will get an HSS signal, $b(n_0 + n) = b(n_0 - 1 - n)$, with center $n_0 - 1/2$ if

$$u(M(n_0 + n) - c) = u(M(n_0 - 1 - n) - c) . \quad (9)$$

The same examples show that (9) holds for all WSS signals, u , if and only if

$$M(n_0 + n) - c = -(M(n_0 - 1 - n) - c) ,$$

or $2Mn_0 - 2c = M$. This constrains M to be even, and

$$c = Mn_0 - M/2 . \quad (10)$$

For $0 \leq c \leq M - 1$, the only solution to (10) is

$$c = M/2 , \quad n_0 = 1/2 . \quad (11)$$

Again, the same result holds when u is WSA and b is HSA.

Let's illustrate briefly how to handle signals with more general centers using (4). Suppose we have a WSS signal centered at $M\nu + M/2$ (i.e., $u * \delta_{M\nu+M/2}$); then

$$b = \downarrow_M (u * \delta_{M\nu+M/2}) = \delta_\nu * (\downarrow_M (\delta_{M/2} * u)) .$$

b is a ν -sample delay of an HSS signal centered, according to (11), at $n_0 - 1/2 = 1/2$, so b is centered at $\nu + 1/2$. Thus, if $\delta_c * u$ is WSS, we've shown that all we need for $b = \downarrow_M (\delta_c * u)$ to be HSS is that M be even and that c be congruent to $M/2 \pmod{M}$:

$$c \equiv M/2 \pmod{M} .$$

This allows us to summarize the above results as follows.

Lemma 1. Let u be a WSS (resp., WSA) signal centered at 0, and let $b = \downarrow_M (\delta_c * u)$. Then b is WSS (resp., WSA) with center at n_0 if and only if c is a multiple of M (i.e., $c \equiv 0 \pmod{M}$), in which case $n_0 = c/M$. Similarly, b is HSS (resp., HSA) if and only if M is even and $c \equiv M/2 \pmod{M}$, in which case b is centered at the half-integer point c/M .

2. Downsampling HS-type channels. Now do the same analysis when u is an HSS signal centered at $-1/2$: $u(n) = u(-1-n)$. Can $b = \downarrow_M (\delta_c * u)$ be WSS with center n_0 : $b(n_0+n) = b(n_0-n)$? This says

$$u(M(n_0+n)-c) = u(M(n_0-n)-c) ; \tag{12}$$

by considering examples like $u = \delta_m + \delta_{-1-m}$ we see that (12) holds for all HSS signals, u , if and only if

$$M(n_0+n)-c = -1 - (M(n_0-n)-c) , \quad \text{or} \quad 2Mn_0 - 2c = -1 .$$

This has no solutions, so b is never WSS (or WSA) if u is HSS (or HSA).

There are phases, c , however, for which b is HSS: $b(n_0+n) = b(n_0-1-n)$. In terms of u ,

$$u(M(n_0+n)-c) = u(M(n_0-1-n)-c) .$$

By the same examples, this requires

$$M(n_0+n)-c = -1 - (M(n_0-1-n)-c) ,$$

or $2Mn_0 - 2c = M - 1$. This means that M must be odd and

$$c = Mn_0 + \frac{1-M}{2} . \tag{13}$$

For $0 \leq c \leq M-1$, the only solution to (13) is

$$c = \frac{M+1}{2} , \quad n_0 = 1 . \tag{14}$$

Table V: Symmetry Properties of Down-Sampled Signals.

Signal y : Sym., Center	Filter h :			Subband Symmetry, Center:		Shift β
	Symmetry	Center γ	M	$u = y * h$	$b = \downarrow_M u$	
WSS, 0	WSS/WSA	$M\nu$	Any	WSS/A, $M\nu$	WSS/A, ν	ν
		$M\nu + M/2$	Even	WSS/A, $M\nu + M/2$	HSS/A, $\nu + 1/2$	$\nu + 1$
	HSS/HSA	$M\nu + M/2$	Odd	HSS/A, $M\nu + M/2$	HSS/A, $\nu + 1/2$	$\nu + 1$
HSS, $-1/2$	HSS/HSA	$M\nu + 1/2$	Any	WSS/A, $M\nu$	WSS/A, ν	ν
		$M\nu + (M + 1)/2$	Even	WSS/A, $M\nu + M/2$	HSS/A, $\nu + 1/2$	$\nu + 1$
	WSS/WSA	$M\nu + (M + 1)/2$	Odd	HSS/A, $M\nu + M/2$	HSS/A, $\nu + 1/2$	$\nu + 1$

Thus, b is HSS with center at $n_0 - 1/2 = 1/2$; the same result holds when u and b are HSA. Including the effect of shifts by multiples of M , we get the following.

Lemma 2. Let u be an HSS (resp., HSA) signal centered at $-1/2$, and let $b = \downarrow_M (\delta_c * u)$. Then b is never WSS or WSA, and b is HSS (resp., HSA) if and only if M is odd and $c \equiv (M + 1)/2 \pmod{M}$, in which case $\delta_c * u$ is centered at $c - 1/2 \equiv M/2 \pmod{M}$, and b is centered at $(2c - 1)/2M$.

Now drop the restriction that “ u ” denotes a signal centered only at 0 or $-1/2$, so we can write $u = y * h$ where the group delay, γ , of h is arbitrary. From Table IV we see that there are two ways in which u can arise as an HS-type signal. The delays, γ , for which u will have a center congruent to $M/2 \pmod{M}$ necessarily differ (by $1/2$) depending on whether y is WSS or HSS. Similarly, there are two ways in which u can arise as a WS-type signal with center congruent to $M/2 \pmod{M}$, for group delays depending on the symmetry of y . In Table V we combine the results of Lemmas 1 and 2 with Table IV to enumerate all combinations of signal symmetry, filter symmetry and phase, and decimation factor that result in a symmetric subband, b . Note that we do not distinguish even- from odd-periodic cases in this table, so the symmetry of b is given generically as, e.g., WSS or WSA (abbreviated “WSS/A”). This should, of course, be interpreted as OPS or OPA when the period of b is odd. The shift factor, β , is the number of samples b needs to be advanced to move its center up to 0 or $-1/2$, a detail that will be explained in the next section.

II-C. Subband Projection and Reconstruction.

Table V lists all conditions under which we obtain a symmetric subband, b , from an extended subband coder with linear phase filters. We now exploit the symmetry of b to reduce data storage requirements. Since N is the period of y , the period of b is $K = N/M$. If b is symmetric, we only need to save one-half of a period of b ; the discarded values can be reconstructed using the appropriate symmetry relation.

Consider an HSA subband, b , of period $K = 2\rho$ (compare Figure 6(d)); if b is centered at $-1/2$, then the values $\{b(0), \dots, b(K/2 - 1)\}$ comprise exactly one-half of a period of b . We define

$$a = P_{K/2} b \text{ ,}$$

where $P_{K/2}$ is the projection or rectangular window, defined in Section II-A, on the first $\rho = K/2$ samples in a period of b . Now we can reconstruct b by extending the ρ -point sequence, a :

$$b = E_a^{(2,2)} a \text{ .}$$

The same procedure works when b is HSS, using the symmetric extension operator $E_s^{(2,2)}$ for reconstruction. When b is WSS with center at 0 (cf. Figure 4(b)), we need to save $\rho = K/2 + 1$ values and so define $a = P_{K/2+1} b$, reconstructing b via $b = E_s^{(1,1)} a$.

The situation when b is WSA with center at 0 (Figure 6(a)) is slightly different because we don't need to save the trivial value $b(0) = 0$. Therefore, b gets advanced by one sample and then projected onto the next $\rho = K/2 - 1$ values:

$$a = P_{K/2-1}(\delta_{-1} * b) \text{ ; } b = E_a^{(1,1)} a \text{ .}$$

Note that the antisymmetric extension operator, $E_a^{(1,1)}$, contains the one-sample delay needed to insert the trivial values $b(0) = 0 = b(K/2)$.

In general, the number of points that need to be saved is given by the dimension of b ; dimensions for the six symmetry classes were given in Table I. When b has dimension ρ and center 0 or $-1/2$, we can save $a = P_\rho b$ provided we advance b by one sample in the case of signals with whole-sample antisymmetry at 0. The dimension, ρ , is thus the same as the rank of the pertinent projection operator, P_ρ . The one-sample advance for signals that are WSA at 0 is given by the shift factor η in Table III. One more concern: if b is not centered at 0 or $-1/2$, we first need to advance b by enough samples to move its center up to either 0 or $-1/2$; the necessary shift factor, β , is given in Table V as a function of signal and filter symmetry.

1. The extended subband coder. We are now able to fill in the missing details in the block diagram for the extended M -channel subband coder, Figure 2. For simplicity, we only display a single channel, and we display the analysis and synthesis banks separately; see Figure 7, which indicates the period of each signal component. In the analysis bank, the input signal, x , of length N_0 is extended by an operator, E_{sys} , from Table II, filtered and decimated, then advanced $\beta_i + \eta_i$ samples before being projected onto coordinates $n = 0, \dots, \rho_i - 1$ to produce the output channel a_i . In the synthesis bank, a_i is extended by the appropriate operator, E_i , delayed β_i samples, upsampled, and filtered. The choice of extension operator, E_i , in the synthesis bank is dictated by the symmetry of b_i , as determined from Table V. The appropriate operators and shift factors, η_i , are given in Table VI for each of the six possible subband symmetries. If the filter bank satisfies condition (PR), then the output will have only constant amplitude distortion: $\hat{x}(n) = Ax(n)$.

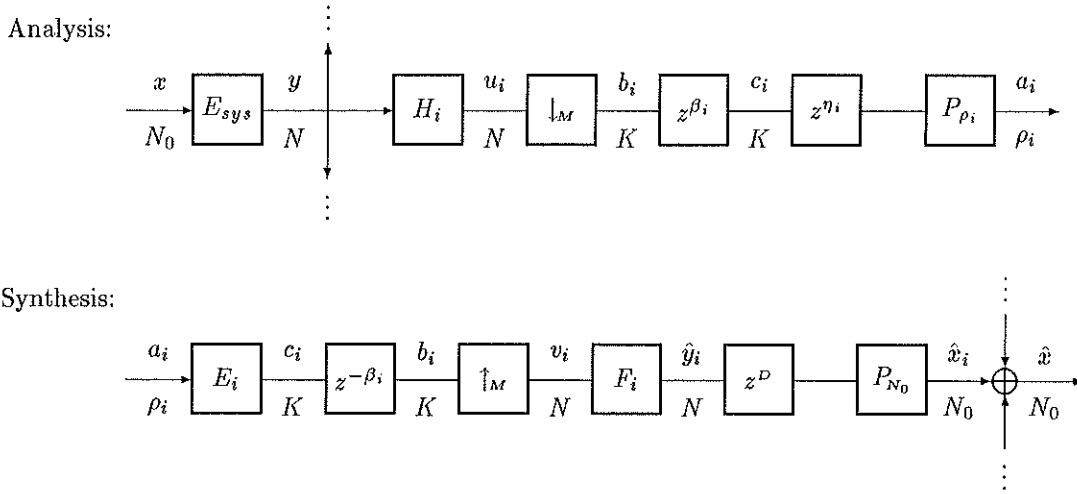


Figure 7: Analysis/Synthesis Channels for Extended Subband Coder.

II-D. Generic Analysis/Synthesis Computations.

It is now simple to write down formulas for direct-form implementation of the SWT analysis and synthesis operations depicted in Figure 7. In the following formulas, we assume x has been extended to a symmetric signal, y , of period N . The filters h_i and f_i are implicitly extended with zeros to period N ; the parameters η_i , β_i , and ρ_i and the synthesis extension E_i are determined from Tables V and VI based on the symmetry of y , the symmetry and phase of h_i , and the decimation, M .

1. *Analysis.* Using equation (4), we can write

$$\begin{aligned} a_i &= P_{\rho_i}(\delta_{-(\beta_i+\eta_i)} * \downarrow_M (y * h_i)) \\ &= P_{\rho_i} \downarrow_M (y * (h_i * \delta_{-M(\beta_i+\eta_i)})) , \end{aligned}$$

so

$$a_i(k) = \sum_{n=0}^{N-1} y(n) h_i(Mk - n + M(\beta_i + \eta_i)) ; k = 0, \dots, \rho_i - 1 . \quad (15)$$

2. *Synthesis.* Note that the definition (6) of the upsampling operator implies the formula

$$((\uparrow_M b_i) * f_i)(n) = \sum_{k=0}^{K-1} b_i(k) f_i(n - Mk) . \quad (16)$$

After forming the synthesis extension, $c_i = E_i a_i$, and simplifying the synthesis calculation using (7),

$$\begin{aligned} \hat{x}_i &= P_{N_0}(\delta_{-D} * f_i * \uparrow_M (\delta_{\beta_i} * c_i)) \\ &= P_{N_0}(\delta_{-D} * f_i * \delta_{M\beta_i} * \uparrow_M c_i) \end{aligned}$$

Table VI: Symmetric Subband Reconstruction.

Subband b :			Shift	Extension
Symmetry	Dim. ρ	Center	η	E
WSS	$K/2 + 1$	0	0	$E_s^{(1,1)}$
WSA	$K/2 - 1$	0	1	$E_a^{(1,1)}$
HSS	$K/2$	$-1/2$	0	$E_s^{(2,2)}$
HSA	$K/2$	$-1/2$	0	$E_a^{(2,2)}$
OPS	$(K + 1)/2$	0	0	$E_s^{(1,2)}$
(K Odd)		$-1/2$	0	$E_s^{(2,1)}$
OPA	$(K - 1)/2$	0	1	$E_a^{(1,2)}$
(K Odd)		$-1/2$	0	$E_a^{(2,1)}$

synthesis can be evaluated using (16):

$$\hat{x}_i(n) = \sum_{k=0}^{K-1} c_i(k) f_i(n - Mk - M\beta_i + D) ; n = 0, \dots, N_0 - 1 . \quad (17)$$

3. Causality. In the papers [33, 34], Martucci points out that perfect reconstruction with zero delay is impossible in a two-channel QMF bank with causal analysis and synthesis filters. He then concludes that this implies non-causal filter bank implementations are necessary when transforming symmetric signal extensions. The real causality issue in the symmetric transform case, however, is the fact that symmetric extensions, $y = E_s^{(i,j)}x$, are *themselves* inherently non-causal processes; i.e., y 's "past," the values $\{y(n) : n < 0\}$, is defined in terms of y 's "future," the values $\{y(n) : n \geq 0\}$. The whole notion of causality is predicated on a source information stream with a time-like orientation, an assumption that no longer holds if one wishes to regard a symmetric, periodized extension of the raw data as the system input.

The crucial consideration for SWT implementations is the precise set of filter output samples that need to be computed to provide a complete, nonredundant half-period of each subband for transmission. Any application employing a symmetric extension technique must provide knowledge of a sufficiently long segment of "future" data to allow the computation of those nonredundant samples whose values depend on the input signal's "past." This means that SWT systems are *never* causal in the temporal, intuitive sense of the word. For this reason, we shall only use the word in its trivial, mathematical sense: a causal filter bank is one whose impulse responses are identically zero for all $n < 0$. We will not talk about "causal SWT systems."

To implement an SWT starting with a given PR QMF bank, one needs to manipulate filter delays because filter banks are frequently designed (either analytically or numerically) with causal impulse responses. According to Figure 7 and Tables V and VI, the effect of filter group delay on an SWT system is the waiting

time specified by the advances, $z^{\beta_i + \eta_i}$, needed before projecting off a complete, nonredundant half-period of the subband b_i . This necessary wait can be eliminated by pushing the advances ahead of the decimator using (4) and advancing the analysis filter:

$$H_i(z) \rightarrow z^{M(\beta_i + \eta_i)} H_i(z) \quad (\text{cf. equation (15)}) \quad .$$

When filters are made noncausal in this manner to eliminate delays in the analysis bank, the system delay is determined by the phase of the synthesis bank; this subject will be treated more thoroughly in Section III for two-channel systems, where we will derive precise relationships between the group delays in the analysis and synthesis banks and the overall system delay. For instance, we show that a PR QMF system with causal analysis filters can have zero delay distortion if the synthesis bank is advanced sufficiently. Thus, noncausality for SWT analysis banks results from the need to compute a complete, nonredundant half-period's worth of samples in each SWT subband rather than from aliasing cancellation requirements.

II-E. SWT Expansiveness.

In Section I-A we defined the dimension, N_1 , of an SWT like the one in Figure 2 to be the sum of the individual subband ranks, $N_1 = \sum \rho_i$, and called an SWT *nonexpansive* if $N_1 = N_0$. We now consider the issue of expansiveness in greater detail. Our analysis will concentrate on maximally decimated PR QMF banks of the type shown in Figure 1, although we will mention briefly how to extend the analysis to SWT's based on nonuniformly downsampled PR QMF banks.

The SWT, $x \mapsto \{a_1, \dots, a_M\}$, is a linear transformation,

$$\text{SWT} : \mathbf{R}^{N_0} \rightarrow \mathbf{R}^{\rho_1} \times \dots \times \mathbf{R}^{\rho_M} \cong \mathbf{R}^{N_1} ; \quad (18)$$

since the transform is invertible, we have $N_1 \geq N_0$. We define the *expansiveness*, ε_{sys} , of an SWT to be

$$\varepsilon_{sys} = N_1 - N_0 \geq 0 . \quad (19)$$

The expansiveness represents the number of additional dimensions (read: "storage registers") needed to hold the transformed signal above and beyond the number required by the original input, x . The nonexpansive case is the one in which $\varepsilon_{sys} = 0$.

The transform (18) will be called an *equal-rank* SWT if all subbands have the same rank: $\rho_1 = \rho_2 = \dots = \rho_M \equiv \rho$. Equal-rank transforms are nice because the details of storing and manipulating output are a little simpler than for systems with variable-length output channels, particularly if the outputs are cascaded back through an SWT analysis bank. The equal-rank condition is not necessary, however, for either perfect reconstruction or, as we will show, for nonexpansiveness. An equal-rank transform has dimension

$$N_1 = \sum_{i=1}^M \rho_i = M\rho ,$$

so to design a nonexpansive equal-rank SWT, $N_0 = N_1 = M\rho$, we must first satisfy the necessary condition $M|N_0$. As a trivial example of the kind of design limitations this condition imposes on us, note that we cannot construct a nonexpansive equal-rank two-channel SWT for odd-length signals. In Section III we will construct nonexpansive two-channel SWT's for odd-length signals using channels with unequal ranks.

The analysis so far of the operations involved in converting a PR QMF bank into an SWT has focused on individual channels in the coder, without requiring any system-wide considerations as are needed in designing filter banks that satisfy condition (PR). While expansiveness is clearly a property of the overall SWT system, we can still define a measure of expansiveness for individual SWT channels to give some indication of the efficiency of a single channel. Define the *channel expansiveness*, ε_i , of the i^{th} channel in an SWT to be

$$\varepsilon_i = \rho_i - N_0/M . \quad (20)$$

This is consistent with definition (19) in the sense that

$$\sum_{i=1}^M \varepsilon_i = \sum_{i=1}^M (\rho_i - N_0/M) = \varepsilon_{sys} , \quad (21)$$

although it is possible, under definition (20), for a channel to have fractional expansiveness. An advantage of this approach is that it is easy to tabulate the expansiveness of individual channels using (20) and then calculate ε_{sys} via (21). It also allows us to use the inequality in (19) to rule out certain channel combinations for linear phase PR QMF banks via the constraint

$$\sum_{i=1}^M \varepsilon_i \geq 0 . \quad (22)$$

If a bank of linear phase filters creates symmetric subbands with $\sum \varepsilon_i < 0$ for some symmetric extension, E_{sys} , then it defines a noninvertible transform of \mathbf{R}^{N_0} and therefore cannot be a PR QMF bank.

1. Nonuniformly downsampled filter banks. Some recent work on symmetric wavelet transform techniques [37, 38] has addressed the possibility of using nonuniformly downsampled linear phase PR QMF banks; i.e., filter banks that differ from the one depicted in Figure 1 by having different decimation ratios, M_i , in different subbands. Such a filter bank is *critically downsampled* if

$$\sum_i M_i^{-1} = 1 .$$

Because of the requirement that $M | N$ in any SWT scheme, nonuniform downsampling imposes more restrictions on the allowable combinations of input signal length and symmetric extension type than does uniform downsampling since we now need $M_i | N$ for all M_i ; see Section IV-A for more on divisibility criteria for M -channel SWT's. Nonetheless, we can still employ the above analysis of expansiveness in the case of nonuniform filter banks.

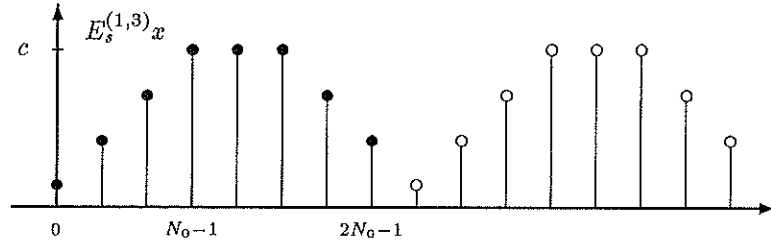


Figure 8: The Symmetric Signal Extension $y = E_s^{(1,3)}x$.

For coding applications in which expansiveness is an issue, it makes sense to restrict attention to critically downsampled filter banks. In the nonuniform case, we define the expansiveness of the i^{th} channel to be

$$\varepsilon_i = \rho_i - N_0/M_i ,$$

generalizing (20). Since $\varepsilon_{sys} = \sum \rho_i - N_0$, the relationship (21) continues to hold for critically downsampled nonuniform filter banks. This means that all of the results we shall present in Section IV concerning individual channels in M -channel SWT's are applicable to the design of nonuniformly downsampled SWT's, including the analysis of expansiveness.

2. Completeness of the classification. The claim was made in Section II-A that the three principal symmetric extensions characterized in Table II and depicted in Figure 4 are the only ones (except for trivial phase shifts of these three, like $E_s^{(2,1)}x$) that are capable of producing nonexpansive SWT's. To make this claim precise, we need to establish the scope of what we are considering to be "symmetric wavelet transforms." Accordingly, we restrict attention to the extensions $E_s^{(i,j)}x$; as mentioned before, these extensions have the advantage that they require no additional computations when extrapolating the signal. Moreover, we only consider algorithms in which the redundancy introduced by symmetric extension is removed by windowing symmetric subbands, b_i . The claim is that the extensions $E_s^{(i,j)}x$ for $(i,j)=(1,1)$, $(1,2)$, or $(2,2)$ are the only ones for which the ranks, ρ_i , of the subbands can add up to exactly N_0 . The reason for this is that if either i or j is greater than or equal to 3, then the resulting transform is equivalent to an (unnecessary) extension of the input, x , to a longer input, x' , followed by a nonexpansive transform of x' .

For instance, consider the extension $y = E_s^{(1,3)}x$ shown in Figure 8. This is equivalent to $E_s^{(1,1)}x'$, where x' is the signal of length $N_0 + 1$ obtained by replicating the last sample, $c = x(N_0 - 1)$. Since we are now transforming a signal of length $N_0 + 1$, the above analysis shows that an SWT based on applying $E_s^{(1,1)}$ to x' will produce subbands having ranks satisfying $\sum \rho'_i \geq N_0 + 1$, unless there is some other way to remove the extra sample's worth of redundancy hidden in the transform of x' . To see that this additional redundancy cannot be removed by simple windowing of the subbands b'_i , note that x' can be obtained by one-sample

zero-padding of x followed by the addition of an impulse, $c\delta_{N_0}$, at $n = N_0$:

$$x'(n) = x^+(n) + c\delta_{N_0}(n) \quad ; \quad n = 0, \dots, N_0 \quad ,$$

where

$$x^+(n) = \begin{cases} x(n), & 0 \leq n < N_0 \\ 0, & n = N_0 \end{cases} .$$

Then $y = y^+ + c\delta_{N_0}$, where $y^+ = E_s^{(1,1)}x^+$, so the result of filtering y by h is

$$y * h = y^+ * h + c\delta_{N_0} * h \quad .$$

The redundancy inherent in the symmetry of $b = \downarrow_M(y * h)$ can be removed by windowing, as described above, but the redundancy resulting from the one-point extension $x \rightarrow x'$ is spread out across the subband by the $(c\delta_{N_0} * h)$ term. In particular, the effects of the one-sample redundancy are *not* manifest in a form (like subband symmetry or endpoint duplication) that can be removed by windowing operations.

This same analysis can be applied to other extensions involving multiple duplications of signal endpoints, leading to the following conclusion.

Theorem 3. All nonexpansive symmetric wavelet transforms based on signal extrapolations of the form $y = E_s^{(i,j)}x$ are based on the three extensions $E_s^{(1,1)}x$, $E_s^{(1,2)}x$, and $E_s^{(2,2)}x$.

Thus, this paper's scope is limited to a classification of SWT's based on these three principal extensions.

III. TWO-CHANNEL SYMMETRIC SUBBAND CODERS.

This section of the paper will give a complete classification of all nonexpansive two-channel SWT's, using the extensions $E_s^{(1,1)}x$ and $E_s^{(2,2)}x$ described in Section II-A. The extension $E_s^{(1,2)}x$ has an odd period and therefore cannot be used in conjunction with 2:1 downsampling. We first tabulate all individual channel symmetries and channel expansiveness figures for both admissible analysis extensions. Next, we give all possible filter symmetry and phase combinations for two-channel SWT filter banks. Finally, we put these possibilities together to classify two-channel SWT's. The particular cases incorporated in the FBI standard [29] for compression of gray-scale fingerprint images are given at the end of this section.

III-A. Two Types of Symmetric Extension.

1. *The extension $y = E_s^{(1,1)}x$.* We will walk through the analysis, listing the results in Table VII. According to Table II, the period of y is $N = 2N_0 - 2$, so the period of an unprojected subband, b_i , is $K = N/2 = N_0 - 1$. Since y is WSS we see from Table V that HSS/HSA filters never produce symmetric subbands when $M = 2$. We therefore will not list any signal/filter combinations in Table VII involving HSS/HSA filters since they

Table VII: (1, 1)-Symmetric Wavelet Transform Channels, $M = 2$.

Signal x : N_0	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ε
Even (K Odd)	WSS	2ν	OPS, ν	$N_0/2$	ν	0	$E_s^{(1,2)}$	0
		$2\nu + 1$	OPS, $\nu + 1/2$	$N_0/2$	$\nu + 1$	0	$E_s^{(2,1)}$	0
	WSA	2ν	OPA, ν	$N_0/2 - 1$	ν	1	$E_a^{(1,2)}$	-1
		$2\nu + 1$	OPA, $\nu + 1/2$	$N_0/2 - 1$	$\nu + 1$	0	$E_a^{(2,1)}$	-1
Odd (K Even)	WSS	2ν	WSS, ν	$(N_0 + 1)/2$	ν	0	$E_s^{(1,1)}$	1/2
		$2\nu + 1$	HSS, $\nu + 1/2$	$(N_0 - 1)/2$	$\nu + 1$	0	$E_s^{(2,2)}$	-1/2
	WSA	2ν	WSA, ν	$(N_0 - 3)/2$	ν	1	$E_a^{(1,1)}$	-3/2
		$2\nu + 1$	HSA, $\nu + 1/2$	$(N_0 - 1)/2$	$\nu + 1$	0	$E_a^{(2,2)}$	-1/2

can never be part of a (1, 1)-SWT. In general, we will not bother listing signal/filter symmetry or phase combinations that do not result in symmetric subbands.

Let N_0 be even (so K is odd), and consider WSS filters, h . From Table V we see there are two distinct cases, depending on whether the group delay of h is even ($\gamma = 2\nu$) or odd ($\gamma = 2\nu + 1$). Both cases work since M is even, resulting in OPS subbands with centers ν and $\nu + 1/2$, respectively, and requiring shift factors, β , of ν and $\nu + 1$. The dimension of an OPS subband of period $K = N_0 - 1$ is, according to Table VI,

$$\rho = \frac{K + 1}{2} = N_0/2 ,$$

so the channel expansiveness is

$$\varepsilon = \rho - N_0/M = 0 .$$

Table VI also gives the shift factor $\eta = 0$ (in both phases) and the synthesis extensions $E_s^{(1,2)}$ and $E_s^{(2,1)}$.

This yields the first two lines in Table VII. Note that these are precisely the design features needed to describe the systems depicted in Figure 7. The next two lines in the table give the analogous results for WSA filters. The second half of Table VII contains the subband design parameters when N_0 is odd.

2. *The extension $y = E_s^{(2,2)}x$.* This is an HSS extension of period $N = 2N_0$, with subband period $K = N_0$. Table V shows that we will not obtain symmetric subbands from $E_s^{(2,2)}x$ if h is WSS or WSA, so Table VIII lists the possible channels obtained using HSS/HSA filters. Derivation is similar to that for Table VII.

Table VIII: (2, 2)-Symmetric Wavelet Transform Channels, $M = 2$.

Signal x : N_0	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ϵ
Even (K Even)	HSS	$2\nu + 1/2$	WSS, ν	$N_0/2 + 1$	ν	0	$E_s^{(1,1)}$	1
		$2\nu + 3/2$	HSS, $\nu + 1/2$	$N_0/2$	$\nu + 1$	0	$E_s^{(2,2)}$	0
	HSA	$2\nu + 1/2$	WSA, ν	$N_0/2 - 1$	ν	1	$E_a^{(1,1)}$	-1
		$2\nu + 3/2$	HSA, $\nu + 1/2$	$N_0/2$	$\nu + 1$	0	$E_a^{(2,2)}$	0
Odd (K Odd)	HSS	$2\nu + 1/2$	OPS, ν	$(N_0 + 1)/2$	ν	0	$E_s^{(1,2)}$	1/2
		$2\nu + 3/2$	OPS, $\nu + 1/2$	$(N_0 + 1)/2$	$\nu + 1$	0	$E_s^{(2,1)}$	1/2
	HSA	$2\nu + 1/2$	OPA, ν	$(N_0 - 1)/2$	ν	1	$E_a^{(1,2)}$	-1/2
		$2\nu + 3/2$	OPA, $\nu + 1/2$	$(N_0 - 1)/2$	$\nu + 1$	0	$E_a^{(2,1)}$	-1/2

III-B. Two-Channel Discrete Wavelet Transforms.

In this section we will determine all possible combinations of filter symmetry and phase for SWT's based on two-channel PR QMF banks. The input-output relationship for such a system (see Figure 1) is

$$\hat{X}(z) = \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z) .$$

The coefficient of $X(-z)$, the ‘‘aliasing term’’ in the system function, can be eliminated by defining the synthesis filters according to the following anti-aliasing relations:

$$F_0(z) = \mp z^{-D_s} H_1(-z) , \quad F_1(z) = \pm z^{-D_s} H_0(-z) . \quad (\text{AA})$$

This produces a linear translation-invariant system with transfer function

$$T(z) \equiv \frac{\hat{X}(z)}{X(z)} = \pm \frac{z^{-D_s}}{2} [H_0(-z)H_1(z) - H_0(z)H_1(-z)] . \quad (23)$$

In the Introduction, we said that a PR QMF bank satisfies the perfect-reconstruction condition,

$$T(z) = Az^{-D} . \quad (\text{PR})$$

Using equation (23) in the two-channel case, condition (PR) reads

$$H_0(-z)H_1(z) - H_0(z)H_1(-z) = 2Az^{-D+D_s} . \quad (24)$$

Since the left-hand side of (24) is an odd function of z , $D_s - D$ must be odd; define D_a to be the difference,

$$D_a \equiv D - D_s = 2m + 1 .$$

The process of designing a two-channel PR QMF bank therefore reduces to the following: given an analysis bank satisfying

$$H_0(-z)H_1(z) - H_0(z)H_1(-z) = 2Az^{-D_a} \quad , \quad D_a \text{ odd}, \quad (\text{PR2})$$

the anti-aliasing relations (AA) define a PR QMF system with transfer function

$$T(z) = \pm Az^{-D} \quad , \quad D = D_a + D_s \quad . \quad (\text{TF2})$$

According to (TF2), we can advance or delay the synthesis bank by an arbitrary factor, D_s , with a concomitant advance or delay of the transfer function. In particular, any PR QMF bank can be made into a zero-delay system by choosing $D_s = -D_a$ for the synthesis bank. From now on, we shall concentrate on DWT analysis banks and assume that the synthesis banks are given by (AA). The precise relationship between D_a and the group delays of the analysis filters will be made clear in the next subsection.

1. Symmetry/phase possibilities: nontrivial filter banks. Consider the possible combinations of filter symmetries that might occur in an SWT analysis bank. While [13] reports the existence of trivial PR QMF pairs with both even- and odd-order (i.e., WS- and HS-type) filters, Table V shows that one or the other of these classes is excluded when $M = 2$, depending on whether the extension $E_{s_y,s}x$ is HSS or WSS. Consequently, the filters in a two-channel SWT must have the same symmetry type (i.e., either both must be HS-type or both must be WS-type). Moreover, it is shown in [13] that the only nontrivial classes of two-channel linear phase FIR PR QMF banks are those containing either a WSS-WSS or an HSS-HSA filter pair.

In the nontrivial cases, an interesting relationship holds between the analysis-bank component of the DWT system delay, z^{-D_a} , and the group delays γ_0, γ_1 of the individual analysis filters:

Theorem 4. *Let $\{h_0, h_1\}$ be a nontrivial linear phase PR QMF bank with analysis delay component z^{-D_a} . Then $D_a = \gamma_0 + \gamma_1$; in particular, $\gamma_0 + \gamma_1$ must be odd.*

Proof. By (2) and (3),

$$H_0(z^{-1}) = z^{2\gamma_0} H_0(z) \quad , \quad H_1(z^{-1}) = (-1)^{2\gamma_1} z^{2\gamma_1} H_1(z) \quad ,$$

so condition (PR2) implies

$$\begin{aligned} 2Az^{D_a} &= H_0(-z^{-1})H_1(z^{-1}) - H_0(z^{-1})H_1(-z^{-1}) \\ &= z^{2\gamma_0+2\gamma_1} [(-1)^{2\gamma_0+2\gamma_1} H_0(-z)H_1(z) - (-1)^{4\gamma_1} H_0(z)H_1(-z)] \\ &= z^{2\gamma_0+2\gamma_1} [H_0(-z)H_1(z) - H_0(z)H_1(-z)] \\ &= 2Az^{-D_a} z^{2\gamma_0+2\gamma_1} \quad , \end{aligned}$$

since $4\gamma_1$ is even and $2\gamma_0 + 2\gamma_1$ is even in the nontrivial cases. Thus, $D_a = \gamma_0 + \gamma_1$. \blacksquare

The crucial assumption that $2\gamma_0 + 2\gamma_1$ is even fails for the trivial mixed-type filter banks mentioned above. As a corollary of Theorem 4, we get the following result characterizing the possible phase shifts for a nontrivial linear phase PR QMF analysis bank.

Corollary 5. *If $\{H_0(z), H_1(z)\}$ is a nontrivial linear phase PR QMF bank and $H'_0(z) = z^{-m_0} H_0(z)$, $H'_1(z) = z^{-m_1} H_1(z)$, then $\{H'_0(z), H'_1(z)\}$ is a PR QMF bank if and only if $m_0 + m_1$ is even, with delay component*

$$D'_a = D_a + m_0 + m_1 .$$

Proof. We are given

$$H_0(-z)H_1(z) - H_0(z)H_1(-z) = 2Az^{-D_a} , \quad D_a \text{ odd},$$

so

$$\begin{aligned} H'_0(-z)H'_1(z) - H'_0(z)H'_1(-z) &= (-z)^{-m_0} H_0(-z)z^{-m_1} H_1(z) - z^{-m_0} H_0(z)(-z)^{-m_1} H_1(-z) \\ &= (-1)^{-m_0} z^{-m_0-m_1} [H_0(-z)H_1(z) - (-1)^{m_0-m_1} H_0(z)H_1(-z)] . \end{aligned} \quad (25)$$

If $m_0 + m_1$ is even then so is $m_0 - m_1$, and (25) reduces to

$$H'_0(-z)H'_1(z) - H'_0(z)H'_1(-z) = \pm 2Az^{-D_a-m_0-m_1} ,$$

which has an odd exponent and thus satisfies (PR2). Conversely, if $\{H'_0(z), H'_1(z)\}$ satisfies (PR2) then, by Theorem 4,

$$D'_a = \gamma'_0 + \gamma'_1 = D_a + m_0 + m_1 ,$$

so $m_0 + m_1$ must be even since both D_a and D'_a are odd. \blacksquare

III-C. Classification of Two-Channel Symmetric Wavelet Transforms.

This section enumerates all possible two-channel SWT's based on the results of Sections III-A and III-B. The transforms will be classified according to extension, E_{sys} , input signal length, N_0 , and analysis filter bank symmetries and phases. Synthesis filter banks will not be discussed, except for the FBI implementations, since they are completely determined by condition (AA) and do not affect our analysis of system expansiveness. Transforms based on the extension $E_s^{(i,j)}$ will be referred to as " (i, j) -SWT's."

1. *(1,1)-Symmetric wavelet transforms.* According to Table VII, the SWT's based on the extension $y = E_s^{(1,1)}x$ all use WS-type filters, so Theorem 4 says that the analysis bank $\{H_0, H_1\}$ must consist of a WSS-WSS pair whose phases have opposite parities, which we denote $\gamma_0 = 2\nu_0$, $\gamma_1 = 2\nu_1 + 1$. There are only two classes of SWT's satisfying these conditions, corresponding to N_0 even or N_0 odd.

N_0 *Even*: This (1,1)-SWT is defined by the first two lines in Table VII. The subband ranks and the channel expansions are

$$\rho_0 = N_0/2 = \rho_1 \quad , \quad \varepsilon_0 = 0 = \varepsilon_1 \quad ,$$

so this is a nonexpansive equal-rank transform.

N_0 *Odd*: The (1,1)-SWT for odd-length signals necessarily has unequal ranks,

$$\rho_0 = (N_0 + 1)/2 \quad , \quad \rho_1 = (N_0 - 1)/2 \quad .$$

Nonetheless, it is nonexpansive: $\varepsilon_0 = 1/2$ and $\varepsilon_1 = -1/2$ so $\varepsilon_{sys} = 0$.

While one could always duplicate an endpoint on an odd-length signal and apply an even-length transform, such a procedure would necessarily be expansive according to Theorem 3; the transform described here, while it does have unequal subband ranks, provides a nonexpansive alternative for odd-length signals.

2. *(2,2)-Symmetric wavelet transforms*. According to Theorem 4, we must use an HSS-HSA pair of analysis filters whose phases satisfy either $\gamma_0 = 2\nu_0 + 1/2$ and $\gamma_1 = 2\nu_1 + 1/2$, or else $\gamma_0 = 2\nu_0 + 3/2$ and $\gamma_1 = 2\nu_1 + 3/2$. Combining these two options with the choice of N_0 even or N_0 odd gives us four distinct cases to consider.

N_0 *Even*, $\gamma_i = 2\nu_i + 1/2$: Let h_0 be HSS and h_1 HSA; Table VIII indicates unequal ranks,

$$\rho_0 = N_0/2 + 1 \quad , \quad \rho_1 = N_0/2 - 1 \quad ,$$

but zero expansiveness, $\varepsilon_{sys} = 0$.

N_0 *Even*, $\gamma_i = 2\nu_i + 3/2$: This SWT has equal subband ranks, $\rho_0 = N_0/2 = \rho_1$, and is nonexpansive.

N_0 *Odd*, *Both Phases*: Both of these nonexpansive choices have subband ranks

$$\rho_0 = (N_0 + 1)/2 \quad , \quad \rho_1 = (N_0 - 1)/2 \quad .$$

The difference between the two is in the extension operators needed in the synthesis bank.

3. *The FBI fingerprint coding standard*. The recently published *WSQ Gray-Scale Fingerprint Image Compression Specification* [29] involves scalar quantization of the subbands resulting from a cascaded two-dimensional (product) filter bank decomposition of digitized fingerprint images. The decomposition involves five levels of cascade, resulting in 64 two-dimensional subbands. The primitive one-dimensional filtering operations are two-channel SWT's based on either WSS-WSS or HSS-HSA PR QMF banks. Since the FBI specification does not include constraints on the exact dimensions of an image (e.g., image dimensions need not be divisible by 2^5), the algorithm incorporates SWT's for both even- and odd-length signals. This ensures that an image of arbitrary dimensions can be transformed nonexpansively by choosing the appropriate SWT algorithm at each stage in the cascade.

The standard dictates a class of SWT decompositions that a decoder must be capable of reconstructing. As of the publication of [29] there is just one approved filter bank for fingerprint image coding—a WSS-WSS lowpass-highpass filter pair with nine and seven impulse response taps, respectively:

$$H_0(z) = \sum_{n=0}^8 h_0(n)z^{-n} \quad ; \quad H_1(z) = \sum_{n=0}^6 h_1(n)z^{-n} \quad .$$

Values for the taps can be found in [29]; the filters correspond to a pair of smooth biorthogonal wavelet bases constructed in [14, 6]. Although the above expressions are for causal filters, according to Corollary 5 they can be advanced by

$$z^{-m_0} = z^4 = z^{-m_1}$$

to form an equivalent PR QMF bank with $\gamma_0 = 0$, $\gamma_1 = -1$. With these advances, the shift factors given in Table VII are $\beta_0 = 0 = \beta_1$ and $\eta_0 = 0 = \eta_1$ for N_0 both even and odd. Recall that noncausal filtering is required to compute a complete, nonredundant half-period of each symmetric subband, as explained in Section II-D. The symmetries and ranks of the resulting subbands and the extensions used in the synthesis bank are given in Table VII. Since the delay of the analysis bank is $D_a = -1$, we set $D_s = 1$ in the anti-aliasing relations (AA) to get synthesis filters that yield a transfer function with zero phase distortion:

$$F_0(z) = -z^{-1}H_1(-z) \quad ; \quad F_1(z) = z^{-1}H_0(-z) \quad . \quad (26)$$

The corresponding impulse response relationships are given in [29].

The standard also allows for the use of HSS-HSA filter banks in conjunction with (2,2)-SWT's. In this case the convention is to set $\gamma_0 = -1/2 = \gamma_1$ so that $D_a = -1$. When N_0 is even, the second and fourth lines in Table VIII give subband ranks $\rho_0 = N_0/2 = \rho_1$, and when N_0 is odd Table VIII gives subband ranks $\rho_0 = (N_0 + 1)/2$ and $\rho_1 = (N_0 - 1)/2$; these agree with the ranks resulting in the case of the (1,1)-SWT. The group delay convention for the analysis bank again ensures that the shift factors are all zero, and we again set $D_s = 1$ and use the anti-aliasing relations (26) to get synthesis filters that yield zero phase distortion.

IV. M-CHANNEL SYMMETRIC SUBBAND CODERS.

We now extend the analysis given in Section III for two-channel SWT's to the case of general M -channel systems. The details of implementing the individual channels in such a subband coder are still based on Figure 7 and are thus entirely analogous to the two-channel case, but at the time of this writing there is no exhaustive classification of the possible M -channel filter symmetry and phase combinations like the classification given in Section III-B for the two-channel case.

The difficulties involved in producing numerically tractable design procedures for linear phase M -channel PR QMF banks are such that efforts to date have focused on special cases or subclasses of linear phase PR QMF banks or on particular values of $M > 2$ [21, 22, 23]. For this reason we cannot enumerate the various

symmetry/phase combinations and system expansiveness values for M -channel SWT's like we did for the two-channel case in Section III-C. Instead, we shall tabulate all possible individual channels for M -channel coders that result in symmetric subbands, as we did for the two-channel case in the tables of Section III-A. It should be clear to the reader, based on the presentation in Section III-C, that it's quite easy to design specific SWT systems using such tables. Channel expansiveness values are included to give some indication of how efficient the channels are and to facilitate computing the expansiveness of specific systems via (21).

There is one special case of interest about which we can draw some conclusions based on our single-channel analysis. We will call a linear phase PR QMF analysis bank, $\{h_1, \dots, h_M\}$, a *concentric* filter bank if all filters have the same center of symmetry. Note that all of the filters in a concentric filter bank must be of the same type, i.e., all must be either HS-type or all must be WS-type filters, and all have the same group delay. The design of concentric filter banks has been studied in [21, 23].

While we do not have a result as comprehensive as Corollary 5 for describing the phase shifts of an M -channel PR QMF bank that result in another PR QMF bank, it is clear that the entire analysis bank can be advanced or delayed in unison by an arbitrary phase since applying the same phase shift to the synthesis bank will result in a new analysis/synthesis system whose output differs from that of the original system by a constant phase shift. It is also easy to see that we can shift any one filter by a multiple of M samples: according to relation (4) in Section II-B, shifting h_i by $M\nu_i$ is equivalent to shifting the downsampled subband, b_i , by ν_i , a process that is easy to compensate for in the synthesis bank. In particular, a linear phase PR QMF bank in which all filters have group delays that are pairwise congruent modulo M can be reduced to a concentric PR QMF bank. We shall use these simple facts when discussing the implementation of PR QMF banks in SWT's.

The most useful result to date on filter symmetries in concentric filter banks is the following theorem from [23] on paraunitary (i.e, orthogonal) systems.

Theorem 6. Let $\{h_1, \dots, h_M\}$ be a concentric paraunitary filter bank.

1. *If M is even, there are $M/2$ symmetric filters and $M/2$ antisymmetric filters.*
2. *If M is odd, there are $(M + 1)/2$ symmetric filters and $(M - 1)/2$ antisymmetric filters.*

This result will allow us to calculate the overall expansiveness, ε_{sys} , for concentric paraunitary systems.

As in the two-channel case, the synthesis bank for an M -channel PR QMF system is determined by anti-aliasing relations in terms of the analysis bank [17, 3]. For M -channel systems it is most convenient to express this relationship in terms of the M -component polyphase decompositions of the analysis and synthesis banks, a topic we have promised to avoid in the present exposition. Note that, as was mentioned previously, there are no nontrivial two-channel paraunitary QMF banks with linear phase FIR filters.

IV-A. Divisibility Constraints on Signal Length.

Unlike the two-channel case, when $M > 2$ there do not always exist SWT's for input signals, x , of arbitrary length. Recall from Section II-B that we imposed the divisibility constraint

$$M \mid N \tag{DIV}$$

to ensure that $M : 1$ circular decimation of y coincides with $M : 1$ linear decimation of the infinite, N -periodic signal. This prevents us from using the $(2N_0 - 1)$ -periodic extension $y = E_s^{(1,2)}x$ when M is even. The periods of the even-length analysis extensions in Table II may also fail to be divisible by certain M for certain N_0 , so let us consider the possibilities carefully.

First, while one might expect to use an M -channel coder if the signal length contains powers of M , $N_0 = J \cdot M^L$, this will only be possible for SWT's based on the $2N_0$ -periodic extension $E_s^{(2,2)}x$ when $M > 2$. The extension $E_s^{(1,2)}x$ of period $2N_0 - 1$ is out since $M \nmid (2JM^L - 1)$, and $E_s^{(1,1)}x$ only works for $M = 2$ because $M \mid (2JM^L - 2)$ implies $M=2$.

To decide just when a given extension has a period divisible by M , we define a number's *residue modulo* M . If $0 \leq r \leq M - 1$ and

$$N_0 \equiv r \pmod{M}$$

then we call r the *residue of* $N_0 \pmod{M}$ and write

$$r = \text{res}(N_0, M) .$$

Since r satisfies the same congruences modulo M as N_0 , we can now classify the extensions satisfying condition (DIV) in terms of $\text{res}(N_0, M)$ by solving the following simple congruences.

$$\begin{aligned} M \mid 2N_0, \text{ or } 2r \equiv 0 \pmod{M}: & \quad r = 0 \text{ (all } M), M/2 \text{ (} M \text{ even)} \\ M \mid 2N_0 - 1, \text{ or } 2r \equiv 1 \pmod{M}: & \quad r = (M + 1)/2 \text{ (} M \text{ odd)} \\ M \mid 2N_0 - 2, \text{ or } 2r \equiv 2 \pmod{M}: & \quad r = 1 \text{ (all } M), 1 + M/2 \text{ (} M \text{ even)} \end{aligned}$$

The corresponding constraints are listed in Table IX and will be used to derive the results in the next subsections. One observation that is immediately clear from Table IX is that while there are admissible SWT extensions for all input lengths when $M=2, 3$, or 4 , there are always excluded congruence classes of input lengths for filter banks with 5 or more channels.

Given a combination satisfying condition (DIV), the next question is whether there exist nonexpansive SWT's for this choice of N_0 , N , and M . As mentioned above, we cannot answer this question in complete generality, but we can say something negative concerning the possibility of nonexpansive SWT's with equal subband ranks. In Section II-E we showed that such transforms are only possible when $M \mid N_0$. This is equivalent to having $\text{res}(N_0, M) = 0$, so from Table IX we see that nonexpansive equal-rank SWT's are only possible using $2N_0$ -periodic extensions. The only hope for nonexpansive (1,1)- and (1,2)-SWT's when $M > 2$

Table IX: Divisibility Constraints on Signal Length, N_0 .

M	N	$\text{res}(N_0, M)$	$K = N/M$
Even	$2N_0$	0	Even
		$M/2$	Odd
	$2N_0 - 2$	1	Even
		$M/2 + 1$	Odd
Odd	$2N_0$	0	Even
	$2N_0 - 1$	$(M+1)/2$	Odd
	$2N_0 - 2$	1	Even

is with unequal subband ranks. In light of the distinctions in Tables V and IX, the classification of channels will be presented in two parts: first for M even, then for M odd.

IV-B. M Even.

For both $2N_0$ - and $(2N_0 - 2)$ -periodic extensions, Table IX indicates two distinct classes of signal lengths, parameterized by $\text{res}(N_0, M)$, that satisfy condition (DIV). As in Section III-A, for each class we shall list only those combinations of filter symmetry and phase that result in symmetric subbands, b_i .

1. *The extension $y = E_s^{(1,1)}x$.* Since M is even, Table V indicates that the $(2N_0 - 2)$ -periodic WSS extension, $E_s^{(1,1)}x$, produces symmetric subbands only when used with WS-type filters, and only for group delays $\gamma = M\nu$ or $\gamma = M\nu + M/2$. Consider the case of signals with $\text{res}(N_0, M) = M/2 + 1$, as given in Table IX; when h is WSS with phase $M\nu$ then, since K is odd in this case, Table V shows that b will be OPS with center ν . From Table VI, the dimension of b is

$$\rho = \frac{K+1}{2} = \frac{N_0-1}{M} + \frac{1}{2},$$

so the expansiveness of this channel is

$$\varepsilon = \rho - \frac{N_0}{M} = \frac{1}{2} - \frac{1}{M}.$$

When $\gamma = M\nu + M/2$, b is again OPS but with center $\nu + 1/2$. The pertinent results for these two cases are given in the first two lines of Table X; derivation of the other lines in that table is similar.

The channel expansiveness figures in Table X indicate some of the limitations of the (1,1)-SWT when $M > 2$. By inequality (22), we cannot have perfect reconstruction unless $\sum \varepsilon_i \geq 0$; this can be used to rule out the existence of concentric WS-type filter banks when M is even. By preceding remarks, such a filter

Table X: (1, 1)-Symmetric Wavelet Transform Channels, M Even.

Signal x : $\text{res}(N_0, M)$	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ϵ
$M/2+1$ (K Odd)	WSS	$M\nu$	OPS, ν	$(N_0 - 1)/M + 1/2$	ν	0	$E_s^{(1,2)}$	$1/2 - 1/M$
		$M\nu + M/2$	OPS, $\nu + 1/2$	$(N_0 - 1)/M + 1/2$	$\nu + 1$	0	$E_s^{(2,1)}$	$1/2 - 1/M$
	WSA	$M\nu$	OPA, ν	$(N_0 - 1)/M - 1/2$	ν	1	$E_a^{(1,2)}$	$-1/2 - 1/M$
		$M\nu + M/2$	OPA, $\nu + 1/2$	$(N_0 - 1)/M - 1/2$	$\nu + 1$	0	$E_a^{(2,1)}$	$-1/2 - 1/M$
1 (K Even)	WSS	$M\nu$	WSS, ν	$(N_0 - 1)/M + 1$	ν	0	$E_s^{(1,1)}$	$1 - 1/M$
		$M\nu + M/2$	HSS, $\nu + 1/2$	$(N_0 - 1)/M$	$\nu + 1$	0	$E_s^{(2,2)}$	$-1/M$
	WSA	$M\nu$	WSA, ν	$(N_0 - 1)/M - 1$	ν	1	$E_a^{(1,1)}$	$-1 - 1/M$
		$M\nu + M/2$	HSA, $\nu + 1/2$	$(N_0 - 1)/M$	$\nu + 1$	0	$E_a^{(2,2)}$	$-1/M$

bank could be advanced or delayed to have its center at $M/2$. If it were then applied in a (1, 1)-SWT to signals of length N_0 , where $\text{res}(N_0, M) = 1$, it would produce a transform with overall expansiveness

$$\epsilon_{sys} = M(-1/M) = -1 ,$$

regardless of the mix of WSS and WSA filters. This contradicts (22), implying

Theorem 7. There are no concentric WS-type FIR PR QMF banks for M even.

This generalizes one of the consequences of Theorem 4 in Section III-A. We do not know if there exist non-concentric WS-type QMF banks for M even, $M > 2$.

2. *The extension $y = E_s^{(2,2)}x$.* Since this is an HSS extension, Table V indicates that we are constrained to using HS-type filters. The possible subband symmetries and expansion factors are listed in Table XI.

Example: ($M = 4$) A four-channel paraunitary filter bank has been constructed in [23] with four HS-type filters, each of length eight. The filter bank is causal, i.e., the filters are supported on the interval $[0, 7]$, with centers at $7/2$. Filters h_0 and h_2 are HSS while h_1 and h_3 are HSA. If the analysis bank is advanced one sample,

$$H'_i(z) = zH_i(z) ,$$

then the filters will have centers $\gamma'_i = 5/2 = (M + 1)/2$ so Table XI shows that we will have a nonexpansive equal-rank (2, 2)-SWT when $\text{res}(N_0, 4)=0$ and a nonexpansive (2,2)-SWT when $\text{res}(N_0, 4)=2$. If the analysis bank is advanced three samples,

$$H''_i(z) = z^3H_i(z) ,$$

Table XI: (2, 2)-Symmetric Wavelet Transform Channels, M Even.

Signal x : $\text{res}(N_0, M)$	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ϵ
0 (K Even)	HSS	$M\nu + 1/2$	WSS, ν	$N_0/M + 1$	ν	0	$E_s^{(1,1)}$	1
		$M\nu + (M + 1)/2$	HSS, $\nu + 1/2$	N_0/M	$\nu + 1$	0	$E_s^{(2,2)}$	0
	HSA	$M\nu + 1/2$	WSA, ν	$N_0/M - 1$	ν	1	$E_a^{(1,1)}$	-1
		$M\nu + (M + 1)/2$	HSA, $\nu + 1/2$	N_0/M	$\nu + 1$	0	$E_a^{(2,2)}$	0
$M/2$ (K Odd)	HSS	$M\nu + 1/2$	OPS, ν	$N_0/M + 1/2$	ν	0	$E_s^{(1,2)}$	$1/2$
		$M\nu + (M + 1)/2$	OPS, $\nu + 1/2$	$N_0/M + 1/2$	$\nu + 1$	0	$E_s^{(2,1)}$	$1/2$
	HSA	$M\nu + 1/2$	OPA, ν	$N_0/M - 1/2$	ν	1	$E_a^{(1,2)}$	$-1/2$
		$M\nu + (M + 1)/2$	OPA, $\nu + 1/2$	$N_0/M - 1/2$	$\nu + 1$	0	$E_a^{(2,1)}$	$-1/2$

the centers will be at $1/2$ and the resulting (2, 2)-SWT will be nonexpansive, with the HSS channels having rank $\rho_i'' = N_0/4 + 1$ and the HSA channels having rank $\rho_i'' = N_0/4 - 1$ when $\text{res}(N_0, 4) = 0$, and ranks $N_0/4 \pm 1/2$ when $\text{res}(N_0, 4) = 2$.

We can generalize this example:

Proposition 8. For M even, any concentric HS-type PR QMF bank yields a nonexpansive equal-rank (2, 2)-SWT for inputs with $\text{res}(N_0, M) = 0$ if we shift the analysis bank so that its center is congruent to $(M + 1)/2 \pmod{M}$. If the analysis bank contains equal numbers of symmetric and antisymmetric filters (e.g., paraunitary filter banks), nonexpansiveness also holds for inputs with $\text{res}(N_0, M) = M/2$ and for analysis banks with centers congruent to $1/2 \pmod{M}$ (for both input residue classes 0 and $M/2$).

When M is even, note that a given filter bank can be used for SWT's on just two distinct residue classes of input lengths, meaning that there will always be excluded input lengths for a given filter bank whenever $M \geq 4$. For $M=4$, the existence of nonexpansive SWT's for signals with $\text{res}(N_0, 4)=1$ or 3 depends on the existence of (non-concentric) WS-type QMF banks for use in (1, 1)-SWT's.

IV-C. M Odd.

The SWT possibilities in this case differ markedly from the situation when M is even. Table IX gives only one residue class of signal lengths satisfying condition (DIV) for each possible extension. For either WSS or HSS signal extensions, Table V indicates that there are symmetric subband coders for both WS- and HS-type filters. This increases the possible combinations of filter symmetries that can be used in SWT's. For instance, in [22] the authors construct a three-channel PR QMF analysis bank with causal filters having

Table XII: (1, 1)-Symmetric Wavelet Transform Channels, M Odd.

Signal x : $\text{res}(N_0, M)$	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ϵ
1 (K Even)	WSS	$M\nu$	WSS, ν	$(N_0 - 1)/M + 1$	ν	0	$E_s^{(1,1)}$	$1 - 1/M$
	WSA	$M\nu$	WSA, ν	$(N_0 - 1)/M - 1$	ν	1	$E_a^{(1,1)}$	$-1 - 1/M$
	HSS	$M\nu + M/2$	HSS, $\nu + 1/2$	$(N_0 - 1)/M$	$\nu + 1$	0	$E_s^{(2,2)}$	$-1/M$
	HSA	$M\nu + M/2$	HSA, $\nu + 1/2$	$(N_0 - 1)/M$	$\nu + 1$	0	$E_a^{(2,2)}$	$-1/M$

the following symmetries.

$$\text{HSS}, \gamma_0 = 27.5 \quad (56 \text{ taps}) \quad ; \quad \text{WSS}, \gamma_1 = 26 \quad (53 \text{ taps}) \quad ; \quad \text{HSA}, \gamma_2 = 27.5 \quad (56 \text{ taps}) \quad . \quad (27)$$

We will discuss the implementation of SWT's based on this example later in this section.

1. *The extension $y = E_s^{(1,1)}x$.* When $N = 2N_0 - 2$, the only residue class satisfying condition (DIV) is $\text{res}(N_0, M) = 1$, but there are symmetric subband coders for all four classes of filters. The possible symmetry/relative phase combinations are listed in Table XII; the derivation is via Tables V and VI, as in preceding sections. An analogue of Theorem 7 for concentric HS-type filter banks when M is odd follows from the last two lines in Table XII: by shifting such a filter bank so it is centered at $M/2$, we could construct a (1, 1)-SWT with expansiveness $\epsilon_{sys} = -1$. Since this is impossible, we conclude

Theorem 9. There are no concentric HS-type FIR PR QMF banks for M odd.

As in Theorem 7, note that this conclusion is not limited to the paraunitary case.

We can take full advantage of the choice of filter symmetries in Table XII to construct a nonexpansive three-channel (1, 1)-SWT using the filter bank (27). Delay the bank by one sample,

$$H'_i(z) = z^{-1}H_i(z) \quad , \quad \gamma'_i = \gamma_i + 1 \quad ,$$

so that $\gamma'_1 = 27 \equiv 0 \pmod{3}$ and $\gamma'_0 = \gamma'_2 = 28.5 \equiv 3/2 \pmod{3}$. Then this new filter bank defines a (1, 1)-SWT when $\text{res}(N_0, 3) = 1$, with expansiveness $\epsilon_{sys} = (1 - 1/3) + 2(-1/3) = 0$.

When M is odd, there is the possibility of using concentric WS-type PR QMF banks. Although we do not have an explicit example of such a filter bank construction (a theoretical classification is given in [23]), we can use Theorem 6 and (21) to calculate the expansiveness of a (1, 1)-SWT based on a concentric paraunitary WS-type filter bank with center congruent to 0 mod M . Using the first two lines in Table XII,

$$\epsilon_{sys} = \left(\frac{M+1}{2}\right) \left(1 - \frac{1}{M}\right) + \left(\frac{M-1}{2}\right) \left(-1 - \frac{1}{M}\right) = 0 \quad .$$

Thus, a concentric paraunitary WS-type (1, 1)-SWT is always nonexpansive when M is odd.

Table XIII: (1, 2)-Symmetric Wavelet Transform Channels, M Odd.

Signal x : $\text{res}(N_0, M)$	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ε
$(M+1)/2$ (K Odd)	WSS	$M\nu$	OPS, ν	$(2N_0 - 1 + M)/2M$	ν	0	$E_s^{(1,2)}$	$(M-1)/2M$
	WSA	$M\nu$	OPA, ν	$(2N_0 - 1 - M)/2M$	ν	1	$E_a^{(1,2)}$	$-(M+1)/2M$
	HSS	$M\nu + M/2$	OPS, $\nu + 1/2$	$(2N_0 - 1 + M)/2M$	$\nu + 1$	0	$E_s^{(2,1)}$	$(M-1)/2M$
	HSA	$M\nu + M/2$	OPA, $\nu + 1/2$	$(2N_0 - 1 - M)/2M$	$\nu + 1$	0	$E_a^{(2,1)}$	$-(M+1)/2M$

Table XIV: (2, 2)-Symmetric Wavelet Transform Channels, M Odd.

Signal x : $\text{res}(N_0, M)$	Filter h :		Subband b :		Shifts:		Exten.	Expan.
	Sym.	Center γ	Sym., Center	Dim. ρ	β	η	E	ε
0 (K Even)	WSS	$M\nu + (M+1)/2$	HSS, $\nu + 1/2$	N_0/M	$\nu + 1$	0	$E_s^{(2,2)}$	0
	WSA	$M\nu + (M+1)/2$	HSA, $\nu + 1/2$	N_0/M	$\nu + 1$	0	$E_a^{(2,2)}$	0
	HSS	$M\nu + 1/2$	WSS, ν	$N_0/M + 1$	ν	0	$E_s^{(1,1)}$	1
	HSA	$M\nu + 1/2$	WSA, ν	$N_0/M - 1$	ν	1	$E_a^{(1,1)}$	-1

2. *The extension $y = E_s^{(1,2)}x$.* The possible symmetric channels for the (1, 2)-SWT, M odd, are listed in Table XIII. The same one-sample delay of the filter bank (27) that was used in a (1,1)-SWT above now furnishes a (1,2)-SWT for inputs with $\text{res}(N_0, 3)=2$; the expansiveness is $\varepsilon_{sys} = 2(1/3) - (2/3) = 0$.

Now compute the expansiveness of a (1,2)-SWT based on a concentric paraunitary WS-type filter bank with center congruent to 0 mod M :

$$\varepsilon_{sys} = \left(\frac{M+1}{2}\right)\left(\frac{M-1}{2M}\right) + \left(\frac{M-1}{2}\right)\left(-\frac{M+1}{2M}\right) = 0.$$

3. *The extension $y = E_s^{(2,2)}x$.* The possible symmetric channels for the (2, 2)-SWT, M odd, are listed in Table XIV. The example (27) can be incorporated into a nonexpansive (2, 2)-SWT without added phase shifts since $\gamma_1 = 26 \equiv 2 \pmod{3}$ and $\gamma_0 = \gamma_2 = 27.5 \equiv 1/2 \pmod{3}$.

Note that both WS-type channels for the (2, 2)-SWT have expansiveness $\varepsilon_i = 0$. This means that a concentric WS-type PR QMF bank—regardless of the mix of symmetric and antisymmetric filters—would furnish a nonexpansive equal-rank (2, 2)-SWT for signals satisfying $\text{res}(N_0, M) = 0$ if the analysis bank's center is congruent to $(M+1)/2 \pmod{M}$. We summarize the possibilities for SWT's based on concentric

WS-type filter banks in

Proposition 10. For M odd, any concentric WS-type PR QMF bank yields a nonexpansive equal-rank $(2, 2)$ -SWT for inputs with $\text{res}(N_0, M) = 0$ if its center is congruent to $(M + 1)/2 \pmod{M}$. If the analysis bank contains $(M + 1)/2$ symmetric filters and $(M - 1)/2$ antisymmetric filters (e.g., paraunitary filter banks) and the center of the analysis bank is congruent to $0 \pmod{M}$, we get a nonexpansive $(1, 1)$ -SWT for inputs satisfying $\text{res}(N_0, M) = 1$ and a nonexpansive $(1, 2)$ -SWT for inputs satisfying $\text{res}(N_0, M) = (M + 1)/2$. In particular, when $M = 3$ a concentric WS-type PR QMF bank with two symmetric filters and one antisymmetric filter furnishes nonexpansive SWT's for all input length residue classes.

V. CONCLUSIONS.

This paper characterizes all possible nonexpansive symmetric wavelet transform algorithms based on a classification of the possible combinations of symmetric signal extensions and linear phase filters that produce symmetric subbands in downsampled subband coders. Detailed algorithms are developed for direct-form implementations of all of these transforms. A general notion of transform expansiveness has been introduced to analyze the extent to which SWT's conserve data storage requirements.

In the case of two-channel PR QMF banks, we have been able to enumerate all possible SWT's explicitly. Nonexpansive SWT's based on $(1, 1)$ - and $(2, 2)$ -symmetric extensions have been shown to exist for both even- and odd-length signals. The specific SWT algorithms employed in the FBI fingerprint image coding specification have been described.

In the general M -channel case, the classification of symmetric subband coder channels has been applied to the analysis of SWT algorithms for recently constructed examples of linear phase PR QMF banks, such as concentric filter banks. Algebraic restrictions on the allowable combinations of input signal length and downsample factor prevent any one symmetric extension method from being applicable to signals of arbitrary length when $M > 2$. By employing different analysis extensions and judicious phase shifts, however, we have shown how certain 3-channel filter banks can provide nonexpansive SWT's for input signals of arbitrary length. We have also shown that a single filter bank never furnishes nonexpansive SWT's for input signals of arbitrary length when $M \geq 4$.

Modulo constraints on input length, we have constructed examples of nonexpansive SWT's for both even and odd values of M . We prove that concentric PR QMF banks always yield nonexpansive equal-rank SWT's for the $(2, 2)$ -symmetric extension in both of the cases M even (with concentric HS-type filter banks) and M odd (with concentric WS-type filter banks). When M is odd, we prove that a concentric paraunitary WS-type PR QMF bank generates a nonexpansive $(1, 1)$ -SWT. We also prove that nonexpansive SWT's can never have equal-rank subbands for $(1, 1)$ - or $(1, 2)$ -symmetric extensions when $M > 2$.

As an interesting corollary of our analysis, we are able to use dimensionality results about SWT's to

prove the nonexistence of certain classes of linear phase PR QMF banks. In particular, we can show that concentric PR QMF banks based on WS-type filters are impossible for even M and that concentric PR QMF banks based on HS-type filters are impossible for odd M .

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