polarized-beam neutron scattering using He-3: transport and data analysis

May 21, 2015

1 Introduction

This is a summary and recapitulation in terms of transfer matrices of some of the information in "Polarized ^{3}He in Neutron Scattering" by T.R. Gentile, and other texts on polarized-neutron beams.

2 Setup

A general polarized beam setup for neutron-scattering spectrometers using He-3 polarization cells, P1 (polarizer) and P2 (analyzer), is represented in the following diagram. Our convention is that + represents the neutron spin-state when the front flipper is OFF.

Typically the incoming beam is unpolarized so that $N_+ = N_- = \frac{1}{2}N$, where N is the total number of neutrons incident on P1. The detector, D, does not discriminate polarization states, and so counts $n = n_+ + n_-$. The detection system, D, may include energy analysis of the scattered neutrons. Here we assume that such energy analysis would have equal efficiencies for the two neutron spin states. In the above diagram, f1 and f2 are higher order wavelength filters, m1a, m1b, m2a and m2b are low efficiency beam intensity monitors with efficiency proportional to wavelength, and F1 and F2 are spin flippers.

3 Transfer Matrices

The detected counts can be calculated using transfer matrices for each device along the beam path that affects the neutron spin, so that

$$\begin{pmatrix} n_+ \\ n_- \end{pmatrix} = T \begin{pmatrix} N_+ \\ N_- \end{pmatrix} = T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{N}{2}. \tag{1}$$

where the transfer matrix for the total beam path is the product of the transfer matrices for each beam component. The detector will count $n_+ + n_-$.

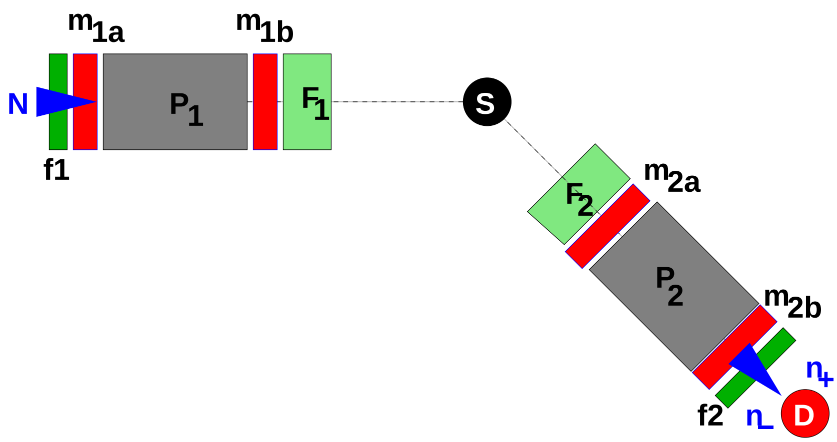


Figure 1: polarized beam triple-axis setup

$$T = AF_A B_A S\, B_P F_P P \tag{2}$$

and where A and P are the transfer matrices for the He-3 analyzer and polarizer, F_A and F_P are the transfer matrices for the flipper on the analyzer and polarizer sides of the sample, B_A and B_P are the transfer matrices for beam transport efficiency on the analyzer and polarizer sides of the sample, and S is the transfer matrix for the sample. It will be shown that the flipper and transport efficiency matrices commute so that the transport loss before the sample can occur anywhere between the polarizer and sample, and the analogous condition applies to the transport loss after the sample. Often the sample transfer matrix, S, also commutes (is symmetric) with the flipper and transport loss matrices, in which case the location of transport loss cannot be determined by neutron intensity measurements any better than to have occured somewhere between the He3 polarizer and analyzer.

Note that in some of what follows we neglect time and spectrometer setting variations of the parameters that determine the transfer matrix. In particular we assume that a set of polarized beam cross-sections is measured in a short enough time to neglect the time dependence of the He-3 transmissions. When doing complete polarized beam data corrections this will often not be the case.

3.1 He-3 polarizer/analyzer

The transmission of the He-3 polarizer is characterized by the two different absorption cross-sections: the neutron interacts with a polarized He-3 atom with its spin (or magnetic moment) z-component aligned with that of the He-3, or the neutron interacts with a polarized He-3 atom with its spin (or magnetic moment) z-component anti-aligned with that of the He-3. The total cross-sections for these two processes are $\sigma_+ = \sigma \uparrow\uparrow \cong 0$, and $\sigma_- = \sigma \uparrow\downarrow = 10666\; barns\; \frac{\lambda}{1.78\text{Å}}$.

Also $\sigma_0 = \frac{1}{2}(\sigma_+ + \sigma_-) \cong \frac{1}{2}\sigma_-$. The effective absorption coefficients (inverse of the absorption length) for each process are $\alpha \uparrow\uparrow = \sigma_+ nf \cong 0$, and $\alpha \uparrow\downarrow = \sigma_- n(1-f)$, where n is the number density of He-3 atoms in the polarizing cell and f measures the fraction of He-3 atoms that have their angular momentum polarized along the neutron spin direction.

The standard preparation of He-3 cells at the NCNR produces polarized neutrons that are in the lowest energy Zeeman state. This is the same for supermirror-transmission or Heusler polarizing devices (This is shown in another document). This means that the He-3 magnetic moment is also prepared in the lower energy Zeeman state (parallel to the holding guide field direction). In order to symmetrize these expressions, define the He-3 polarization (a number in the inclusive range -1 to 1) as

$$P_{He3} = \frac{n_{He3}\uparrow - n_{He3}\downarrow}{n_{He3}\uparrow + n_{He3}\downarrow} \tag{3}$$

where $n_{He3}\uparrow$is the number density of He-3 magnetic moments aligned with the He3 guide-field quantization axis. With this convention the standard setup will take the He-3 polarization to have $P_{He3} > 0$. This means

$$n_{He3}\uparrow,\downarrow = \frac{1}{2}(1 \pm P_{He3})n$$

where $n = n_{He3}\uparrow + n_{He3}\downarrow$ is the He-3 number density.

Then if a neutron attempts to transit the He-3 polarizer with its spin $\downarrow$(magnetic moment $\uparrow$aligned with the cell guide-field), it is only abosrbed by He-3 atoms with magnetic-moments anti-aligned with the cell guide-field, so that the effective absorption coefficient for this +state neutron is $\alpha_+ = \sigma_- n_{He3}\downarrow = n\sigma_0(1-P_{He3})$. On the other hand, if a neutron attempts to transit the He-3 polarizer with its spin $\uparrow$(magnetic moment $\downarrow$anti-aligned with the cell guide-field), it is only abosrbed by He-3 atoms with magnetic-moments aligned with the cell guide-field, so the effective absorption coefficient for this $-$ state neutron is $\alpha_- = \sigma_- n_{He3}\uparrow = n\sigma_0(1+P_{He3})$.

The ideal gas calculation of $n\sigma_0$ yields 0.07404 cm^{-1}x cell-pressure(bars at 293K) x neutron-wavelength in Angstroms. Then the transmission of the two neutron spin states is

$$t_\pm = t_E \exp(-\alpha_\pm L) = t_E \exp(-\tau_0[1 \mp P_{He3}]) \tag{4}$$

where L is the path length through the He-3 gas in the cell, t_E is the transmission of an empty cell, and

$$\tau_0 = n\sigma_0 L. \tag{5}$$

The wavelength dependence of the absorption coeficient is linear to a very good approximation so that

$$\tau = n\sigma_0(\lambda_0)L\frac{\lambda}{\lambda_0} = \tau_0\frac{\lambda}{\lambda_0} = \tau_0\widetilde{\lambda} = n\sigma_0 L\widetilde{\lambda},$$

where as above we take $\lambda_0 = 1\, Angstrom$ and

$$t_{\pm} = t_E \exp\left(-\tau[1 \mp P_{He3}]\right) \tag{6}$$

$$\tau_{\pm} = \tau\left(1 \mp P_{He3}\right).$$

A typical cell is roughly 10 cm in length so a typical value for τ_{-M} at $P_{He3} = 0.75$ is $\tau_{-M} \cong 1.295{*}bars{*}\lambda\,(Angstroms)$, while $\tau_{+M} \cong 0.185{*}bars{*}\lambda\,(Angstroms)$.

The above analysis remains valid if the He-3 polarization is made negative (which is functionally equivalent but not necessarily numerically equivalent to flipping the neutron spin-state if the flipping efficiency is not unity).

3.1.1 Wavelength and pathlength variation

There are wavelength and pathlength variations for the neutrons traversing the He-3 spin-filter. A zero order approximation to the averaged transmission over all neutrons is just the transmission evaluated at the average values for the pathlength and wavelength. Striclty, we should average the product of pathlength and wavelength, but here we assume that the two are statistically independent.

$$< t_{\pm} >_0 \cong t_{\pm 0} = t_E \exp\left(-\langle\tau\rangle\left(1 \mp P_{He3}\right)\right) = t_E \exp\left(-\tilde{\tau}_{\pm}\right)$$

$$\langle\tau\rangle = n\sigma_0 \langle L\rangle \left\langle\widetilde{\lambda}\right\rangle$$

Now if we include the variations of wavelength and pathlength about the averages, we can improve out estimation of the averaged transmission by performing a Gaussian average for the transmission. First write the varying τ as

$$\tau = \langle\tau\rangle + \epsilon_{\tau} = n\sigma_0 \langle L\rangle \left(1 + \epsilon_{\langle L\rangle}\right) \left\langle\widetilde{\lambda}\right\rangle \left(1 + \epsilon_{\widetilde{\lambda}}\right).$$

Note that $\epsilon_{\langle L\rangle} = \epsilon_L / \langle L\rangle$ is dimensionless, as is $\epsilon_{\widetilde{\lambda}} = \epsilon_{\lambda}/\lambda_0$. We drop the modifiers that indicate dimensionless quantities in the following. Now the varying transmission is

$$t_{\pm}(\epsilon_{\langle L\rangle}, \epsilon_{\widetilde{\lambda}}) = t_E \exp\left(-\left(1 + \epsilon_{\langle L\rangle}\right)\left(1 + \epsilon_{\widetilde{\lambda}}\right)\tilde{\tau}_{\pm}\right)$$

and its Gaussian average is

$$< t_{\pm} > = \frac{t_{\pm 0}}{2\pi\sigma_{\lambda}\sigma_L} \int \exp -\frac{1}{2}\left[\left(\frac{\epsilon_{\lambda}}{\sigma_{\lambda}}\right)^2 + \left(\frac{\epsilon_L}{\sigma_L}\right)^2 + 2\tilde{\tau}_{\pm}\left(\epsilon_{\lambda} + \epsilon_L + \epsilon_{\lambda}\epsilon_L\right)\right] d\epsilon_{\lambda} d\epsilon_L$$

where σ_{λ} and σ_L are the standard deviations for the Gaussian distributions for normalized wavelength and pathlength, and we use

$$\int \exp\left(-\frac{1}{2}\left[\mathbf{X}^t \bullet \mathbf{A} \bullet \mathbf{X} + \mathbf{B}^t \bullet \mathbf{X}\right]\right) d\mathbf{X}^n \quad = \quad (2\pi)^{n/2} |\mathbf{A}|^{-1/2} \exp\left(\frac{1}{8}\mathbf{B}^t \bullet \mathbf{A}^{-1} \bullet \mathbf{B}\right) \qquad .$$

The Gaussian approximation numerically works provided $d_{\lambda L} = 1-\sigma_\lambda^2\sigma_L^2\tilde{\tau}_\pm^2 > 0$. If both σ_λ and σ_L reach 0.15 the Gaussian distribution works up to $\tilde{\tau}_\pm \cong 44$ which (using the example above) corresponds to a wavelength of $15 Angstroms$ for $\tilde{\tau}_-$. Then

$$< t_\pm >= C_\pm t_{\pm 0}$$

$$C_\pm = d_{\lambda L}^{-1/2} \exp \frac{1}{2} \frac{\sigma_\lambda^2 + \sigma_L^2 - 2\sigma_\lambda^2\sigma_L^2\tilde{\tau}_\pm}{d_{\lambda L}} \tilde{\tau}_\pm^2$$

Typically at least one of the distribution widths is small enough that $d_{\lambda L} \cong 1$ and

$$C_\pm \cong \exp \frac{1}{2}\left(\sigma_\lambda^2 + \sigma_L^2\right)\tilde{\tau}_\pm^2.$$

We will use this last approximation to $C_\pm$ for correcting the derivatives with respect to He-3 parameters, for example,

$$\frac{d < t_\pm >}{d\tilde{\tau}_\pm} \frac{1}{< t_\pm >} \frac{d\tilde{\tau}_\pm}{dP_{He3}} \cong \left[\left(\sigma_\lambda^2 + \sigma_L^2\right)\tilde{\tau}_\pm - 1\right] \frac{d\tilde{\tau}_\pm}{dP_{He3}}.$$

The averaging brackets are removed in the notation that follows. We point out that strictly speaking the wavelength and pathlength variations may not be independently distributed. For example, when neutrons are scattered from a crystal, the scattering angle and wavelength are correlated by Bragg's law. That is σ_λ^2will depend on the crystal setting angle. However, crystals are typically used with Soller collimators that will wash out this effect on the pathlength, when averaged over the He3 cell.

3.1.2 Time dependence

Once a He-3 cell is polarized and removed from the optical pumping system that produced the polarization, that polarization begins to decay exponentially with a characteristic time constant, t_C, that depends on the homogeneity of the magnetic-field on the cell (among other things). Thus

$$P_{He3}(t) = P_{He3}(t=0)\exp\left(-t/t_C\right) = P_0 \exp\left(-t/t_C\right),$$

with

$$\tilde{\sigma}^2_{P_{He3}}(t) = \tilde{\sigma}^2_{P_0} + \left(\frac{t}{t_C}\right)^2 \tilde{\sigma}^2_{t_C}.$$

If only the initial and final He-3 polarization measurements are available, then the proper parameterization is

$$P_{He3}(t) = P_0 \exp\left(-\ln\left(P_0/P_f\right)\left[t/t_f\right]\right),$$

with

$$t_C = t_f/\ln\left(P_0/P_f\right).$$

Then

$$\tilde{\sigma}^2_{P_{He3}}(t) = \tilde{\sigma}^2_{P_0}\left(1-\frac{t}{t_f}\right)^2 + \left(\frac{t}{t_f}\right)^2 \tilde{\sigma}^2_{P_f}. \qquad (7)$$

This time dependence is an important consideration when checking transport efficiencies and performing data analysis so that it is necessary that it is measured. This is accomplished by measuring the total transmission of an unpolarized neutron beam through the He3 cell, both when it is polarized and unpolarized. It is important that these measurements are performed without higher order wavelength contamination present in the neutron beam (correcting for the higher order contamination is difficult and introduces additional uncertainty).

The total transmission for an incident unpolarized neutron beam will be

$$t_0(P_{He3}) = \frac{C_H}{C_0} = \frac{1}{2} t_E \left[\widetilde{C}_+ \exp\left(-\tilde{\tau}_+\right) + \widetilde{C}_- \exp\left(-\tilde{\tau}_-\right)\right]. \qquad (8)$$

where $C_H$is the observed count rate with the He-3 cell in the beam, and C_0 is the count rate without the cell. As mentioned previously, we can arrange that $\tilde{\tau}_\pm = \langle\tau\rangle\left(1 \mp P_{He3}\right)$. Then,

$$t_0(P_{He3}) = t_E \exp\left(-\langle\tau\rangle\right)\left[< C > \cosh(\langle\tau\rangle P_{He3}) + \Delta \sinh(\langle\tau\rangle P_{He3})\right],$$

where $< C >= (\widetilde{C}_+ + \widetilde{C}_-)/2 \cong 1$ and $\Delta = (\widetilde{C}_+ - \widetilde{C}_-)/2$ with $|\Delta| \ll 1$. If the He-3 cell is unpolarized

$$t_0(0) = t_{00} = \frac{C_u}{C_0} =< C > t_E \exp\left(-\tilde{\tau}\right)$$

where C_u is the count rate with the unpolarized He-3 cell in the beam. $\tilde{\tau}$ can be determined as

$$\tilde{\tau} = \langle\tau\rangle = \ln\left(\frac{< C > t_E}{t_{00}}\right) = \ln\left(\frac{t_E}{t_{00}}\right) + \ln\left(< C >\right),$$

with the squared relative uncertainty given as

$$\sigma_{\tilde{\tau}}^2 = \tilde{\sigma}_{t_E}^2 + \tilde{\sigma}_{C_u}^2 + \tilde{\sigma}_{C_0}^2.$$

In general the squared relative uncertainty for any measured variable is

$$\tilde{\sigma}_V^2 = \frac{\sigma_V^2}{V^2}.$$

Once $\tilde{\tau}$ is determined, the ratio, r, of the polarized cell transmission to the unpolarized cell transmission can be used to determine the He-3 polarization, P_{He3}, with $\Delta / < C > \cong \hat{\Delta}$

$$r(P_{He3}) = \frac{t_0(P_{He3})}{t_{00}} = \frac{C_H}{C_u} = \cosh(\langle\tau\rangle \, P_{He3}) + \hat{\Delta} \sinh(\langle\tau\rangle \, P_{He3}). \qquad (9)$$

This can be inverted to give

$$\langle\tau\rangle \, P_{He3} = \ln \left[\left(r + \sqrt{r^2 - \left(1 + \hat{\Delta}\right)\left(1 - \hat{\Delta}\right)} \right) / \left(1 + \hat{\Delta}\right) \right].$$

Error analysis on the determination of the He3 polarization gives

$$\tilde{\sigma}_{P_{He3}}^2 = \tilde{\sigma}_{\tilde{\tau}}^2 + (\tilde{\tau} P_{He3})^{-2} \frac{r^2}{r^2 - 1} \tilde{\sigma}_r^2,$$

where

$$\tilde{\sigma}_r^2 = \frac{\sigma_r^2}{r^2} = \tilde{\sigma}_{C_H}^2 + \tilde{\sigma}_{C_u}^2,$$

More often, one measures the polarized-cell transmission and uses knowledge of $\langle\tau\rangle$, $\langle C\rangle$ and Δ (which depend on the measurement conditions, beam size etc.) to determine P_{He3}. Then

$$\cosh(\langle\tau\rangle \, P_{He3}) \cong \frac{t_0(P_{He3})}{< C > t_E \exp\left(- \langle\tau\rangle\right)} \left(1 - \frac{\Delta}{< C >}\right) = \tilde{r}$$

$$\tilde{\tau} P_{He3} = \ln\left(\tilde{r} + \sqrt{\tilde{r}^2 - 1}\right).$$

Once the He-3 polarization is measured, its uncertainty increases with time because the polarization decays with time, so that (compare with 7)

$$\tilde{\sigma}_{P_{He3}}^2(t) = \tilde{\sigma}_{P_{He3}}^2 + \left(\frac{t}{t_C}\right)^2 \tilde{\sigma}_{t_C}^2.$$

Care must be taken when measuring transmissions with a detector where the efficiency is wavelength dependent and higher order wavelength contamination is present. A typical beam monitor is a fission detector where the efficiency is proportional to wavlength, $\epsilon = \epsilon_0 \lambda$. Then, neglecting beam pathlength and wavelength variation corrections, the measured transmission is

$$t_0(P_{He3}) = t_E \sum_{n=1} a_n \frac{1}{n} \exp\left(-\frac{1}{n}\langle\tau\rangle\right) \cosh\left(\frac{1}{n}\langle\tau\rangle P_{He3}\right) / \sum_{n=1} a_n \frac{1}{n}$$

$$t_{00} = t_E \sum_{n=1} a_n \frac{1}{n} \exp\left(-\frac{1}{n}\langle\tau\rangle\right) / \sum_{n=1} a_n \frac{1}{n},$$

where a_n are the wavelength order fractions. It is obvious that wavelength contamination is problematic for using transmission measurements to determine the He-3 cell properties, especially if using a wavelength dependent detector.

The outgoing neutron polarization, $-1 \leq P_n \leq 1$, after an incident unpolarized beam passes through a polarized He3 cell is

$$P_{neutron} = \frac{n_+ - n_-}{n_+ + n_-} = \tanh\left(\langle\tau\rangle P_{He3}\right) + \frac{\Delta}{\cosh^2(\langle\tau\rangle P_{He3})}. \qquad (10)$$

The time dependent transfer matrices for polarizer or analyzer are

$$P, A = t_E \begin{bmatrix} \widetilde{C}_+ \exp\left(-\tilde{\tau}_+\right) & 0 \\ 0 & \widetilde{C}_- \exp\left(-\tilde{\tau}_-\right) \end{bmatrix} = \begin{bmatrix} t_{+P,A} & 0 \\ 0 & t_{-P,A} \end{bmatrix}. \qquad (11)$$

$$\widetilde{C}_\pm = \hat{C}_\pm \left\{1 + \sum_{n=2} a_n K_{\pm n}\right\}$$

$$\langle C\rangle = \frac{1}{2}\left(\widetilde{C}_+ + \widetilde{C}_-\right)$$

$$\Delta = \frac{1}{2}\left(\widetilde{C}_+ - \widetilde{C}_-\right)$$

$$\hat{C}_\pm = 1 + \frac{1}{2}\left(\tilde{\tau}_\pm \frac{\sigma_\lambda}{\lambda_M}\right)^2 - \frac{1}{2}\tilde{\tau}_\pm P\left(\sigma_\gamma^2 + \sigma_\delta^2\right)$$

$$P = 1 - \frac{L}{2R}\left(1 - \frac{\langle\rho^2\rangle}{LR}\right)$$

$$K_{\pm n} = \exp\left[\left(1 - \frac{1}{n}\right)\tilde{\tau}_\pm\right] - 1$$

$$\tilde{\tau}_{\pm} = \tau_{\pm}\left(1 - \frac{\left\langle \rho^2 \right\rangle}{LR}\right)$$

$$\tau_{\pm} = \tau\left(1 \mp P_{He3}(t)\right)$$

$$\tau = n\sigma_0(\lambda_0)L\frac{\lambda}{\lambda_0} = \tau_0\frac{\lambda}{\lambda_0}$$

$$\tilde{\tau} = \tau\left(1 - \frac{\left\langle \rho^2 \right\rangle}{LR}\right)$$

See the section on wavelength and pathlength corrections for definitions of the symbols in these expressions. When convenient the correction factors, $\widetilde{C}_{\pm}$, can be approximated as unity.

Transmission measurements are used to extract the P_{He3} , and a general expression for transmission of an incident unpolarized beam of neutrons is

$$t_{up} = \sum_{n=1} a_n \left[\frac{1}{n}\right] \frac{1}{2}\left(t_{+n} + t_{-n}\right) / \sum_{n=1} a_n \left[\frac{1}{n}\right]$$

where the $\left[\frac{1}{n}\right]$ is inserted when using a monitor with order contamination, and

$$t_{\pm n} = C_{\pm n}\left(\tilde{\tau}_{\pm n}\right) t_E \exp\left(-\tilde{\tau}_{\pm n}\right).$$

n just divides the wavelength.

$$\tilde{\tau}_{\pm n} = \frac{1}{n}\left\langle \tau \right\rangle \left(1 \mp P_{He3}\right)$$

with $\langle \tau \rangle$ evaluated at the primary wavelength.

The measured transmission will have a counting statistics uncertainty that will lead to the corresponding uncertainty in the solution for P_{He3}. That is the slope of the transmission as a function of P_{He3} at the solution will be σ_{ts}/σ_{Ps}, where σ_{ts} is the uncertainty in the transmission due to counting statistics, and σ_{Ps} is the corresponding uncertainty in the polarization solution. If only relative changes in transmission are important (as when determining the polarization lifetime), the counting statistics uncertainty is appropriate. However, when absolute polarization is required, one should include the effect of the absolute uncertainty in the transmission. This absolute uncertainty derives from uncertainties in the glass transmission, σ_{tE}, and the value of $\langle \tau \rangle$ as σ_{τ}.

3.2 spin-rotation flipper

Excluding transport losses, the transfer matrix for a spin-rotation flipper when that flipper is ON, can be written in terms of the flipping efficiency, e_F (a number in the inclusive range 0 to 1), as

$$F_{P,A} = \begin{bmatrix} 1 - e_F & e_F \\ e_F & 1 - e_F \end{bmatrix}_{P,A} \tag{12}$$

Of course when the flipper is OFF this transfer matrix is the identity matrix which can be fudged by an effective flipping efficiency of $e_F = 0$. So the flipper-state dependent transfer matrix can be written

$$F_{P,A}^{\alpha=\pm} = \begin{bmatrix} 1 - e_F^\alpha & e_F^\alpha \\ e_F^\alpha & 1 - e_F^\alpha \end{bmatrix}_{P,A} \tag{13}$$

With $\alpha = -1$ indicating the flipper-ON state (many use the opposite convention),

$$e_F^\alpha = \delta_{\alpha,-1} e_F = \frac{1}{2}(1-\alpha)\, e_F.$$

The efficiency of a standard spin-rotation (Mezei) flipper depends on the exact angle of the neutron spin as it exits the precession coil. It is assumed that the guide field outside the precession coil is precisely in the z-direction, and the neutron spin enters the precession coil with its spin precisely along this z-direction which is parallel to the wires carrying the precession coil current. It is also assumed that that current has been set so that neutrons with wavelength, λ_M, will precess by exactly π radians as it crosses the flipper on a path that is in the x-direction perpendicular to the precession coil wire surface. Variations in λ or the path direction will result in variations in the precession angle as the neutron leaves the precession coil. In the "sudden" approximation the probability that the neutron spin has flipped is just the modulus of the overlap of the exit spinor with the z-direction state. The rotation axis of the neutron spin is the +y-axis, so that if θ is the rotation angle of the neutron spin after it crosses the precession coil, then the exit spinor in the z-up coordinate system is

$$\chi_{exit} = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}.$$

The probability that this spinor state will be spin-down (flipped) is then just $\sin^2(\frac{\theta}{2})$. The actual precession angle of any given neutron just depends on the time, t, it spends inside the precession coil, since the precession rate is fixed by the uniform magnetic field inside the coil. Thus, if t_M is the optimum time in the coil that produces a π flip, and L_M is the corresponding minimum pathlength, the precession angle for a neutron with actual time in the coil, t, over pathlength, L, can be written as

$$\theta = \frac{t}{t_M}\pi = \frac{L\lambda}{L_M\lambda_M}\pi = (1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2)(1+x)\pi = \pi + \epsilon,$$

where the actual pathlength as a function of horizontal and vertical deviation angles from the optimum perpendicular to coil direction is

$$L = L_M(1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2),$$

and $\lambda/\lambda_M = 1 + x$, so that $\epsilon = \left(x + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2\right)\pi$ to second order in the deviations. Then the flipper efficiency is

$$e_F = \sin^2(\frac{\pi + \epsilon}{2}) \cong 1 - \frac{1}{4}\epsilon^2 \cong 1 - \frac{\pi^2}{4}\left[x^2 + x(\gamma^2 + \delta^2) + \frac{1}{4}(\gamma^4 + 2\gamma^2\delta^2 + \delta^4)\right].$$

Averaging over independent Gaussian probability distributions for the angle and wavelength deviations yields

$$e_F \cong 1 - \frac{\pi^2}{4}\left\{\left(\frac{\sigma_\lambda}{\lambda_M}\right)^2 + \frac{3}{4}\left(\sigma_\gamma^4 + \sigma_\delta^4\right) + \frac{1}{2}\sigma_\gamma^2\sigma_\delta^2\right\}.$$

Note that the angle deviations contibute to fourth order while the relative wavelength deviation contributes to second order. If $\sigma_\lambda/\lambda_M = 0.02$ the flipper efficiency is about 0.999.

3.3 transport losses

In order to account for transport losses in terms of a transport efficiency, e_t, use the matrix

$$B_{P,A} = \frac{1}{2}\begin{bmatrix} 1 + e_{tP,A} & 1 - e_{tP,A} \\ 1 - e_{tP,A} & 1 + e_{tP,A} \end{bmatrix} \tag{14}$$

If transport loss, $\epsilon_t = 1 - e_t$, is used as the parameter, then the matrix is

$$B_{P,A} = \begin{bmatrix} 1 - \frac{1}{2}\epsilon_t & \frac{1}{2}\epsilon_t \\ \frac{1}{2}\epsilon_t & 1 - \frac{1}{2}\epsilon_t \end{bmatrix}$$

Transport losses are assumed to produce neutrons that have equal probability of being spin-up or spin-down (depolarized), although more complicated cases can occur. Note that multiplying two transport loss matrices results in a transport loss matrix where the transport efficiency is just the product of the two separate efficiencies. Also the product of a spin-flip matrix and a beam transport loss matrix is

$$F^\alpha B = \frac{1}{2}\begin{bmatrix} 1 - e_t(2e_F^\alpha - 1) & 1 + e_t(2e_F^\alpha - 1) \\ 1 + e_t(2e_F^\alpha - 1) & 1 - e_t(2e_F^\alpha - 1) \end{bmatrix} \tag{15}$$

The symmetric matrices for the spin-flipper and transport losses commute. This means that without loss of generality we can combine the transport loss matrices with the corresponding polarizer and analyzer matrices. Thus

$$B_P P \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{N}{2} = \begin{bmatrix} 1-\frac{1}{2}\epsilon_{tP} & \frac{1}{2}\epsilon_{tP} \\ \frac{1}{2}\epsilon_{tP} & 1-\frac{1}{2}\epsilon_{tP} \end{bmatrix} \begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{N}{2} =$$

$$\begin{bmatrix} t_{+P} - \frac{1}{2}\left(t_{+P} - t_{-P}\right)\epsilon_{tP} & 0 \\ 0 & t_{-P} + \frac{1}{2}\left(t_{+P} - t_{-P}\right)\epsilon_{tP} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{N}{2},$$

so that the beam transport loss on the polarizer side can be effectively absorbed into the polarizer matrix. For the scattered beam, suppose that $n_{f\pm}$ are the number of scattered neutrons in the $\pm$ neutron spin channels after the analyzer flipper. Then the detected neutrons are

$$\begin{pmatrix} n_+ \\ n_- \end{pmatrix} = AB_A \begin{pmatrix} n_{f+} \\ n_{f-} \end{pmatrix} = \begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\epsilon_{tA} & \frac{1}{2}\epsilon_{tA} \\ \frac{1}{2}\epsilon_{tA} & 1-\frac{1}{2}\epsilon_{tA} \end{bmatrix} \begin{pmatrix} n_{f+} \\ n_{f-} \end{pmatrix} =$$

$$\begin{bmatrix} t_{+A}\left(1-\frac{1}{2}\epsilon_{tA}\right) & t_{+A}\frac{1}{2}\epsilon_{tA} \\ t_{-A}\frac{1}{2}\epsilon_{tA} & t_{-A}\left(1-\frac{1}{2}\epsilon_{tA}\right) \end{bmatrix} \begin{pmatrix} n_{f+} \\ n_{f-} \end{pmatrix} =$$

$$\begin{bmatrix} t_{+A} - \frac{1}{2}\left(t_{+A} - t_{-A}\right)\epsilon_{tA} & 0 \\ 0 & t_{-A} + \frac{1}{2}\left(t_{+A} - t_{-A}\right)\epsilon_{tA} \end{bmatrix} \begin{pmatrix} n_{f+} \\ n_{f-} \end{pmatrix},$$

where the last line is valid since the detector measures $n_+ + n_-$. Thus we can include the transport loss on the analyzer side of the spectrometer into the analyzer matrix. This simplifies the total transport matrix to

$$T = A_t F_A S\, F_P P_t, \tag{16}$$

where the subscript t indicates that beam transport loss has been included.

How does adiabatic transport loss occur? Consider a neutron travelling in a guide magnetic field along the z-direction that encounters a magnetic field perturbation. Take the magnetic field to vary in the reference frame of the neutron as

$$\overrightarrow{B}(t) = B_z\hat{z} + B_x G(t)\hat{x}$$

so that the time dependence is in the magnetic field component along the $\hat{x}$ direction. Take, for example, a Gaussian time perturbation of B_x,

$$G(t) = \exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\tau}\right)^2\right].$$

If $B_x$is sufficiently small, a time-dependent perturbation solution based on an expansion in terms of the eigenstates when B_x is zero can be used. Such an expansion is

$$\overrightarrow{\chi}(t) = c_+(t)\exp(i\omega_z t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_-(t)\exp(-i\omega_z t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $\omega_z = \tilde{\gamma_n} B_z$ ($\tilde{\gamma_n} = 0.916x10^8 s^{-1}T^{-1}$is half the neutron gyromagnetic ratio). Substituting this solution into the spinor Schroedinger equation yields

$$\dot{c}_+ = i\omega_x G(t) c_- \exp(-2i\omega_z t)$$

$$\dot{c}_- = i\omega_x G(t) c_+ \exp(+2i\omega_z t)$$

where $\omega_x = \tilde{\gamma_n} B_x$. Satisfying the initial condition that $\overrightarrow{\chi}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, means that $c_+(0) = 1$ and $c_-(0) = 0$. Then the approximation is that c_+remains near 1 and c_-remains near zero during the perturbation, so solve only

$$\dot{c}_- = i\omega_x G(t) \exp(+2i\omega_z t).$$

Thus

$$c_-(T) = i\omega_x \int_0^T \exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\tau}\right)^2\right] \exp(+2i\omega_z t)\, dt.$$

or

$$c_-(\infty) = i\sqrt{2\pi}\omega_x \tau \, \exp\left[-2(\omega_z\tau)^2 + 2i\omega_z t_0\right].$$

Then the probability that the neutron ends up in the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ state is

$$|c_-(\infty)|^2 = 2\pi\left(\omega_x\tau\right)^2 \exp\left[-4(\omega_z\tau)^2\right].$$

This result shows that spin transport loss increases as the square of the magnetic field perturbation, and decreases exponentially with the square of the number of Larmor precessions the neutron makes during the time of the perturbation (i.e. large guide field magnitude is better for this term). This indicates why spin transport may be problematic, since the magnitude of field perturbations may be proportional to the magnitude of the guide field. The conclusion is that to keep the depolarization minimal we have the competing conditions, $\omega_z\tau \gg 1$ and $|B_x/B_z| \ll 1$.

3.4 sample transfer matrix

The transfer matrix for the sample is

$$S = \begin{bmatrix} S^{++} & S^{+-} \\ S^{-+} & S^{--} \end{bmatrix} \tag{17}$$

where S^{++}refers to the cross-section for scattering a neutron from a spin-up state to a spin-up state, and S^{+-}refers to the cross-section for scattering a neutron from a spin-up state to a spin-down state (spin-flip scattering). It is important to note that in general $S^{++} \neq S^{--}$and $S^{+-} \neq S^{-+}$, so that the sample transfer matrix does not commute with the spin-flipper and transport loss matrices (which do commute with one another). This creates a real problem for computation if there is depolarization in the sample where it can't be determined if the depolarization occurs before the scattering event or after the scattering event (or if there may be multiple scattering and depolarization events). The effective sample transfer matrix for depolarization before the scattering event is

$$\left[\begin{array}{cc} S^{++} & S^{+-} \\ S^{-+} & S^{--} \end{array}\right]\left[\begin{array}{cc} 1-\frac{1}{2}\epsilon_t & \frac{1}{2}\epsilon_t \\ \frac{1}{2}\epsilon_t & 1-\frac{1}{2}\epsilon_t \end{array}\right]=\left[\begin{array}{cc} \left(1-\frac{1}{2}\epsilon_t\right)S^{++}+\frac{1}{2}\epsilon_t S^{+-} & \left(1-\frac{1}{2}\epsilon_t\right)S^{+-}+\frac{1}{2}\epsilon_t S^{++} \\ \left(1-\frac{1}{2}\epsilon_t\right)S^{-+}+\frac{1}{2}\epsilon_t S^{--} & \left(1-\frac{1}{2}\epsilon_t\right)S^{--}+\frac{1}{2}\epsilon_t S^{-+} \end{array}\right]$$

On the other hand if the depolarization occurs after the scattering event the sample transfer is

$$\left[\begin{array}{cc} 1-\frac{1}{2}\epsilon_t & \frac{1}{2}\epsilon_t \\ \frac{1}{2}\epsilon_t & 1-\frac{1}{2}\epsilon_t \end{array}\right]\left[\begin{array}{cc} S^{++} & S^{+-} \\ S^{-+} & S^{--} \end{array}\right]=\left[\begin{array}{cc} \left(1-\frac{1}{2}\epsilon_t\right)S^{++}+\frac{1}{2}\epsilon_t S^{-+} & \left(1-\frac{1}{2}\epsilon_t\right)S^{+-}+\frac{1}{2}\epsilon_t S^{--} \\ \left(1-\frac{1}{2}\epsilon_t\right)S^{-+}+\frac{1}{2}\epsilon_t S^{++} & \left(1-\frac{1}{2}\epsilon_t\right)S^{--}+\frac{1}{2}\epsilon_t S^{+-} \end{array}\right]$$

One can sensibly assume that the depolarization occurs with equal probability before or after the scattering event, so that one can use the everage of these two results as

$$\left\langle\left[\begin{array}{cc} S^{++} & S^{+-} \\ S^{-+} & S^{--} \end{array}\right]\right\rangle=\left(1-\frac{1}{2}\epsilon_t\right)\left[\begin{array}{cc} S^{++} & S^{+-} \\ S^{-+} & S^{--} \end{array}\right]+\frac{1}{4}\epsilon_t\left[\begin{array}{cc} S^{+-}+S^{-+} & S^{++}+S^{--} \\ S^{++}+S^{--} & S^{+-}+S^{-+} \end{array}\right].$$

Thus if the scattering matrix has been solved without assuming any sample depolarization and with matrix elements

$$\left[\begin{array}{cc} \hat{S}^{++} & \hat{S}^{+-} \\ \hat{S}^{-+} & \hat{S}^{--} \end{array}\right]$$

then with the postulation of a sample depolarization probability of $\epsilon_t < 1$ and from the combinations

$$\tilde{S}^{++}=\hat{S}^{++}-\frac{\frac{1}{4}\epsilon_t}{1-\frac{1}{2}\epsilon_t}\left(\hat{S}^{-+}-\hat{S}^{+-}\right)$$

$$\tilde{S}^{+-}=\hat{S}^{+-}-\frac{\frac{1}{4}\epsilon_t}{1-\frac{1}{2}\epsilon_t}\left(\hat{S}^{--}-\hat{S}^{++}\right)$$

one can extract the cross-sections corrected for sample depolarization as

$$\left(\begin{array}{c} S^{++} \\ S^{+-} \end{array}\right)=\frac{1}{1-\epsilon_t}\left[\begin{array}{cc} 1-\frac{1}{2}\epsilon_t & -\frac{1}{2}\epsilon_t \\ -\frac{1}{2}\epsilon_t & 1-\frac{1}{2}\epsilon_t \end{array}\right]\left(\begin{array}{c} \tilde{S}^{++} \\ \tilde{S}^{+-} \end{array}\right)$$

and

$$S^{-+}=S^{+-}+\frac{1}{1-\frac{1}{2}\epsilon_t}\left(\hat{S}^{--}-\hat{S}^{++}\right)$$

$$S^{--}=S^{++}+\frac{1}{1-\frac{1}{2}\epsilon_t}\left(\hat{S}^{-+}-\hat{S}^{+-}\right)$$

There are cases where the depolarization is itself caused by a scattering event, for example when spin-incoherent scattering from hydrogen is the culprit. Then the simple minded procedure above is invalid. There are also models for depolarization due to ferromagnetic domains.

3.5 total transfer matrix

Combining all of the above transfer matrices, the time dependent detected counts for the up and down spin channels, which depend on the flipper states, α and β can be written,

$$\begin{bmatrix} n_+(t) \\ n_-(t) \end{bmatrix}^{\alpha\beta} = A(t)F_A^\beta B_A S B_P F_P^\alpha P(t) \begin{bmatrix} N/2 \\ N/2 \end{bmatrix} = T^{\alpha\beta}(t) \begin{bmatrix} N/2 \\ N/2 \end{bmatrix} \tag{18}$$

where $T^{\alpha\beta}(t)$ is a 2x2 matrix. Recall that the product of flip and transport efficiency matrices can be written

$$\begin{aligned} (F^\alpha B_,)_{P,A} &= \frac{1}{2} \begin{bmatrix} 1 - e_t(2e_F^\alpha - 1) & 1 + e_t(2e_F^\alpha - 1) \\ 1 + e_t(2e_F^\alpha - 1) & 1 - e_t(2e_F^\alpha - 1) \end{bmatrix}_{P,A} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2} e_{P,A}^\alpha \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \end{aligned}$$

where $e_{P,A}^\alpha = \{e_t(2e_F^\alpha - 1)\}_{P,A}$. The following matrix products are then required

$$\begin{aligned} \begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} &= \\ \sigma_{++++} \begin{bmatrix} t_{+A}t_{+P} & t_{+A}t_{-P} \\ t_{-A}t_{+P} & t_{-A}t_{-P} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} S \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} &= \\ \sigma_{+--+} \begin{bmatrix} t_{+A}t_{+P} & -t_{+A}t_{-P} \\ -t_{-A}t_{+P} & t_{-A}t_{-P} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} &= \\ \sigma_{+-+-} \begin{bmatrix} t_{+A}t_{+P} & -t_{+A}t_{-P} \\ t_{-A}t_{+P} & -t_{-A}t_{-P} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} t_{+A} & 0 \\ 0 & t_{-A} \end{bmatrix} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} S \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t_{+P} & 0 \\ 0 & t_{-P} \end{bmatrix} &= \\ \sigma_{++--} \begin{bmatrix} t_{+A}t_{+P} & t_{+A}t_{-P} \\ -t_{-A}t_{+P} & -t_{-A}t_{-P} \end{bmatrix}. \end{aligned}$$

Here $\sigma_{\pm\pm\pm\pm}$ refers to a sum of the four cross-sections, $S^{\pm\pm}$, with the sign of each term given by the corresponding $\pm$ index of σ. For example, $\sigma_{++++} =$

$S^{++}+S^{+-}+S^{-+}+S^{--}$. The four elements ($\mu=\pm1, \nu=\pm1$) of the flipper-state-dependent total transfer matrix then follow as

$$T_{\mu\nu}^{\alpha\beta}(t)=E_{\mu\nu}^{\alpha\beta}t_{\mu A}^{\alpha\beta}t_{\nu P}^{\alpha\beta},$$

where

$$E_{\mu\nu}^{\alpha\beta}=\frac{1}{4}\Sigma_{\mu'\nu'}S^{\mu'\nu'}\left(1-\mu'\mu e_A^{\alpha}\right)\left(1-\nu'\nu e_P^{\beta}\right).$$

The $\alpha\beta$ indices are added to the He-3 transmission factors to indicate that they must be evaluated at the time that the $\alpha\beta$ count-rate is measured. The total detected counts for each combination of polarizer and analyzer flipper states, $\alpha=1$ for analyzer flipper OFF, $\alpha=-1$ for analyzer flipper ON, $\beta=1$ for polarizer flipper OFF and $\beta=-1$ for polarizer flipper ON, are

$$Counts^{\alpha\beta}=C^{\alpha\beta}=n_+^{\alpha\beta}+n_-^{\alpha\beta}=\frac{N}{2}\Sigma_{\mu\nu}T_{\mu\nu}^{\alpha\beta}=\frac{N}{2}\Sigma_{\mu\nu}E_{\mu\nu}^{\alpha\beta}t_{\mu A}^{\alpha\beta}t_{\nu P}^{\alpha\beta}.$$

Recall that $t_{\pm A,P}=\widetilde{C}_{\pm A,P}t_E\exp\left(-\tilde{\tau}_{\pm A,P}\right)$ are the transmission factors from the He-3 analyzer and polarizer. Now the expected count rates can be written as a linear function of the four polarized beam cross-sections

$$C^{\alpha\beta}=\frac{N}{2}\Sigma_{\mu\nu}c_{\mu\nu}^{\alpha\beta}S^{\mu\nu}$$

where the 4x4 matrix of coefficients is

$$c_{\mu\nu}^{\alpha\beta}=\frac{1}{4}\sum_{\mu'}\left(1-\mu'\mu e_A^{\alpha}\right)t_{\mu'A}^{\alpha\beta}\sum_{\nu'}\left(1-\nu'\nu e_P^{\beta}\right)t_{\nu'P}^{\alpha\beta}. \tag{19}$$

so that each matrix element is the product of factors from before and after the sample. Recall that the efficiency coefficients, $e_{A,P}^{\alpha}$, are given by,

$$e_{A,P}^{\alpha}=e_{tA,P}(2e_{FA,P}^{\alpha}-1)$$

which is a product involving the transport and spin-flip efficiencies. P refers to before the sample and A refers to after the sample.

In order to understand the matrix of coefficients, examine the simplest case. Assume that the transport and flipping efficiencies are unity so that $e_{A,P}^{\alpha}=-\alpha$ (since $e_F^{\alpha}=\frac{1}{2}\left(1-\alpha\right)e_F$). Then

$$c_{\mu\nu}^{\alpha\beta}=\frac{1}{4}\sum_{\mu'}\left(1+\mu'\mu\alpha\right)t_{\mu'A}^{\alpha\beta}\sum_{\nu'}\left(1+\nu'\nu\beta\right)t_{\nu'P}^{\alpha\beta}.$$

Now

$$\sum_{\mu'}\left(1+\mu'\mu\alpha\right)t_{\mu'A}=t_{+A}+t_{-A}+\mu\alpha\left(t_{+A}-t_{-A}\right)=2t_{(\mu\alpha)A}.$$

so that the matrix elements have simplified to

$$c_{\mu\nu}^{\alpha\beta} = t_{(\mu\alpha)A} t_{(\nu\beta)P}.$$

For example, if both flippers are off, $c_{\mu\nu}^{++} = t_{\mu A} t_{\nu P}$ and the total detected counts are

$$\begin{aligned} BothFlippersOFFCounts &= C^{++} = \\ \left(S^{++} t_{+A} t_{+P} + S^{+-} t_{+A} t_{-P} + S^{-+} t_{-A} t_{+P} + S^{--} t_{-A} t_{-P}\right) \frac{N}{2}. \end{aligned}$$

The He-3 transmission factors for the perferred spin-states, $t_{+A}$and $t_{+P}$are typically much larger than the transmission factors for the non-preferred states, so one approximately measures S^{++}.

$$C^{++} \cong \left(S^{++} t_{+A} t_{+P}\right) \frac{N}{2}.$$

Now, turning on the polarizer flipper gives $c_{\mu\nu}^{+-} = t_{\mu A} t_{-\nu P}$, and

$$\begin{aligned} PolarizerFlipperONCounts = C^{+-} &= \\ \left(S^{++} t_{+A} t_{-P} + S^{+-} t_{+A} t_{+P} + S^{-+} t_{-A} t_{-P} + S^{--} t_{-A} t_{+P}\right) \frac{N}{2}. \end{aligned}$$

$S^{+-}$is multiplied by the largest transmission factors so that

$$C^{+-} \cong \left(S^{+-} t_{+A} t_{+P}\right) \frac{N}{2}.$$

Similarly

$$AnalyzerFlipperONCounts = C^{-+} \cong \left(S^{-+} t_{+A} t_{+P}\right) \frac{N}{2},$$

and

$$BothFlippersONCounts = C^{--} \cong \left(S^{--} t_{+A} t_{+P}\right) \frac{N}{2}.$$

These expressions have to be corrected since the transmission factors for the non-preferred states, $t_{-A}$and t_{-P} are likely not zero.

Once more, it must be emphasized that the separate count-rates for the different cross-sections are measured at different times, so that the He-3 transmission coeficients must be evaluated for each different count-rate.

4 correcting polarized beam data

Since neutron polarizing, flipping and transport devices may not be perfectly efficient, it is necessary to examine the corrections that need to be made to raw polarized beam data in order to extract the the cross-sections that produce observed count rates. In this section it is assumed that the efficiencies of the polarized beam transport have already been determined (that determination is discussed in section 6). Recall from the section that derived the transfer matrix, that the expected count rates can be written as a linear function of the cross-sections

$$C^{\alpha\beta} = \frac{N}{2}\Sigma_{\mu\nu} c^{\alpha\beta}_{\mu\nu} S^{\mu\nu}$$

where the elements of the matrix of coefficients are

$$c^{\alpha\beta}_{\mu\nu} = \frac{1}{4}\sum_{\mu'}\left(1-\mu'\mu e^{\alpha}_{A}\right) t^{\alpha\beta}_{\mu' A}\sum_{\nu'}\left(1-\nu'\nu e^{\beta}_{P}\right) t^{\alpha\beta}_{\nu' P}.$$

To be precise, each row $(\alpha\beta)$ can be associated with a time at which the count-rate, $C^{\alpha\beta}$, is measured, so that the transmission coeficients, $t^{\alpha\beta}_{\mu' A}$ and $t^{\alpha\beta}_{\nu' P}$, are labelled with the row index, $\alpha\beta$. Otherwise we are assuming that all of the count-rates are measured near enough to each other in time to avoid the necessity for this lablelling. Do the sums on μ' and ν' by defining

$$\sum_{\mu'} t_{\mu' X} = t_{+X} + t_{-X} = t_{sX}$$

$$\sum_{\mu'} \mu' t_{\mu' X} = t_{+X} - t_{-X} = t_{aX}.$$

From the expressions for the transmission coefficients, t_{sX} and t_{aX} can be expanded to

$$t_{sX} = 2t_{EX} C_{sX}\exp\left(-\tilde{\tau}_X\right)\cosh\left(\tilde{\tau}_X P_{He3X}\right)$$

$$t_{aX} = 2t_{EX} C_{aX}\exp\left(-\tilde{\tau}_X\right)\sinh\left(\tilde{\tau}_X P_{He3X}\right)$$

where

$$C_{sX} = \langle C_X\rangle\left\{1+\frac{\Delta_X}{\langle C_X\rangle}\tanh\left(\tilde{\tau}_X P_{He3X}\right)\right\}$$

$$C_{aX} = \langle C_X\rangle\left\{1+\frac{\Delta_X}{\langle C_X\rangle}\coth\left(\tilde{\tau}_X P_{He3X}\right)\right\}.$$

See the section on correcting the transmission for wavelength and pathlength deviations, 7, for the definitions of $\langle C_X \rangle \cong 1$ and $\Delta_X \ll 1$. Then the matrix elements are

$$c_{\mu\nu}^{\alpha\beta} = \frac{1}{4}\left(t_{sA} - \mu e_A^\alpha t_{aA}^{\alpha\beta}\right)\left(t_{sP} - \nu e_P^\beta t_{aP}^{\alpha\beta}\right).$$

In general the matrix of these coefficients will require numerical inversion to solve for the cross-sections corresponding to observed count rates. In order to determine the uncertainty in the coefficients, it is reasonable to assume that the uncertainty in the transmission factors, $t_{\pm M} = \tilde{C}_{\pm M} t_{EM} \exp\left(-\tilde{\tau}_{\pm M}\right)$, is due to the uncertainty in $\tilde{\tau}_{\pm M}$. This works because the empty cell transmissions, t_{EM}, occur in all of the the coefficients, so that the uncertainty in t_{EM} can be considered to just affect the uncertainty in an overall scale factor when solving for the underlying cross-sections $S^{\mu\nu}$. Also, the uncertainty in the correction coefficient, $\tilde{C}_{\pm M}$, arises primarily from the uncertainty in $\tilde{\tau}_{\pm M}$, as can be seen by looking at a typical formula for $\tilde{C}_{\pm M}$ and realizing that the instrumental factors are typically on the order of 0.01. Thus it is easy to show that

$$\sigma_{\widetilde{C}_{\pm M}} = \left|\frac{\partial \widetilde{C}_{\pm M}}{\partial \tilde{\tau}_{\pm M}}\right| \sigma_{\tilde{\tau}_{\pm M}},$$

but the partial derivative coefficient is much less than one, and $\sigma_{\widetilde{C}_{\pm M}}$ can be neglected compared to $\sigma_{\tilde{\tau}_{\pm M}}$. Thus we can write to a good approximation that

$$\sigma_{t_{\pm M}} \cong t_{\pm M}\sigma_{\tilde{\tau}_{\pm M}}.$$

From the definition of $\tilde{\tau}_{\pm M}$ it follows that

$$\sigma_{\tilde{\tau}_{\pm M}}^2 = \tilde{\tau}_M^2 \sigma_{PHe3,M}^2 + \sigma_{\tilde{\tau}_M}^2.$$

This means that the relative uncertainty in the transmission coefficient, $\tilde{\sigma}_{t_{\pm M}} = \sigma_{t_{\pm M}}/t_{\pm M} = \sigma_{\tilde{\tau}_{\pm M}}$, is independent of the spin state. The computation of $\sigma_{PHe3,M}^2$ and $\sigma_{\tilde{\tau}_M}^2$ is described in the previous section 3.1. Trivially we also have

$$\sigma_{t_{sM}}^2 = \sigma_{t_{aM}}^2 = \sigma_{t_{+M}}^2 + \sigma_{t_{-M}}^2.$$

The partial derivatives of the coefficients, $c_{\mu\nu}^{\alpha\beta}$, with respect to the transmission factors, t_{sM} and t_{aM} are

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial t_{sA}} = \frac{1}{4}\left(t_{sP} - \nu e_P^\beta t_{aP}\right)$$

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial t_{sP}} = \frac{1}{4}\left(t_{sA} - \mu e_A^\alpha t_{aA}\right)$$

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial t_{aA}} = -\frac{1}{4}\mu e_A^{\alpha}\left(t_{sP} - \nu e_P^{\beta} t_{aP}\right)$$

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial t_{aP}} = -\frac{1}{4}\nu e_P^{\beta}\left(t_{sA} - \mu e_A^{\alpha} t_{aA}\right).$$

The partial derivatives of the coefficients, $c_{\mu\nu}^{\alpha\beta}$, with respect to the transport efficiencies, e_{tM}, and flipper efficiencies, e_{FM}, are

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial e_{tA}} = -\frac{1}{4}\mu t_{aA}\left(t_{sP} - \nu e_P^{\beta} t_{aP}\right)\left(2e_{FA}^{\alpha} - 1\right)$$

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial e_{tP}} = -\frac{1}{4}\nu t_{aP}\left(t_{sA} - \mu e_A^{\alpha} t_{aA}\right)\left(2e_{FP}^{\beta} - 1\right)$$

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial e_{FA}} = -\frac{1}{4}\mu t_{aA}\left(t_{sP} - \nu e_P^{\beta} t_{aP}\right)\left(1-\alpha\right) e_{tA}$$

$$\frac{\partial c_{\mu\nu}^{\alpha\beta}}{\partial e_{FP}} = -\frac{1}{4}\nu t_{aP}\left(t_{sA} - \mu e_A^{\alpha} t_{aA}\right)\left(1-\beta\right) e_{tP}.$$

Combining all the terms for the uncertainty in $c_{\mu\nu}^{\alpha\beta}$ results in contributions from both the analyzer and polarizer groups,

$$\sigma_{c_{\mu\nu}^{\alpha\beta}}^2 = \sigma_A^2 + \sigma_P^2,$$

where

$$\sigma_A^2 = \frac{1}{16}\left(t_{sP} - \nu e_P^{\beta} t_{aP}\right)^2 f(A,\alpha)$$

$$\sigma_P^2 = \frac{1}{16}\left(t_{sA} - \mu e_A^{\alpha} t_{aA}\right)^2 f(P,\beta),$$

and where

$$f(M,\gamma) = \left[1 + \left(e_M^{\gamma}\right)^2\right]\left[\sigma_{t+M}^2 + \sigma_{t-M}^2\right] + \left[t_{aM}\right]^2\left[\left(2e_{FM}^{\gamma} - 1\right)^2\sigma_{e_{tM}}^2 + \left(1-\gamma\right)^2 e_{tM}^2\sigma_{e_{FM}}^2\right].$$

Given the coefficients, $c_{\mu\nu}^{\alpha\beta}$, and their uncertainties, $\sigma_{c_{\mu\nu}^{\alpha\beta}}^2$, it is then possible to solve for the underlying cross-sections, $S^{\mu\nu}$, and propagate the errors to $\sigma_{S^{\mu\nu}}$. Methods for doing the inversion and propagating errors are discussed in another document, *PBcorrect*. Sometimes in polarized beam experiments the number of unknown cross-sections to be determined is reduced by contraints, for example when $S^{+-} = S^{-+}$and/or $S^{++} = S^{--}$. One must be able to put these constraints into the master equation and there should also be a method to reduce the number of equations if more of the count rates, $C^{\alpha\beta}$, have been measured than there are independent equations. One can, of course, discard equations, or combine by adding or subtracting equations. For example, if $S^{+-} = S^{-+}$, then one simple way to bring the number of equations down if both $C^{+-}$and C^{-+} were measured, is to add the two equations for $C^{+-}$and C^{-+}taking care to propagate the errors. A discussion of the overdetermined problem where there are constraints so that there are more equations than unknowns will require additional discussion in terms of the *Totla Least Squares* problem.

4.1 determining count rates

Note that the formulae for the observed count rates depend linearly on the neutron flux into the He-3 polarizer. If this neutron flux is time indepedent, then the count rates can be used directly in the correction formulae. If the neutron flux varies with time, which or course happens if the incident neutron energy is varied, then a low efficiency beam monitor before the polarizer could be used to determine relative changes in the incident neutron flux. This is usually accomplished by counting until a fixed number of beam monitor counts is recorded and correcting for the higher-order wavelength contamination (Ei dependent) counted by the beam monitor. The correction factor that multiplies the counts, in terms of the wavelength order fractions, a_n, (where $\sum_n a_n = 1$) is

$$correctionFactor = \frac{monitorRateAllOrders}{monitorRatePrimary} = \left(\sum_n \frac{a_n}{n}\right)/a_1.$$

This correction to the count rate is greater than unity when the incident beam is contaminated by higher orders since the monitor counts the higher orders as well as the primary measuring wavelength.

A more involved correction is required if the incident flux has to be tracked using a beam-monitor placed after the polarizer. Now the time and wavelength dependence of the polarizer transmission has to be taken into account to determine relative fluxes of the primary and higher order wavelength neutrons. In this case with the beam-monitor after the polarizer, the correction factor is given by

$$correctionFactor = \left(\sum_n \frac{a_n}{n}(t_{\pm n} + t_{-n})\right)/(a_1(t_{+1} + t_{-1})).$$

Here $t_{\pm n}$ are the transmission factors for the preferred and nonpreffered spins states for the nth order wavelength.

4.2 spin-flip and non-spin-flip cross-sections only

One important special case is when the cross-sections have the often occurring symmetry that $S^{++} = S^{--} = S^{nsf}$, and $S^{+-} = S^{-+} = S^{sf}$. In this case the master equation for the expected count rates reduces to

$$C^{\alpha\beta}/\frac{N}{2} = \left(c_{++}^{\alpha\beta} + c_{--}^{\alpha\beta}\right) S^{nsf} + \left(c_{+-}^{\alpha\beta} + c_{-+}^{\alpha\beta}\right) S^{sf}.$$

Cross terms cancel when the coefficients are added in this way (in the equal time approximation for the He-3 transmission factors), so that

$$C^{\alpha\beta}/\frac{N}{2} = \frac{1}{2}\left(t_+ + e_A^{\alpha} e_P^{\beta} t_-\right) S^{nsf} + \frac{1}{2}\left(t_+ - e_A^{\alpha} e_P^{\beta} t_-\right) S^{sf},$$

where

$$t_+ = t_{sA} t_{sP} = t_{+A} t_{+P} + t_{-A} t_{-P} + t_{+A} t_{-P} + t_{-A} t_{+P}$$

$$t_- = t_{aA} t_{aP} = t_{+A} t_{+P} + t_{-A} t_{-P} - t_{+A} t_{-P} - t_{-A} t_{+P}.$$

The expansions of t_+ and t_- follow simply from the expansions of t_{sX} and t_{aX}. These equations can easily be inverted to obtain the cross-sections as a function of the count-rates. This is typically accomplished by measuring the non-spin-flip counts as $C^{++} = C^{nsf}$, and using one flipper to measure either C^{+-} or C^{-+} as the spin-flip count rate, C^{sfX}, where X indicates which flipper is used. This system of equations is

$$\begin{pmatrix} C^{++} \\ C^{sfX} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\begin{pmatrix} t_+ + e_t A t_- & t_+ - e_t A t_- \\ t_+ - e_t B t_- & t_+ + e_t B t_- \end{pmatrix}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix}$$

where $e_t = e_{tA} e_{tP}$ is the aggregate beam transport efficiency, $A = 1$, $B = 2e_{FX} - 1$, and $X = P, A$ depending on which flipper is used. In the case that these equations are used to solve for count rates when the beam is $\lambda/2$, a current-flipper set to flip λ will depolarize the beam when activated. This can be handled by setting $B = 0$, or $e_{FX} = 1/2$. The determinant of the matrix is $4e_t e_{FX} t_+ t_-$ so that matrix inversion gives the result

$$\frac{N}{2}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{1}{2e_t e_{FX}}\begin{pmatrix} a_{+B} & -a_{-A} \\ -a_{-B} & a_{+A} \end{pmatrix}\begin{pmatrix} C^{++} \\ C^{sfX} \end{pmatrix}$$

where the elements of the inverted matrix are found from

$$a_{\pm A} = \frac{t_+ \pm e_t A t_-}{t_+ t_-}$$

$$a_{\pm B} = \frac{t_+ \pm e_t B t_-}{t_+ t_-}.$$

In order to do the error analysis on this solution, write

$$S^{nsf} e_t e_{FX} N = \left(\frac{e_t B}{t_+} + \frac{1}{t_-}\right) C^{++} + \left(\frac{e_t A}{t_+} - \frac{1}{t_-}\right) C^{sfX}$$

$$S^{sf} e_t e_{FX} N = \left(\frac{e_t B}{t_+} - \frac{1}{t_-} \right) C^{++} + \left(\frac{e_t A}{t_+} + \frac{1}{t_-} \right) C^{sfX}$$

which is

$$NS^{nsf} = K^{++} C^{++} + K^{+-} C^{sfX}$$

$$NS^{sf} = K^{-+} C^{++} + K^{--} C^{sfX}$$

where

$$K^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta}$$

and

$$A^{\alpha\beta} = \frac{\alpha\beta}{e_t e_{FX}} \frac{1}{t_-}$$

$$B^{\alpha\beta} = \frac{2\delta_{\beta+}}{t_+}$$

$$C^{\alpha\beta} = \frac{-\beta}{e_{FX}} \frac{1}{t_+}.$$

The partial derivatives of t_+ and t_- are

$$\frac{\partial t_\pm}{\partial \tilde{\tau}_{P,A}} = t_\pm \left[P_{He3P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3} \right)_{P,A} - 1 \right]$$

$$\frac{\partial t_\pm}{\partial P_{He3P,A}} = t_\pm \left[\tilde{\tau}_{P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3} \right)_{P,A} \right].$$

Then the error propagation for the coefficients is

$$\sigma^2_{K^{\alpha\beta}} = \left(A^{\alpha\beta} \right)^2 \tilde{\sigma}^2_{e_t} + \left(A^{\alpha\beta} + C^{\alpha\beta} \right)^2 \tilde{\sigma}^2_{e_{FX}} + \sum_{X = \tilde{\tau}_A \tilde{\tau}_P P_A P_P} \left(W^{\alpha\beta}_X \right)^2 \sigma^2_X$$

where

$$W^{\alpha\beta}_X = A^{\alpha\beta} \left[\bar{x} \coth \left(x\overline{x} \right) - \delta_{x\tau} \right] + \left(B^{\alpha\beta} + C^{\alpha\beta} \right) \left[\bar{x} \tanh \left(x\overline{x} \right) - \delta_{x\tau} \right]$$

and $\bar{x}$ is the partner variable for x in the pairs $\tilde{\tau}_A P_{He3A}$ and $\tilde{\tau}_P P_{He3P}$. The error propagation for the X variables $\tilde{\tau}$ and P is in section 3.1. Also, in general

$$\tilde{\sigma}^2_X = \frac{\sigma^2_X}{X^2}.$$

The final error propagation to the cross-sections is

$$\sigma^2_{NS^{nsf}} = \left(C^{++}\sigma_{K^{++}}\right)^2 + \left(C^{sfX}\sigma_{K^{+-}}\right)^2 + \left(K^{++}\sigma_{C^{++}}\right)^2 + \left(K^{+-}\sigma_{C^{sfX}}\right)^2$$

$$\sigma^2_{NS^{sf}} = \left(C^{++}\sigma_{K^{-+}}\right)^2 + \left(C^{sfX}\sigma_{K^{--}}\right)^2 + \left(K^{-+}\sigma_{C^{++}}\right)^2 + \left(K^{--}\sigma_{C^{sfX}}\right)^2$$

If count rates are measured for all four cross-sections then the counts in the spin-flip and non-spin flip channels can be added so that

$$\frac{1}{2}\begin{pmatrix} C^{++}+C^{--} \\ C^{+-}+C^{-+} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\begin{pmatrix} t_+ + e_t A t_- & t_+ - e_t A t_- \\ t_+ - e_t B t_- & t_+ + e_t B t_- \end{pmatrix}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix}$$

where

$$A = \frac{1}{2}\left[1 + (2e_{FA}-1)(2e_{FP}-1)\right]$$

$$B = \frac{1}{2}\left[(2e_{FA}-1) + (2e_{FP}-1)\right]$$

The determinant of this matrix is $4e_t e_{FA} e_{FP} t_+ t_-$ so that the inversion becomes

$$\frac{N}{2}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{1}{2e_t e_{FA} e_{FP}}\begin{pmatrix} a_{+B} & -a_{-A} \\ -a_{-B} & a_{+A} \end{pmatrix}\begin{pmatrix} \langle C^{nsf}\rangle \\ \langle C^{sf}\rangle \end{pmatrix}$$

where

$$a_{\pm A} = \frac{t_+ \pm e_t A t_-}{t_+ t_-}$$

$$a_{\pm B} = \frac{t_+ \pm e_t B t_-}{t_+ t_-}.$$

The error analysis proceeds as before by separating the solution coefficients into terms that depend on transport and flipper efficiencies,

$$S^{nsf} e_t e_{FA} e_{FP} N = \left(\frac{e_t B}{t_+} + \frac{1}{t_-}\right)\langle C^{nsf}\rangle + \left(\frac{e_t A}{t_+} - \frac{1}{t_-}\right)\langle C^{sf}\rangle$$

$$S^{sf} e_t e_{FA} e_{FP} N = \left(\frac{e_t B}{t_+} - \frac{1}{t_-}\right)\langle C^{nsf}\rangle + \left(\frac{e_t A}{t_+} + \frac{1}{t_-}\right)\langle C^{sf}\rangle,$$

which as before is

$$NS^{nsf} = K^{++}C^{++} + K^{+-}C^{sfX}$$

$$NS^{sf} = K^{-+}C^{++} + K^{--}C^{sfX}$$

where

$$K^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta} + E^{\alpha\beta}$$

and

$$A^{\alpha\beta} = \frac{\alpha\beta}{e_t e_{FA} e_{FP}} \frac{1}{t_-}$$

$$B^{\alpha\beta} = \frac{2\delta_{\beta-}}{t_+}$$

$$C^{\alpha\beta} = \frac{\beta}{e_{FA}} \frac{1}{t_+}$$

$$D^{\alpha\beta} = \frac{\beta}{e_{FP}} \frac{1}{t_+}$$

$$E^{\alpha\beta} = -\frac{\beta}{e_{FA} e_{FP}} \frac{1}{t_+}.$$

Then the error propagation for the coefficients is

$$\sigma^2_{K^{\alpha\beta}} = \left(A^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_t} + \left(W^{\alpha\beta}_{e_{FA}}\right)^2 \tilde{\sigma}^2_{e_{FA}} + \left(W^{\alpha\beta}_{e_{FP}}\right)^2 \tilde{\sigma}^2_{e_{FP}} + \sum_{X=\tilde{\tau}_A \tilde{\tau}_P P_A P_P} \left(W^{\alpha\beta}_X\right)^2 \sigma^2_X$$

where

$$W^{\alpha\beta}_{e_{FA}} = A^{\alpha\beta} + C^{\alpha\beta} + E^{\alpha\beta}$$

$$W^{\alpha\beta}_{e_{FP}} = A^{\alpha\beta} + D^{\alpha\beta} + E^{\alpha\beta}$$

$$W^{\alpha\beta}_X = A^{\alpha\beta}\left[\bar{x}\coth\left(x\overline{x}\right) - \delta_{x\tau}\right] + \left(B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta} + E^{\alpha\beta}\right)\left[\bar{x}\tanh\left(x\overline{x}\right) - \delta_{x\tau}\right]$$

The final error propagation to the cross-sections is

$$\sigma^2_{NS^{nsf}} = \left(\left\langle C^{nsf}\right\rangle \sigma_{K^{++}}\right)^2 + \left(\left\langle C^{sf}\right\rangle \sigma_{K^{+-}}\right)^2 + \left(K^{++}\sigma_{\langle C^{nsf}\rangle}\right)^2 + \left(K^{+-}\sigma_{\langle C^{sf}\rangle}\right)^2$$

$$\sigma^2_{NS^{sf}} = \left(\left\langle C^{nsf}\right\rangle \sigma_{K^{-+}}\right)^2 + \left(\left\langle C^{sf}\right\rangle \sigma_{K^{--}}\right)^2 + \left(K^{-+}\sigma_{\langle C^{nsf}\rangle}\right)^2 + \left(K^{--}\sigma_{\langle C^{sf}\rangle}\right)^2$$

4.2.1 simplification using a flipping ratio measurement

Returning to the solution for the cross-sections, this is a good approach if the ratio S^{sf}/S^{nsf}(or its inverse) is of interest, since this ratio depends only on the measured counts, the measured flipping ratio and a small correction for flipping efficiency. However, in the following it will be shown that the formula and error analysis is even simpler in terms of the cross-section asymmetry and count-rate asymmetry. Using these results, the solution for the ratio S^{nsf}/S^{sf} is (assuming that $S^{sf} > 0$),

$$s_{nsf} = \frac{S^{nsf}}{S^{sf}} = \frac{C^{nsf} - \frac{1}{R_n} C^{sf} - \epsilon \left(C^{nsf} - C^{sf} \right)}{C^{sf} - \frac{1}{R_n} C^{nsf}}$$

with squared relative error

$$\tilde{\sigma}^2_{s_{nsf}} = W_R^2 \tilde{\sigma}_R^2 + W_{e_F}^2 \tilde{\sigma}^2_{e_F} + W_C^2 \left(\tilde{\sigma}^2_{C^{nsf}} + \tilde{\sigma}^2_{C^{sf}} \right)$$

where

$$W_R = R_n \frac{\left(C^{nsf}\right)^2 - \left(C^{sf}\right)^2}{\left(R_n C^{nsf} - C^{sf}\right)\left(R_n C^{sf} - C^{nsf}\right)}$$

$$W_{e_F} = \frac{C^{nsf} - C^{sf}}{R_n C^{nsf} - C^{sf}} \frac{R_n - 1}{e_F}$$

$$W_C = \frac{C^{nsf} C^{sf} \left(R_n^2 - 1\right)}{\left(R_n C^{nsf} - C^{sf}\right)\left(R_n C^{sf} - C^{nsf}\right)}.$$

If the ratio S^{sf}/S^{nsf} is of interest, simply invert the formula above and interchange C^{nsf} and C^{sf} in the error analysis. $\tilde{\sigma}_R^2$ is defined in the section on flipping ratios 6.

Now the remaining time dependence is in K_e which is

$$K_e = \frac{e_t e_F t_-}{1 + e_t t_- / t_+} \tag{20}$$

Making the same replacement for t_-/t_+ in terms of R, $K_e$can be rewritten as

$$K_e = \frac{1}{2} e_t \left(2 e_F - 1 + \frac{1}{R_n} \right) t_-$$

where recall that $t_-$is

$$t_- = 4 t_{EA} t_{EP} C_{\Delta -} \exp\left(-\tilde{\tau}_A\right) \exp\left(-\tilde{\tau}_P\right) \sinh\left(\tilde{\tau}_A P_{He3A}\right) \sinh\left(\tilde{\tau}_P P_{He3P}\right).$$

Be aware that the transport efficiency may be angle dependent. To be precise, the transport and flipping efficiencies should be measured at the same spectrometer settings and guide field settings used to measure C^{nsf} and C^{sf} and those efficiencies should then be used to make the corrections to obtain S^{nsf} and S^{sf}. A classic case is the use of the neutron polarization direction to vary the amount of magnetic scattering that contributes to the spin-flip and non-spin-flip channels. This dependence arises from the fact that spin-flip magnetic scattering is due to the neutron scattering from sample magnetic moment components that are perpendicular to the neutron polarization direction, and conversely the non-spin-flip magnetic scattering is due to the neutron scattering from sample magnetic moment components that are parallel to the neutron polarization direction. This dependence is utilised experimentally by controlling the neutron polarization direction at the sample with either a vertical (to scattering plane) or horizontal guide field. If the horizontal sample guide field is aligned along the scattering vector, Q, then all magnetic scattering must be in the spin-flip channel, since the neutron spin scatters only from sample magnetic moment components that are perpendicular to Q, and these same sample magnetic moments are also perpendicular to the neutron spin. Since there are other possible contributions to the scattering in the spin-flip channel, the usual procedure is to subtract off the spin-flip scattering observed when the sample guide field is vertical. This vertical field spin-flip scattering will have a different amount of magnetic scattering but all the other scattering processes will be the same as in the horizontal field case. Since the transport efficiencies may be different between the vertical and horizontal field cases it is important to correct the observed counts (using the efficiencies) to obtain the true cross-sections before making such a subtraction.

Now finally we return to consider the contamination of a flipping ratio measurement by spin-flip scattering. We want to use information about the contamination obtained from a background measurement to calculate the correction necessary to give the true non-spin-flip flipping ratio. Using the formula for the spin-flip scattering measured in the background (after fast-background subtraction)

$$S_{bg}^{sf} N K_e = C_{bg}^{sf} - \frac{1}{R_n} C_{bg}^{nsf}$$

Measuring at the Bragg peak, we assume that the spin-flip cross-section from the background, S_{bg}^{sf}, is the only source of spin-flip scattering at the Bragg peak. Then

$$S_{bg}^{sf} N K_e = C_{Bragg}^{sf} - \frac{1}{R_n} C_{Bragg}^{nsf}$$

and then

$$R_n = \left(C_{Bragg}^{nsf} - C_{bg}^{nsf} \right) / \left(C_{Bragg}^{sf} - C_{bg}^{sf} \right)$$

4.2.2 cross-section asymmetry solution

When the ratio of cross-sections is the quantity of interest, the ideal analysis is in terms of the cross-section asymmetry, $-1 \leq s \leq 1$, defined as

$$s = \frac{S^{nsf} - S^{sf}}{S^{nsf} + S^{sf}}$$

and the count-rate asymmetry, $-1 \le c \le 1$, defined as

$$c = \frac{C^{nsf} - C^{sf}}{C^{nsf} + C^{sf}},$$

$$(1 - c)\, C^{nsf} = (1 + c)\, C^{sf}.$$

The result for s in terms of c in the equal time approximation is

$$s = \frac{t_+}{t_-} \frac{c/e_t}{e_F - (1 - e_F)c} = \frac{R_n\,(2e_F - 1) + 1}{R_n - 1} \frac{c}{e_F - (1 - e_F)c} \cong \frac{R_n + 1}{R_n - 1} \frac{c}{e_F},$$

where the approximation is for e_F near to unity. All of the time dependence and beam transport efficiency is now contained in the measured flipping ratio, R_n. This result becomes quite simple when the flipper efficiency can be assumed to be unity. This can be combined with

$$S^{nsf} + S^{sf} = \frac{1}{t_+} \frac{(2e_F - 1)C^{nsf} + C^{sf}}{e_F},$$

if it is necessary to extract the individual values S^{nsf} and S^{sf}.

The error analysis on s produces

$$\sigma_s^2 = W_R^2 \tilde{\sigma}_R^2 + W_{e_F}^2 \tilde{\sigma}_{e_F}^2 + W_C^2 \left(\tilde{\sigma}_{C^{nsf}}^2 + \tilde{\sigma}_{C^{sf}}^2\right)$$

where

$$W_R = s \frac{2e_F R_n}{[R_n - 1]\,[R_n\,(2e_F - 1) + 1]} \cong s \frac{2e_F R_n}{R_n^2 - 1}$$

$$W_{e_F} = s \frac{2e_F R_n}{[R_n\,(2e_F - 1) + 1]} - s \frac{e_F\,(1 + c)}{e_F - (1 - e_F)c} \cong s \frac{2e_F R_n}{R_n + 1} - s\,(1 + c)$$

$$W_C = 2s \frac{e_F}{e_F - (1 - e_F)c} \frac{C^{nsf} C^{sf}}{\left(C^{nsf}\right)^2 - \left(C^{sf}\right)^2} \cong 2 \frac{R_n + 1}{R_n - 1} \frac{C^{nsf} C^{sf}}{\left(C^{nsf} + C^{sf}\right)^2}.$$

Note that σ_R is just the error for the flipping ratio measurement, which simply depends on the count rates measured in obtaining R_n as shown in a following section.

4.3 all four cross-sections with perfect flippers

In the case that the spin flipper efficiencies are unity, the full transfer matrix can be solved algebraically for all four cross-sections when we can approximate that all count-rates are measured at the same time. Recall that the formula for the transfer matrix coefficients in

$$C^{\alpha\beta} = \frac{N}{2}\Sigma_{\mu\nu} c^{\alpha\beta}_{\mu\nu} S^{\mu\nu}$$

was

$$c^{\alpha\beta}_{\mu\nu} = \frac{1}{4}\left(t_{sA} - \mu e^{\alpha}_{A} t^{\alpha\beta}_{aA}\right)\left(t_{sP} - \nu e^{\beta}_{P} t^{\alpha\beta}_{aP}\right).$$

Also, $e^{\alpha}_{P,A} = \{e_t(2e^{\alpha}_{F} - 1)\}_{P,A}$ can be rewritten as

$$e^{\alpha}_{P,A} = -\alpha\left(1 - \epsilon^{\alpha}_{P,A}\right)$$

where the small transport inefficiency parameter, $\epsilon^{\alpha}_{P,A}$, is

$$\epsilon^{+} = 1 - e_t$$

and

$$\epsilon^{-} = 1 - e_{tF} = 1 - e_t\left(2e_F - 1\right)$$

If the spin-flipper efficiency is unity then $\epsilon^{+}_{P,A} = \epsilon^{-}_{P,A} = \epsilon_{P,A} = 1 - e_{tA,P}$ is independent of α so that $e^{\alpha}_{P,A} = -\alpha\left(1 - \epsilon_{P,A}\right) = -\alpha e_{tA,P}$. Then when count-rates are measured at the same time so that the transmission factors are independent of α and β,

$$4c^{\alpha\beta}_{\mu\nu} = \left[t_{sA} + \mu\alpha e_{tA} t_{aA}\right]\left[t_{sP} + \nu\beta e_{tP} t_{aP}\right],$$

so that μ and α appear only as their product, and the same for ν and β. This means

$$c^{\alpha\beta}_{\mu\nu} = c^{\bar{\alpha}\beta}_{\bar{\mu}\nu} = c^{\alpha\bar{\beta}}_{\mu\bar{\nu}} = c^{\bar{\alpha}\bar{\beta}}_{\bar{\mu}\bar{\nu}}$$

where $\bar{\alpha} = -\alpha$, so that there are only four distinct elements in the matrix. These four distinct elements can be generated by fixing $\mu = \nu = +1$, and they are

$$4c^{++} = \left[t_{sA} + e_{tA} t_{aA}\right]\left[t_{sP} + e_{tP} t_{aP}\right] = t_{+} + e_{tA} e_{tP} t_{-} + e_{tA} t_{A} + e_{tP} t_{P}$$

$$4c^{--} = \left[t_{sA} - e_{tA} t_{aA}\right]\left[t_{sP} - e_{tP} t_{aP}\right] = t_{+} + e_{tA} e_{tP} t_{-} - e_{tA} t_{A} - e_{tP} t_{P}$$

$$4c^{+-} = \left[t_{sA} + e_{tA} t_{aA}\right]\left[t_{sP} - e_{tP} t_{aP}\right] = t_{+} - e_{tA} e_{tP} t_{-} + e_{tA} t_{A} - e_{tP} t_{P}$$

$$4c^{-+} = [t_{sA} - e_{tA}t_{aA}]\,[t_{sP} + e_{tP}t_{aP}] = t_+ - e_{tA}e_{tP}t_- - e_{tA}t_A + e_{tP}t_P,$$

where

$$t_A = t_{aA}t_{sP} = (t_{+A} - t_{-A})\,(t_{+P} + t_{-P})$$

$$t_P = t_{sA}t_{aP} = (t_{+A} + t_{-A})\,(t_{+P} - t_{-P})\,.$$

Now form the symmetric two by two matrices

$$\mathbf{c}_n = \begin{pmatrix} c^{++} & c^{--} \\ c^{--} & c^{++} \end{pmatrix}$$

and

$$\mathbf{c}_f = \begin{pmatrix} c^{+-} & c^{-+} \\ c^{-+} & c^{+-} \end{pmatrix},$$

with the corresponding two component vectors

$$\mathbf{C}_n = \begin{pmatrix} C^{++} \\ C^{--} \end{pmatrix}$$

$$\mathbf{C}_f = \begin{pmatrix} C^{+-} \\ C^{-+} \end{pmatrix}$$

$$\mathbf{S}_n = \begin{pmatrix} S^{++} \\ S^{--} \end{pmatrix}$$

$$\mathbf{S}_f = \begin{pmatrix} S^{+-} \\ S^{-+} \end{pmatrix}$$

and the system of equations can then be written

$$\begin{pmatrix} \mathbf{C}_n \\ \mathbf{C}_f \end{pmatrix} = \frac{N}{2}\begin{pmatrix} \mathbf{c}_n & \mathbf{c}_f \\ \mathbf{c}_f & \mathbf{c}_n \end{pmatrix}\begin{pmatrix} \mathbf{S}_n \\ \mathbf{S}_f \end{pmatrix}.$$

Because the matrices $\mathbf{c}_n$ and $\mathbf{c}_f$ are symmetric and thus commute with each other, the inversion of this matrix problem is

$$\begin{pmatrix} \mathbf{c}_n & -\mathbf{c}_f \\ -\mathbf{c}_f & \mathbf{c}_n \end{pmatrix}\begin{pmatrix} \mathbf{C}_n \\ \mathbf{C}_f \end{pmatrix} = \frac{N}{2}\begin{pmatrix} \mathbf{c}_n^2 - \mathbf{c}_f^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_n^2 - \mathbf{c}_f^2 \end{pmatrix}\begin{pmatrix} \mathbf{S}_n \\ \mathbf{S}_f \end{pmatrix}$$

which can be checked by substituting the solution for $(\mathbf{C}_n, \mathbf{C}_f)$ from the previous equation. Letting $\mathbf{c}_n^2 - \mathbf{c}_f^2 = \mathbf{c}_d$, which is another symmetric matrix, the solution becomes

$$\left(\begin{array}{cc}\mathbf{s}_n & \mathbf{s}_f \\ \mathbf{s}_f & \mathbf{s}_n\end{array}\right)\left(\begin{array}{c}\mathbf{C}_n \\ \mathbf{C}_f\end{array}\right)=\left(\begin{array}{cc}\mathbf{c}_d^{-1}\mathbf{c}_n & -\mathbf{c}_d^{-1}\mathbf{c}_f \\ -\mathbf{c}_d^{-1}\mathbf{c}_f & \mathbf{c}_d^{-1}\mathbf{c}_n\end{array}\right)\left(\begin{array}{c}\mathbf{C}_n \\ \mathbf{C}_f\end{array}\right)=\frac{N}{2}\left(\begin{array}{c}\mathbf{S}_n \\ \mathbf{S}_f\end{array}\right).$$

With

$$\mathbf{c}_d=\left(\begin{array}{cc}e & f \\ f & e\end{array}\right),$$

$$e=\left(c^{++}\right)^2+\left(c^{--}\right)^2-\left(c^{+-}\right)^2-\left(c^{-+}\right)^2$$

$$f=2c^{++}c^{--}-2c^{+-}c^{-+}$$

$$\mathbf{c}_d^{-1}=\left(e^2-f^2\right)^{-1}\left(\begin{array}{cc}e & -f \\ -f & e\end{array}\right)$$

then

$$\mathbf{c}_d^{-1}\mathbf{c}_n=\mathbf{s}_n=\left(e^2-f^2\right)^{-1}\left(\begin{array}{cc}u & v \\ v & u\end{array}\right)$$

where

$$u=c^{++}\left[\left(c^{++}\right)^2-\left(c^{--}\right)^2-\left(c^{+-}\right)^2-\left(c^{-+}\right)^2\right]+2c^{--}c^{+-}c^{-+}$$

$$v=c^{--}\left[\left(c^{--}\right)^2-\left(c^{++}\right)^2-\left(c^{+-}\right)^2-\left(c^{-+}\right)^2\right]+2c^{++}c^{+-}c^{-+}.$$

Similarly,

$$-\mathbf{c}_d^{-1}\mathbf{c}_f=\mathbf{s}_f=\left(e^2-f^2\right)^{-1}\left(\begin{array}{cc}x & y \\ y & x\end{array}\right)$$

where

$$-x=c^{+-}\left[\left(c^{++}\right)^2+\left(c^{--}\right)^2-\left(c^{+-}\right)^2+\left(c^{-+}\right)^2\right]-2c^{-+}c^{++}c^{--}$$

$$-y=c^{-+}\left[\left(c^{++}\right)^2+\left(c^{--}\right)^2+\left(c^{+-}\right)^2-\left(c^{-+}\right)^2\right]-2c^{+-}c^{++}c^{--}$$

Calculation shows that

$$e^2-f^2=e_{tA}^2e_{tP}^2t_+t_-t_At_P$$

$$\frac{4}{e_t}u=\left(t_++e_tt_-\right)t_At_P+t_+t_-\left(e_{tA}t_A+e_{tP}t_P\right)$$

$$\frac{4}{e_t}v = (t_+ + e_t t_-)\, t_A t_P - t_+ t_- \left(e_{tA} t_A + e_{tP} t_P\right)$$

$$-\frac{4}{e_t}x = (t_+ - e_t t_-)\, t_A t_P + t_+ t_- \left(e_{tA} t_A - e_{tP} t_P\right)$$

$$-\frac{4}{e_t}y = (t_+ - e_t t_-)\, t_A t_P - t_+ t_- \left(e_{tA} t_A - e_{tP} t_P\right)$$

and as usual $e_t = e_{tA} e_{tP}$ is the aggregate beam transport efficiency. Thus the sub-matrix solutions are

$$\mathbf{s}_n = \frac{+1}{4e_t} \begin{pmatrix} a_+ + b_+ & a_+ - b_+ \\ a_+ - b_+ & a_+ + b_+ \end{pmatrix}$$

$$\mathbf{s}_f = \frac{-1}{4e_t} \begin{pmatrix} a_- + b_- & a_- - b_- \\ a_- - b_- & a_- + b_- \end{pmatrix}$$

where

$$a_\pm = \frac{t_+ \pm e_t t_-}{t_+ t_-} = \frac{1}{t_-} \pm \frac{e_t}{t_+}$$

$$b_\pm = \frac{e_{tA} t_A \pm e_{tP} t_P}{t_A t_P} = \frac{e_{tA}}{t_P} \pm \frac{e_{tP}}{t_A}.$$

If at this point it is found that $S^{++} = S^{--} = S^{nsf}$ and $S^{+-} = S^{-+} = S^{sf}$ then the pairs of equations in the solution can be added to reproduce the previous result in terms of average count-rates, but now with the flipper efficiency set to unity,

$$\frac{N}{2} \begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{1}{2e_t} \begin{pmatrix} a_+ & -a_- \\ -a_- & a_+ \end{pmatrix} \begin{pmatrix} \langle C^{nsf} \rangle \\ \langle C^{sf} \rangle \end{pmatrix}.$$

The system of equations for the general solution with flipper efficiency unity can now be written

$$N S^{\alpha\beta} = K^{\alpha\beta}_{\mu\nu} C^{\mu\nu},$$

where

$$K^{\alpha\beta}_{\mu\nu} = \frac{1}{2e_t} \nu\beta \left(\mu\alpha a_{(\mu\alpha\nu\beta)} + b_{(\mu\alpha\nu\beta)}\right).$$

The partial derivatives of t_+ and t_- are

$$\frac{\partial t_\pm}{\partial \tilde{\tau}_{P,A}} = t_\pm \left[P_{He3P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3} \right)_{P,A} - 1 \right]$$

$$\frac{\partial t_\pm}{\partial P_{He3P,A}} = t_\pm \left[\tilde{\tau}_{P,A} \tanh^{\pm 1} \left(\tilde{\tau} P_{He3} \right)_{P,A} \right].$$

The partial derivatives of t_A and t_P are

$$\frac{\partial t_X}{\partial \tilde{\tau}_Y} = t_X \left[P_{He3Y} \tanh^{-XY} \left(\tilde{\tau} P_{He3} \right)_Y - 1 \right]$$

$$\frac{\partial t_X}{\partial P_{He3Y}} = t_X \left[\tilde{\tau}_Y \tanh^{-XY} \left(\tilde{\tau} P_{He3} \right)_Y \right],$$

where $X = P, A$ and $Y = P, A$ and A is equivalent to -1, while P is equivalent to $+1$ for the purpose of calculating the tanh exponent. In order to do the error propagation divide $K^{\alpha\beta}_{\mu\nu}$ into the contributions from a and b to find

$$2K^{\alpha\beta}_{\mu\nu} = \frac{1}{t_+} + \frac{\nu\beta\mu\alpha}{e_{tA} e_{tP} t_-} + \frac{\nu\beta}{e_{tP} t_P} + \frac{\mu\alpha}{e_{tA} t_A}.$$

As before this takes on only four distinct values that can be generated by choosing $\mu = \nu = +1$, so that

$$K^{\alpha\beta}_{\mu\nu} = K^{\bar{\alpha}\beta}_{\bar{\mu}\nu} = K^{\alpha\bar{\beta}}_{\mu\bar{\nu}} = K^{\bar{\alpha}\bar{\beta}}_{\bar{\mu}\bar{\nu}}$$

and

$$K^{(\alpha)(\beta)}_{\mu\nu} = K^{(\alpha\mu)(\beta\nu)}_{++}$$

can be used to generate all the other coefficients. Then write

$$2K^{\alpha\beta}_{++} = A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta} + D^{\alpha\beta},$$

where

$$A^{\alpha\beta} = \frac{1}{t_+}$$

$$B^{\alpha\beta} = \frac{\beta\alpha}{e_{tA} e_{tP} t_-}$$

$$C^{\alpha\beta} = \frac{\beta}{e_{tP} t_P}$$

$$D^{\alpha\beta} = \frac{\alpha}{e_{tA}t_A}.$$

Then

$$4\sigma^2_{K^{\alpha\beta}_{++}} = \left(B^{\alpha\beta} + D^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_{tA}} + \left(B^{\alpha\beta} + C^{\alpha\beta}\right)^2 \tilde{\sigma}^2_{e_{tP}} + \sum_{Y=\tilde{\tau}_A,\tilde{\tau}_P,P_A,P_P} W_Y^2 \sigma_Y^2$$

where

$$W_Y = \sum_{X=A,B,C,D} X^{\alpha\beta} V_Y^X$$

and

$$V_Y^{X=A,B} = \bar{Y}\tanh^X\left(Y\bar{Y}\right) - \delta_{Y\tau}$$

$$V_Y^{X=C,D} = \bar{Y}\tanh^{-XY}\left(Y\bar{Y}\right) - \delta_{Y\tau}.$$

$\bar{Y}$ is the partner variable of Y in the pairs $\tilde{\tau}P_{He3}$. The A, B, C and D coefficients (X) are equivalent to $+1$, -1, $+1$ and -1 respectively, and the A and P subscripts (Y) of $\tilde{\tau}$ and P_{He3} are equivalent to -1 and $+1$, for the purposes of obtaining the exponent of tanh in these equations. From the equation for $NS^{\alpha\beta}$, the final error propagation is

$$\sigma^2_{NS^{\alpha\beta}} = \sum_{\mu\nu} \left(C^{\mu\nu}\right)^2 \sigma^2_{K^{\alpha\beta}_{\mu\nu}} + \left(K^{\alpha\beta}_{\mu\nu}\right)^2 \sigma^2_{C^{\alpha\beta}}.$$

4.4 single spin-flip cross-section only

Suppose that only $S^{+-} \neq 0$ or $S^{-+} \neq 0$, as might be the case for spin-wave scattering. Then the equations for count-rates as a function of the single cross-section S^{+-} are

$$C^{++}/\frac{N}{8} = \left(t_{sA} + e_{tA}t_{aA}\right)\left(t_{sP} - e_{tP}t_{aP}\right)S^{+-}$$

$$C^{--}/\frac{N}{8} = \left(t_{sA} - e_{tA}At_{aA}\right)\left(t_{sP} + e_{tP}Pt_{aP}\right)S^{+-}$$

$$C^{+-}/\frac{N}{8} = \left(t_{sA} + e_{tA}t_{aA}\right)\left(t_{sP} + e_{tP}Pt_{aP}\right)S^{+-}$$

$$C^{-+}/\frac{N}{8} = \left(t_{sA} - e_{tA}At_{aA}\right)\left(t_{sP} - e_{tP}t_{aP}\right)S^{+-},$$

where as usual $A = 2e_{FA} - 1$ and $P = 2e_{FP} - 1$. Then

$$r_{SW}^{+} = \frac{C^{+-}}{C^{++}} = \frac{1 + e_{tP} P t_{aP}/t_{sP}}{1 - e_{tP} t_{aP}/t_{sP}} = \frac{C^{--}}{C^{-+}}.$$

This could be used to extract the polarizer transmission factor ratio t_{aP}/t_{sP} as

$$e_{tP} \frac{t_{aP}}{t_{sP}} = \frac{r_{SW}^{+} - 1}{r_{SW}^{+} + P},$$

where

$$\frac{t_{aP}}{t_{sP}} = \frac{C_{aP}}{C_{sP}} \tanh\left(\tilde{\tau}_P P_{He3P}\right).$$

Once the tanh is calculated, then sinh and coshcan be separately extracted to obtain the individual transmission factors t_{aP} and t_{sP},

$$\cosh\left(\tilde{\tau}_P P_{He3P}\right) = \frac{1}{\sqrt{1 - \tanh^2\left(\tilde{\tau}_P P_{He3P}\right)}}$$

$$\sinh\left(\tilde{\tau}_P P_{He3P}\right) = \frac{\tanh\left(\tilde{\tau}_P P_{He3P}\right)}{\sqrt{1 - \tanh^2\left(\tilde{\tau}_P P_{He3P}\right)}}$$

$$t_{aP} = 2t_{EP} C_{aP} \exp\left(-\tilde{\tau}_P\right) \sinh\left(\tilde{\tau}_P P_{He3P}\right)$$

$$t_{sP} = 2t_{EP} C_{sP} \exp\left(-\tilde{\tau}_P\right) \cosh\left(\tilde{\tau}_P P_{He3P}\right).$$

Similarly, if $S^{-+}$is the only non-zero cross-section contributing, then

$$r_{SW}^{-} = \frac{C^{-+}}{C^{++}} = \frac{1 + e_{tA} A t_{aA}/t_{sA}}{1 - e_{tA} t_{aA}/t_{sA}} = \frac{C^{--}}{C^{+-}}.$$

This could be used to extract the analyzer transmission factor ratio t_{aA}/t_{sA} as

$$e_{tP} \frac{t_{aA}}{t_{sA}} = \frac{r_{SW}^{-} - 1}{r_{SW}^{-} + A},$$

where

$$\frac{t_{aA}}{t_{sA}} = \frac{C_{aA}}{C_{sA}} \tanh\left(\tilde{\tau}_A P_{He3A}\right).$$

It should be noted that if $S^{+-}$and $S^{-+}$are from inelastic cross-sections (spin-waves) then the count rates may be too low to make this analysis possible. One must then rely on separate transmission measurements of the He-3 cells along with any correction for time dependence to obtain the individual transmission factors necessary to extract the cross-sections from the count rates.

4.5 both spin-flip cross-sections contribute

Suppose that only $S^{+-} \neq 0$ and $S^{-+} \neq 0$. Recall that the formula for the transfer matrix coefficients in

$$C^{\alpha\beta} = \frac{N}{2}\Sigma_{\mu\nu}c^{\alpha\beta}_{\mu\nu}S^{\mu\nu}$$

was in equal time approximation

$$c^{\alpha\beta}_{\mu\nu} = \frac{1}{4}\left(t_{sA} - \mu e^{\alpha}_{A}t_{aA}\right)\left(t_{sP} - \nu e^{\beta}_{P}t_{aP}\right),$$

where $e^{\alpha}_{P,A} = \{e_t(2e^{\alpha}_{F} - 1)\}_{P,A}$. Then the equations for count-rates as a function of the two cross-section S^{+-} andS^{-+} are

$$C^{++}/\frac{N}{8} = \left(t_{sA} + e_{tA}t_{aA}\right)\left(t_{sP} - e_{tP}t_{aP}\right)S^{+-}+\left(t_{sA} - e_{tA}t_{aA}\right)\left(t_{sP} + e_{tP}t_{aP}\right)S^{-+}$$

$$C^{--}/\frac{N}{8} = \left(t_{sA} - e_{tA}At_{aA}\right)\left(t_{sP} + e_{tP}Pt_{aP}\right)S^{+-}+\left(t_{sA} + e_{tA}At_{aA}\right)\left(t_{sP} - e_{tP}Pt_{aP}\right)S^{-+}$$

$$C^{+-}/\frac{N}{8} = \left(t_{sA} + e_{tA}t_{aA}\right)\left(t_{sP} + e_{tP}Pt_{aP}\right)S^{+-}+\left(t_{sA} - e_{tA}t_{aA}\right)\left(t_{sP} - e_{tP}Pt_{aP}\right)S^{-+}$$

$$C^{-+}/\frac{N}{8} = \left(t_{sA} - e_{tA}At_{aA}\right)\left(t_{sP} - e_{tP}t_{aP}\right)S^{+-}+\left(t_{sA} + e_{tA}At_{aA}\right)\left(t_{sP} + e_{tP}t_{aP}\right)S^{-+},$$

where as usual $A = 2e_{FA} - 1$ and $P = 2e_{FP} - 1$. It is easy to invert a pair of these equations to solve for S^{+-} and S^{-+} in terms of the count rates.

Assume the common case that the flippers are perfect. Then the coefficients have the symmetry $c^{\alpha\beta}_{\mu\nu} = c^{\beta\alpha}_{\nu\mu}$. Then in each pair of equations above there are only two different coefficients. For the last pair of equations the two coefficients are

$$c^{-+}_{+-} = c^{+-}_{-+} = \frac{1}{4}(t_{sA} - e_{tA}t_{aA})(t_{sP} - e_{tP}t_{aP})$$

$$c^{+-}_{+-} = c^{-+}_{-+} = \frac{1}{4}(t_{sA} + e_{tA}t_{aA})(t_{sP} + e_{tP}t_{aP})$$

As in the case of sf and nsf scattering only, we can define the cross-section asymmetry, $-1 \leq s \leq 1$, as

$$s = \frac{S^{+-} - S^{-+}}{S^{+-} + S^{-+}}$$

and the count-rate asymmetry, $-1 \leq c \leq 1$, as

$$c = \frac{C^{+-} - C^{-+}}{C^{+-} + C^{-+}},$$

$$(1 - c)\, C^{+-} = (1 + c)\, C^{-+},$$

$$(1 - s)\, S^{+-} = (1 + s)\, S^{-+}.$$

Recall that in the case of sf and nsf scattering only the result for the cross-section asymmetry, s, in terms of the count-rate asymmetry, c, for perfect flipper was

$$s = \frac{R+1}{R-1} c,$$

where R was the pure Bragg peak nsf flipping ratio, which again for perfect flipper is

$$R = \frac{t_{sA} t_{sP} + e_t t_{aA} t_{aP}}{t_{sA} t_{sP} - e_t t_{aA} t_{aP}}.$$

In the present case, using the equations for C^{+-} and C^{-+} it can be shown that

$$\frac{s/c+1}{s/c-1} = \frac{c_{+-}^{+-}}{c_{+-}^{-+}} = \frac{(t_{sA} + e_{tA} t_{aA})(t_{sP} + e_{tP} t_{aP})}{(t_{sA} - e_{tA} t_{aA})(t_{sP} - e_{tP} t_{aP})} = \frac{t_{sA} t_{sP} + e_t t_{aA} t_{aP} + e_{tA} t_{aA} t_{sP} + e_{tP} t_{sA} t_{aP}}{t_{sA} t_{sP} + e_t t_{aA} t_{aP} - e_{tA} t_{aA} t_{sP} - e_{tP} t_{sA} t_{aP}}.$$

This can be rewritten in a similar form as before,

$$s = \frac{R_{sf}+1}{R_{sf}-1} c$$

where

$$R_{sf} = \frac{c_{+-}^{+-}}{c_{+-}^{-+}}.$$

Note that typically $R_{sf} \gg R$ as can be seen by inspection of terms in the ratio c_{+-}^{+-}/c_{+-}^{-+}. In terms of the He-3 parameters

$$R_{sf} \cong \frac{1 + e_{tA} \tanh(\tilde{\tau}_A P_{He3A})}{1 - e_{tA} \tanh(\tilde{\tau}_A P_{He3A})} \frac{1 + e_{tP} \tanh(\tilde{\tau}_P P_{He3P})}{1 - e_{tP} \tanh(\tilde{\tau}_P P_{He3P})}.$$

For comparison the equivalent expression for R (with $e_t = e_{tA} e_{tP}$) is

$$R \cong \frac{1 + e_t \tanh(\tilde{\tau}_A P_{He3A}) \tanh(\tilde{\tau}_P P_{He3P})}{1 - e_t \tanh(\tilde{\tau}_A P_{He3A}) \tanh(\tilde{\tau}_P P_{He3P})}.$$

For example if $e_{tA} \tanh(\tilde{\tau}_A P_{He3A}) = e_{tP} \tanh(\tilde{\tau}_P P_{He3P}) = 0.95$ then $R = 19.5$ and $R_{sf} = 1521$. Or if the terms are 0.9 then $R = 9.52$ and $R_{sf} = 361$. Thus, for sf only scattering $s = c$ is typically a good approximation.

4.6 saturated ferromagnet

The case of $S^{+-} = S^{-+} = 0$ and $S^{++} \neq S^{--}$ is treated in a following section

5 inelastic scattering

All of the analysis can be applied to inelastic scattering by simply scaling $\tilde{\tau}$ by λ/λ_0, where λ is the actual nominal wavelength of the incoming or scattered neutrons, and $\lambda_0$is the wavelength at which $\tilde{\tau}$ was originally calculated. $\tilde{\tau}$ occurs in all the He-3 transmission factors and also in the correction coefficients for the He-3 transmission.

6 flipping ratios and efficiency measurements

6.1 spin-flip and non-spin-flip cross-sections only

In order to examine the performance of a polarized beam setup, it is required that the cross-sections be known and fairly simple. One useful case is where $S^{++} = S^{--} = S^{nsf}$ and $S^{+-} = S^{-+} = S^{sf}$, so that the cross-section asymmetry can be defined as

$$s = \frac{S^{nsf} - S^{sf}}{S^{nsf} + S^{sf}}.$$

Note that in this case where the scattering matrix commutes with the transport and flipper matrices, e_A and e_P only appear as the product $e_A e_P$, and there is no way to separate the effects of the transport efficiency before the sample from the transport efficiency after the sample. To perform that separation would require $S^{++} \neq S^{--}$ or $S^{+-} \neq S^{-+}$. Examples of the S^{nsf}, S^{sf} case are pure non-spin-flip scattering, $s = 1$, pure spin-flip scattering, $s = -1$ and spin-incoherent scattering, $s = -1/3$. These cross-sections should be free of multiple scattering and produce count rates that are in the linear range of the detector electronics. Then expressions for the flipping ratios using the polarizer flipper or analyzer flipper can be used to determine transport and flipping efficiencies. These flipping ratios are given by $R_{P,A}(s) = CountsFlipperOFF^{++}/CountsFlipperON^{+-,-+}$. Thus

$$R_{P,A}(s) = \frac{t_+ + e_t s\, t_-}{t_+ - e_t(2e_{FP,A} - 1)s\, t_-} = \frac{1 + e_t s\, t_-/t_+}{1 - e_t(2e_{FP,A} - 1)s\, t_-/t_+}$$

Recalling the expressions for t_+ and t_-, the ratio t_-/t_+ is

$$\frac{t_-}{t_+} = P_n = \frac{C_{-\Delta}}{C_{+\Delta}} \tanh\left(\tilde{\tau}_P P_{He3P}\right) \tanh\left(\tilde{\tau}_A P_{He3A}\right) = \frac{R_{0,nsf} - 1}{R_{0,nsf} + 1}$$

($R_{0,nsf}$ is defined in the following) is approximately the product of the neutron polarizations produced by the two He3 cells. Here $e_{tF} = e_t(2e_F - 1)$ depends on the product of the transport and flipper efficiencies. If the transport and flipper efficiencies are unity then the expected flipping ratios are

$$\begin{aligned} R_0(s &= 1) = R_{0,nsf} = \frac{1+t_-/t_+}{1-t_-/t_+} = \frac{t_{++}}{t_{+-}} = \frac{t_{+A}t_{+P}+t_{-A}t_{-P}}{t_{+A}t_{-P}+t_{-A}t_{+P}} \\ &= \frac{\hat{C}_{+\Delta}}{\hat{C}_{-\Delta}} \frac{\cosh\left(\tilde{\tau}_A P_{He3A}+\tilde{\tau}_P P_{He3P}\right)}{\cosh\left(\tilde{\tau}_A P_{He3A}-\tilde{\tau}_P P_{He3P}\right)} \end{aligned}$$

$$\hat{C}_{\pm\Delta} = 1+\left(\frac{\Delta_A}{\langle C_A\rangle} \pm \frac{\Delta_P}{\langle C_P\rangle}\right)\tanh\left(\tilde{\tau}_A P_{He3A} \pm \tilde{\tau}_P P_{He3P}\right)$$

$$R_0(s=-1) = R_{0,sf} = \frac{1}{R_0(1)}$$

$$R_0(s=-1/3) = R_{0,inc} = \frac{R_0(1)+2}{2R_0(1)+1}.$$

If the flipping efficiency is unity then the expected flipping ratios are

$$R(e_F=1,s) = \frac{1+e_t s\, t_-/t_+}{1-e_t s\, t_-/t_+},$$

and in particular

$$R(e_F=1,s=1) = \frac{1+e_t t_-/t_+}{1-e_t t_-/t_+} = \frac{R_{0,nsf}+\epsilon_t}{\epsilon_t R_{0,nsf}+1}$$

where $\epsilon_t = (1-e_t)\,/\,(1+e_t)$ is the transport loss. Thus, when the flipping efficiency is assumed to be unity, then the transport efficiency can be determined as

$$e_t = \frac{1}{s}\left(\frac{R(s)-1}{R(s)+1}\right)\frac{t_+}{t_-} = \frac{1}{s}\frac{R_{0,nsf}+1}{R_{0,nsf}-1}\left(\frac{R(s)-1}{R(s)+1}\right).$$

or for nsf scattering the transport loss is given by

$$\epsilon_t = \frac{R_{0,nsf}-R_{nsf}}{R_{0,nsf}R_{nsf}-1} \cong \frac{1}{R_{nsf}} - \frac{1}{R_{0,nsf}}$$

If both transport and flipping efficiencies are unknown then they cannot be determined separately by a single flipping ratio measurement. One of the efficiencies can be found in terms of the other for a single flipping ratio measurement as

$$e_t = \frac{1}{s}\frac{R(s)-1}{R(s)\left(2e_{FP,A}-1\right)+1}\frac{t_+}{t_-} \tag{21}$$

6.1.1 using two different cross-section asymmetries to measure efficiencies

One way to uniquely determine the efficiencies is to make flipping ratio measurements for two different types of cross-sections (different known s values). Then

$$e_t = \frac{f(s_1, s_2)}{s_1 s_2 \left[R(s_1) - R(s_2)\right]} \frac{t_+}{t_-}$$

and

$$2e_F - 1 = \frac{s_2 R(s_1) - s_1 R(s_2) + (s_1 - s_2)}{f(s_1, s_2)}$$

where

$$f(s_1, s_2) = R(s_1)R(s_2)(s_1 - s_2) + s_2 R(s_2) - s_1 R(s_1).$$

Another measurement that can be made is the ratio of observed counts when both flippers are OFF to when both flippers are ON. This yields

$$R_{P+A}(s) = \frac{t_+ + e_t s\, t_-}{t_+ + e_t(2e_{FA} - 1)(2e_{FP} - 1)s\, t_-}$$

If $R_{P+A}(s) \equiv 1$, and it can be assumed that the transport is the same for either flipper state, then this is a good indication that the flipper efficiencies are unity (Note that $R_{P+A}(s) \equiv 1$ if the flipper efficiency is zero also). In general it is expected that this flipping ratio is near unity. By measuring both $R_{P+A}(s)$ and $R_{P,A}(s)$ two equations are generated but the product of the two flipper efficiencies appears in one of the equations. If it can be assumed that the flipper efficiencies are equal (as might be suggested if $R_P(s) = R_A(s)$) then a quadratic equation can be found for the flipper efficiency,

$$\left(1 - \frac{1}{R_{P,A}}\right) X^2 + \left(1 - \frac{1}{R_{P+A}}\right) X - \left(\frac{1}{R_{P+A}} - \frac{1}{R_{P,A}}\right) = 0$$

where $X = 2e_F - 1$. Because $R_{P+A}(s)$ is near unity and $R_{P,A}$is not, an approximate solution is

$$X = 2e_F - 1 \cong 1 - \frac{R_{P+A} - 1}{1 - 1/R_{P,A}}$$

or

$$e_F \cong 1 - \frac{1}{2} R_{P,A} \frac{R_{P+A} - 1}{R_{P,A} - 1}$$

This solution for the flipper efficiency can then be used to solve for the transport efficiency21.

6.1.2 using polarizer and analyzer flippers to measure efficiencies

More commonly when both polarizer and analyzer spin flippers are available, the efficiencies can be determined by measuring all four polarized beam cross-sections and the He-3 cell transmissions. This is usually done with pure non-spin-flip scattering, although for any cross-section asymmetry, s, the observed counts for the four cross-sections are in equal time approximation

$$C^{++} = K(1 + se_t t_-/t_+)$$

$$C^{+-} = K(1 - se_t P t_-/t_+)$$

$$C^{-+} = K(1 - se_t A t_-/t_+)$$

$$C^{--} = K(1 + se_t A P t_-/t_+)$$

where K is some proportionality constant, $e_t = e_{tA}e_{tP}$ is the total transport efficiency, $P = 2e_{FP} - 1$ and $A = 2e_{FA} - 1$. Note that if $S^{sf} = S^{nsf}$, then the counts for all four polarized beam cross-sections are identical and independent of beam transport and flipping efficiencies. For $s \neq 0$ it is easy to show that

$$P = 2e_{FP} - 1 = \frac{C^{--} - C^{+-}}{C^{++} - C^{-+}}$$

$$A = 2e_{FA} - 1 = \frac{C^{--} - C^{-+}}{C^{++} - C^{+-}}.$$

The error propagation produces

$$\tilde{\sigma}_P^2 = \frac{\sigma_P^2}{P^2} = \frac{\sigma_{C^{++}}^2 + \sigma_{C^{-+}}^2}{\left(C^{++} - C^{-+}\right)^2} + \frac{\sigma_{C^{--}}^2 + \sigma_{C^{+-}}^2}{\left(C^{--} - C^{+-}\right)^2}$$

$$\tilde{\sigma}_A^2 = \frac{\sigma_A^2}{A^2} = \frac{\sigma_{C^{++}}^2 + \sigma_{C^{+-}}^2}{\left(C^{++} - C^{+-}\right)^2} + \frac{\sigma_{C^{--}}^2 + \sigma_{C^{-+}}^2}{\left(C^{--} - C^{-+}\right)^2}.$$

and where $\sigma_{e_F} = \frac{1}{2}\sigma_{P,A}$.

If one of the flipper efficiencies is known to much greater accuracy than the other, then the unknown flipper efficiency can be found by measuring just 3 cross-sections. For example, suppose $P \neq 0$ is known to some accuracy σ_P. Then

$$P\left(C^{++} - C^{-+}\right) = C^{--} - C^{+-}$$

$$C^{++} - \left(C^{--} - C^{+-}\right)/P = C^{-+}$$

$$A = 2e_{FA} - 1 = \frac{C^{--} - C^{++} - \left(C^{--} - C^{+-}\right)/P}{C^{++} - C^{+-}}.$$

$$A = \frac{C^{--}\left(1 - 1/P\right) - C^{++} + C^{+-}/P}{C^{++} - C^{+-}}.$$

The transport efficiency can also be obtained from

$$se_t \frac{t_-}{t_+} = \frac{C^{++} - C^{+-}}{PC^{++} + C^{+-}} = \frac{C^{++} - C^{-+}}{AC^{++} + C^{-+}} = \frac{\left(C^{++} - C^{+-}\right)\left(C^{++} - C^{-+}\right)}{C^{++}C^{--} - C^{+-}C^{-+}},$$

which is symbolically

$$e_t = \frac{1}{sP_n} \frac{N^{+-}N^{-+}}{D}.$$

The transport efficiency can only be obtained as a function of the cross-section asymmetry and the He-3 transmission factor, t_-/t_+, where

$$P_n = \frac{t_-}{t_+} = \frac{C_{-\Delta}}{C_{+\Delta}} \tanh\left(\tilde{\tau}_A P_{He3A}\right) \tanh\left(\tilde{\tau}_P P_{He3P}\right)$$

and the correction coefficient is

$$\frac{C_{-\Delta}}{C_{+\Delta}} = C_R \cong 1 + \sum_{m=P,A} \frac{\Delta_m}{\langle C_m \rangle} \frac{2}{\sinh\left(2\tilde{\tau_m} P_{He3m}\right)}.$$

The error propagation for the beam transport efficiency measurement thus depends on uncertainties in $\tilde{\tau}$ and P_{He3}, in addition to the uncertainties in the measured count-rates, and is then given by

$$\tilde{\sigma}_{e_t}^2 = \sum_{\alpha\beta} W_{\alpha\beta}^2 \sigma_{C^{\alpha\beta}}^2 + \sum_{m=P,A} W_m^2 \left(\tilde{\sigma}_{P_{He}}^2 + \tilde{\sigma}_{\tilde{\tau}}^2\right)_m$$

where

$$W_{++} = \frac{1}{N^{+-}} + \frac{1}{N^{-+}} - \frac{C^{--}}{D}$$

$$W_{--} = \frac{-C^{++}}{D}$$

$$W_{+-} = \frac{-1}{N^{+-}} + \frac{C^{-+}}{D}$$

$$W_{-+} = \frac{-1}{N^{-+}} + \frac{C^{+-}}{D}$$

$$W_{m=P,A} = \left[\frac{2\tilde{\tau} P_{He3}}{\sinh\left(2\tilde{\tau} P_{He3}\right)}\right]_m.$$

See section 3.1 for an explanation of the calculation of the errors $\tilde{\sigma}_{P_{He}}^2$ and $\tilde{\sigma}_{\tilde{\tau}}^2$.

6.1.3 checking He-3 cell polarization during an experiment

Up to this point it has been assumed that the full time dependence of the He-3 cell transmissions must be taken into account, especially in the error propagation. If a flipping ratio measurement is available at the approximate time of data collection, then the uncertainty produced by the time dependences can be reduced. This procedure will work for elastic-scattering data, since the flipping ratio is measured under elastic scattering conditions (although, as shown later, it is possible to apply this procedure to inelastic data by making a wavelength dependence correction to the flipping ratio). To this end, rewrite the correction formulae in terms of the flipping ratio measured at the same time as the data. Here it is assumed that $C^{nsf} = C^{++}$and the spin-flip counts can be collected with either flipper. The expression for the non-spin-flip flipping ratio (there is absolutely no spin-flip scattering), that is $R_{nsf} = C^{++}/C^{+-}$, is

$$R_{nsf} = \frac{t_+ + e_t t_-}{t_+ - e_t B t_-} = \frac{1 + e_t t_-/t_+}{1 - e_t B t_-/t_+}.$$

where $B = 2e_F - 1$. Solving for t_-/t_+ in terms of R_{nsf}

$$\frac{t_-}{t_+} = \frac{1}{e_t} \frac{R_{nsf} - 1}{R_{nsf} B + 1}.$$

If the time dependence of t_-/t_+ is known, then it can be checked against a measurement of R_{nsf}. However, it may turn out that measurements of R_{nsf} are not clean. That is there may be spin-flip contributions to the cross-section. This is often a background contamination from magnetic inhomogeneity or from hydrogen spin-incoherent scattering. Then, by measuring the spin-flip and non-spin-flip count rates in the background, one can make a reasonable correction to the flipping ratio as shown in the following.

Now the cross-section solutions for spin-flip and non-spin-flip can be written in terms of R_{nsf},

$$S^{nsf} N K_e = \left(\frac{1 + e_{tF} t_-/t_+}{1 + e_t t_-/t_+}\right) C^{nsf} - \frac{1}{R_{nsf}} \left(\frac{1 - e_t t_-/t_+}{1 - e_{tF} t_-/t_+}\right) C^{sf} \qquad (22)$$

or

$$S^{nsf} N K_e = \left[\left(2 - \frac{1}{e_F}\right) + \frac{1}{R_{nsf}} \left(\frac{1}{e_F} - 1\right)\right] C^{nsf} - \left[\frac{1}{R_{nsf}} \frac{1}{e_F} - \left(\frac{1}{e_F} - 1\right)\right] C^{sf}.$$

Separate out the flipper efficiency dependence by using $1/e_F = 1 + (1 - e_F)/e_F$, to find

$$S^{nsf} N K_e = C^{nsf} - \frac{1}{R_{nsf}} C^{sf} - \epsilon \left(C^{nsf} - C^{sf}\right),$$

where

$$\epsilon = \left(\frac{1-e_F}{e_F}\right)\left(\frac{R_{nsf}-1}{R_{nsf}}\right).$$

The formula for the spin-flip cross-section in terms of the pure non-spin-flip flipping ratio is

$$S^{sf} N K_e = C^{sf} - \frac{1}{R_{nsf}} C^{nsf}, \tag{23}$$

and the formula for S^{nsf} takes this same simple form when the flipping efficiency is unity.

If the flipping-ratio is going to be used to track P_{He3}, the measurement is simplest using a Bragg-peak with no spin-flip scattering. If there is spin-flip scattering, a case that is straightforward to treat is that the spin-flip scattering appears as a flat background, as for spin-incoherent scattering (hydrogen) or random magnetic impurities. Then one can measure a flipping-ratio at the Bragg-peak and off the Bragg-peak far enough to be in the background. These two measurements yield

$$S^{sf}_{Bragg} N K_e = C^{sf}_{Bragg} - \frac{1}{R_{nsf}} C^{nsf}_{Bragg}$$

$$S^{sf}_{bg} N K_e = C^{sf}_{bg} - \frac{1}{R_{nsf}} C^{nsf}_{bg}$$

Now if $S^{sf}_{Bragg} = S^{sf}_{bg}$ (flat background) then we can extract the non-spin-flip flipping-ratio as

$$R_{nsf} = \left(C^{nsf}_{Bragg} - C^{nsf}_{bg}\right) / \left(C^{sf}_{Bragg} - C^{sf}_{bg}\right)$$

This formula corrects for any flat background including any fast neutron background, which clearly gets cancelled by the subtractions in the formula. Thus it is often useful to make the flipping ratio measurement on a Bragg peak and in the background.

The formula for observed counts applies to the case that the neutron wavelength is $\lambda/2$, with the caveat that if a current-flipper is used it will depolarize the beam (if it is set to flip λ) so that $e_F = 1/2$, $B = 2e_F - 1 = 0$.

$$\begin{pmatrix} C^{nsf} \\ C^{sf} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\begin{pmatrix} t_+ + e_t t_- & t_+ - e_t t_- \\ t_+ - e_t B t_- & t_+ + e_t B t_- \end{pmatrix}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\mathbf{M}\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix}$$

If the beam is mixed with λ fraction a_1 and $\lambda/2$ fraction a_2 reaching the detector, and the cross-sections for λ and $\lambda/2$ are equal, then we can write equations for the combined count-rate as

$$\begin{pmatrix} C^{nsf} \\ C^{sf} \end{pmatrix} = \frac{N}{2}\frac{1}{2}\left(a_1 \mathbf{M}_\lambda + a_2 \mathbf{M}_{\lambda/2}\right)\begin{pmatrix} S^{nsf} \\ S^{sf} \end{pmatrix}$$

In the case that a He3-flipper is used, the combined matrices have the same form as above, so that we can simply replace t_+ and t_- in the solution,

$$t_+ \to \hat{t}_+ = a_1 t_{+\lambda} + a_2 t_{+\lambda/2}$$

$$t_- \to \hat{t}_- = a_1 t_{-\lambda} + a_2 t_{-\lambda/2}$$

In the case that a current-flipper is used, we must also replace B in the solution,

$$B \to \hat{B} = B a_1 t_{-\lambda}/\hat{t}_-$$

Thus we have shown that flipping ratio measurements can be used to track the time-dependence of the He3-cell transmissions, even when there is background or $\lambda/2$ contamination. If the user is going to run the experiment with $\lambda/2$ contamination, it helps to build confidence if the relation between R_{nsf} and the transmissions is checked by comparing the clean beam case to the $\lambda/2$ contamination case. This comparison can be used to make sure that a_1 and a_2 make sense.

The best way to keep track of the polarization of the He-3 cells is to use beam monitors as shown in the diagram at the start of this document, and measure the transmissions as a function of time. If this is not possible, measurements of the non-spin-flip flipping ratio can be used to monitor the polarized beam performance. Also, as will be shown in the following section, these flipping ratio measurements aid in correcting polarized beam data. Typically, previously measured values of transport and flipping efficiencies are assumed to remain in effect, and the flipping ratio measurement is used to check on the expected polarizing efficiency of the He-3 cells. The solution for the polarizing efficiency, P_n, in terms of the measured non-spin-flip flipping ratio and the transport efficiencies is

$$P_n = \frac{t_-}{t_+} = \frac{1}{e_t} \frac{R_{nsf} - 1}{R_{nsf}(2e_F - 1) + 1}.$$

The error propagation for measuring this polarizing efficiency int terms of the flipping ratio is

$$\tilde{\sigma}^2_{P_n} = \tilde{\sigma}^2_{e_t} + \left(\frac{2e_F R_{nsf}}{R_{nsf}(2e_F - 1) + 1} \right)^2 \tilde{\sigma}^2_{e_F} + \left(\frac{2e_F R^2_{nsf}}{[R_{nsf} - 1]\,[R_{nsf}(2e_F - 1) + 1]} \right)^2 \tilde{\sigma}^2_{R_{nsf}}$$

where

$$\tilde{\sigma}^2_{R_{nsf}} = \tilde{\sigma}^2_{Cnsf} + \tilde{\sigma}^2_{Csf},$$

and C_{nsf} and C_{sf} are the count rates that determine the flipping ratio. Recall that the expected value of P_n is

$$\bar{P}_n = \frac{t_-}{t_+} = C_R \tanh\left(\tilde{\tau}_A P_{He3A}\right) \tanh\left(\tilde{\tau}_P P_{He3P}\right),$$

and the error propagation for this expected value was calculated previously as

$$\tilde{\sigma}^2_{\bar{P}_n} = \sum_{m=P,A} \left[\frac{2\tilde{\tau}P_{He3}}{\sinh\left(2\tilde{\tau}P_{He3}\right)}\right]^2_m \left(\tilde{\sigma}^2_{P_{He}} + \tilde{\sigma}^2_{\tilde{\tau}}\right)_m .$$

The values for $\tilde{\sigma}^2_{P_{He}}(t)$ and $\tilde{\sigma}^2_{\tilde{\tau}}$ are given in section 3.1.

Of course if there is already confidence in the expected value of $\bar{P}_n$, then the flipping ratio measurement can be used to check on the transport efficiency.

A beam monitor after the He3 polarizer can be used to help track its polarization. The monitor rate as a function of time, due to the decay of the He3 polarization can be written as,

$$m(E,s) = M(E)\sum_n \frac{1}{n}a_n(E)\frac{1}{2}\left[t_{P+\lambda/n}(s) + t_{P-\lambda/n}(s)\right]/\sum_n \frac{1}{n}a_n(E)$$

Here s is the time in seconds, $M(E)$ is the energy dependent monitor rate with no cell in the beam, and $t_{P\pm\lambda/n}(s)$ are the transmissions of the He3 polarizer for preferred and non-preferred states at the given wavelength order and time. Normally when cell transmissions are measured we calculated $M(E)$ based on expected order fractions, and then compare with the measured $M(E)$ with the cell out of the beam. The He3 polarization is known at start time s_i, so that when we measure a monitor-rate at later time, s_f, then

$$\frac{m(E,s_f)}{M(E)} = \sum_n \frac{1}{n}a_n(E)\frac{1}{2}\left[t_{P+\lambda/n}(s_f) + t_{P-\lambda/n}(s_f)\right]/\sum_n \frac{1}{n}a_n(E)$$

This allows a calculation of the He3 polarization at time s_f.

6.2 saturated ferromagnet

Another set of cross-sections that can be useful in characterizing a polarized beam setup, has the conditions that $S^{+-} = S^{-+} = S^{sf} = 0$ and $S^{++} \neq S^{--}$. For example, these cross-sections apply to a saturated ferromagnet Bragg peak. It is important that complete saturation is reached, otherwise there will be contributions from spin-flip scattering or beam depolarization from ferromagnetic domains. In this case the cross terms in the expression for the transfer matrix elements do not cancel. This cancellation had simplified these matrix elements in the case of spin-flip and non-spin-flip scattering symmetry, so that there was no dependence on solely the pre-sample or post-sample side of the beam path transport. Breaking this symmetry complicates the expressions, but does allow extraction of the separate beam transport efficiencies. The expressions for the expected count-rates are

$$C^{\alpha\beta}/\frac{N}{2} = c^{\alpha\beta}_{++}S^{++} + c^{\alpha\beta}_{--}S^{--}.$$

Explicitly writing these out

$$C^{++}/\frac{N}{8} = (t_{sA} + e_{tA}t_{aA})(t_{sP} + e_{tP}t_{aP})\,S^{++} + (t_{sA} - e_{tA}t_{aA})(t_{sP} - e_{tP}t_{aP})\,S^{--}$$

$$C^{--}/\frac{N}{8} = (t_{sA} - e_{tA}At_{aA})(t_{sP} - e_{tP}Pt_{aP})\,S^{++} + (t_{sA} + e_{tA}At_{aA})(t_{sP} + e_{tP}Pt_{aP})\,S^{--}$$

$$C^{+-}/\frac{N}{8} = (t_{sA} + e_{tA}t_{aA})(t_{sP} - e_{tP}Pt_{aP})\,S^{++} + (t_{sA} - e_{tA}t_{aA})(t_{sP} + e_{tP}Pt_{aP})\,S^{--}$$

$$C^{-+}/\frac{N}{8} = (t_{sA} - e_{tA}At_{aA})(t_{sP} + e_{tP}t_{aP})\,S^{++} + (t_{sA} + e_{tA}At_{aA})(t_{sP} - e_{tP}t_{aP})\,S^{--},$$

where recall that $A = 2e_{FA} - 1$ and $P = 2e_{FP} - 1$. Now define the following combinations of count rates,

$$d_P = C^{++} - C^{+-}$$

$$d_A = C^{--} - C^{-+}$$

$$s_P = PC^{++} + C^{+-}$$

$$s_A = C^{--} + PC^{-+},$$

and also the cross-section asymmetry,

$$s = \left(\frac{S^{++} - S^{--}}{S^{++} + S^{--}}\right).$$

Then if $P > 0$ (else the expected count-rate differences d_P and $d_A$would be zero) and $A > 0$ (else $d_P + d_A = 0$) and $e_{tA} > 0$,

$$\frac{1}{e_{tA}}\frac{t_{sA}}{t_{aA}}s = \frac{Ad_P - d_A}{d_P + d_A}. \tag{24}$$

However, even if $P = 0$ the following equation holds true provided $A > 0$ (else $s_P = s_A$),

$$e_{tA}\frac{t_{aA}}{t_{sA}}s = \frac{s_P - s_A}{As_P + s_A}. \tag{25}$$

Also if $P > 0$ and $A > 0$ there is the result

$$s^2 = \left(\frac{Ad_P - d_A}{d_P + d_A}\right)\left(\frac{s_P - s_A}{As_P + s_A}\right) = \frac{N_1N_2}{D_1D_2},$$

which is independent of transport efficiency (except that $e_{tA} > 0$) and independent of the time dependence of the He-3 transmission. These formulae allow determination of the beam transport efficiency on the analyzer side, or a measurement of the cross-section asymmetry, s, or a check on the He-3 transmission factor t_{aA}/t_{sA}. Note that the second equation, 25, that holds true even if $P = 0$ indicates that s can be measured even with an unpolarized incident beam provided that $A > 0$ (otherwise $s_P = s_A$) and $e_{tA} > 0$. This is due to the fact that by the nature of the sample cross-sections, the scattered beam is polarized ($S^{++} \neq S^{--}$). For the error propagation on the analyzer-side beam-transport-efficiency, e_{tA}, the transmission factor, t_{aA}/t_{sA}, is required. Recall that this is

$$\frac{t_{aA}}{t_{sA}} = \frac{C_{aA}}{C_{sA}} \tanh\left(\tilde{\tau}_A P_{He3A}\right)$$

where the ratio of correction coefficients is

$$\frac{C_{aA}}{C_{sA}} = 1 + \frac{\Delta_A}{\langle C_A \rangle}\left[\frac{2}{\sinh\left(2\tilde{\tau}_A P_{He3A}\right)}\right].$$

Using the first equation, 24, to measure e_{tA}

$$e_{tA} = s\frac{t_{sA}}{t_{aA}}\frac{d_P + d_A}{Ad_P - d_A},$$

and the error propagation for e_{tA} is then

$$\tilde{\sigma}^2_{e_{tA}} = \sum_{\alpha\beta} W^2_{\alpha\beta}\sigma^2_{C^{\alpha\beta}} + W^2_A\tilde{\sigma}^2_A + W^2_t\left(\tilde{\sigma}^2_{P_{He}} + \tilde{\sigma}^2_{\tilde{\tau}}\right)_A + \tilde{\sigma}^2_s,$$

where

$$W_{\alpha\beta} = \frac{\alpha\beta}{d_P + d_A} - \frac{\beta A^{\delta_{\alpha +}}}{Ad_P - d_A}$$

$$W_A = \frac{Ad_P}{Ad_P - d_A}$$

$$W_t = \frac{2\tilde{\tau}_A P_{He3A}}{\sinh\left(2\tilde{\tau_A} P_{He3A}\right)}.$$

Using the first equation, 24, to measure s

$$s = e_{tA}\frac{t_{aA}}{t_{sA}}\frac{Ad_P - d_A}{d_P + d_A},$$

the same error propagation applies so that

$$\tilde{\sigma}^2_s = \sum_{\alpha\beta} W^2_{\alpha\beta}\sigma^2_{C^{\alpha\beta}} + W^2_A\tilde{\sigma}^2_A + W^2_t\left(\tilde{\sigma}^2_{P_{He}} + \tilde{\sigma}^2_{\tilde{\tau}}\right)_A + \tilde{\sigma}^2_{e_{tA}}.$$

Similarly the error propagation for $s^2$is given by

$$\tilde{\sigma}^2_{s^2} = \sum_{\alpha\beta} W^2_{\alpha\beta}\sigma^2_{C^{\alpha\beta}} + \sum_{X=A,P} W^2_X \sigma^2_X,$$

where

$$W_{++} = \left(\frac{A}{N_1} - \frac{1}{D_1}\right) + \left(\frac{1}{N_2} - \frac{A}{D_2}\right) P$$

$$W_{--} = \left(\frac{1}{N_1} + \frac{1}{D_1}\right) + \left(\frac{1}{N_2} + \frac{1}{D_2}\right)$$

$$W_{+-} = \left(\frac{A}{N_1} - \frac{1}{D_1}\right) - \left(\frac{1}{N_2} - \frac{A}{D_2}\right)$$

$$W_{-+} = \left(\frac{1}{N_1} + \frac{1}{D_1}\right) - \left(\frac{1}{N_2} + \frac{1}{D_2}\right) P$$

$$W_A = \frac{d_P}{N_1} - \frac{s_P}{D_2}$$

$$W_P = \frac{C^{++} - C^{-+}}{N_2} - \frac{AC^{++} + C^{-+}}{D_2}.$$

7 wavelength and path-length variation of He-3 transmission

In order to account for wavelength dependence in the He-3 transmission, take

$$\tau = \tau(\lambda) = n\sigma_0(\lambda_0)L_0\frac{\lambda}{\lambda_0} = \tau_0\frac{\lambda}{\lambda_0}.$$

where λ_0 serves as a reference wavelength for σ_0. We have already shown that the averaged transmission in the Gaussian approximation for the distributions of wavelength and pathlength deviations from average, can be expressed as

$$< t_\pm >= C_\pm t_{\pm 0}$$

$$t_{\pm 0} = t_E \exp\left(- \langle\tau\rangle \left(1 \mp P_{He3}\right)\right) = \exp\left(-\tilde{\tau}_\pm\right)$$

$$C_\pm = d_{\lambda L}^{-1/2} \exp\frac{1}{2}\frac{\sigma^2_\lambda + \sigma^2_L - 2\sigma^2_\lambda\sigma^2_L\tilde{\tau}_\pm}{d_{\lambda L}}\tilde{\tau}^2_\pm$$

$$d_{\lambda L} = 1 - \sigma_\lambda^2 \sigma_L^2 \tilde{\tau}_\pm^2 > 0$$

$$\langle \tau \rangle = n\sigma_0 \langle L \rangle \left\langle \widetilde{\lambda} \right\rangle$$

t_E is roughly wavelength independent, and is about 0.86 for the cells used at the NCNR made of GE180 glass. $L_0$is the path length along the center line of the He-3 cell. Also recall that σ_λ is the dimensionless distribution standard-deviation for λ/λ_0, and σ_L is the dimensionless distribution standard-deviation for $L/\langle L \rangle$.

As a simple example, the neutron path-length through the He-3 may vary due to beam divergence or variation in the separation of the cell walls. If the beam divergence can be treated by assuming parallel cells walls so that the angle dependence of the path length is

$$L(\phi) = L_0/\cos(\phi) \cong L_0(1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2)$$

where L_0 is the minimal He-3 thickness for a beam perpendicular to the cell flat walls (this is the value of L that goes into $\tau_\pm$), ϕ is the neutron path divergence angle with respect to the perpendicular to the cell walls, and γ and δ are the corresponding divergence angles in the scattering plane and perpendicular to the scattering plane respectively. Then using $\left\langle x^4 \right\rangle = 3\sigma_x^4$ for Gaussian distributions, we can compute the average pathlength and estimate its distribution width,

$$\langle L/L_0 \rangle = 1 + \frac{1}{2}\left(\sigma_\gamma^2 + \sigma_\delta^2\right)$$

$$\left\langle (L/L_0)^2 \right\rangle = 1 + \sigma_\gamma^2 + \sigma_\delta^2 + \frac{3}{4}\left(\sigma_\gamma^4 + \sigma_\delta^4\right) + \frac{1}{2}\sigma_\gamma^2\sigma_\delta^2$$

$$\left\langle (L/L_0)^2 \right\rangle - \langle L/L_0 \rangle^2 = \frac{1}{2}\left(\sigma_\gamma^4 + \sigma_\delta^4\right) \cong \sigma_L^2$$

7.1 Triple-Axis Case with flat end-windows

Consider the case that the incident neutrons have been scattered by a monochromating crystal, so that the incident and outgoing deviation angles in the scattering plane, γ_0 and γ_1, are correlated vai the wavelength according to Bragg's law. The transmission probability function (TPF) depends on the crystal mosaics and collimations before and after the crystal. The scattering plane TPF can be derived in terms of the deviation angles (measured positive with respect to nominal in the clockwise from above direction), collimations before and after the crystal, $\alpha_0$and α_1, and crystal scattering-plane mosaic, η_H, as

$$P_H(\gamma_0,\gamma_1) = N_H \exp\left\{-\frac{1}{2}\left[\left(\frac{\gamma_0}{\alpha_0}\right)^2 + \left(\frac{\gamma_0+\gamma_1}{2\eta_H}\right)^2 + \left(\frac{\gamma_1}{\alpha_1}\right)^2\right]\right\} d\gamma_0 d\gamma_1.$$

The Bragg's law correlation gives

$$\gamma_1 = \gamma_0 + 2\frac{\Delta\lambda}{\lambda}\tan(\omega_M)$$

where ω_M is the Bragg angle of the crystal and $\Delta\lambda = \lambda - \lambda_M$, with $\lambda_M = 2d_M \sin(\omega_M)$. Of course, d_M is the crystal d-spacing for the reflecting atomic planes. Thus the in-plane TPF can be written in terms of γ_1 and $x = \frac{\Delta\lambda}{\lambda}$ as

$$P_H = N_H \exp\left\{-\frac{1}{2}\left[\left(\frac{\gamma_1 - 2x\tan(\omega_M)}{\alpha_0}\right)^2 + \left(\frac{\gamma_1 - x\tan(\omega_M)}{\eta_H}\right)^2 + \left(\frac{\gamma_1}{\alpha_1}\right)^2\right]\right\} d\gamma_1 dx$$

or

$$P_H(\gamma_1, x) = N_H \exp\left\{-\frac{1}{2}\left[A\gamma_1^2 - 2B\gamma_1 x + Cx^2\right]\right\} d\gamma_1 dx$$

where

$$N_H = \frac{1}{2\pi}\left(AC - B^2\right)^{1/2} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\alpha_0^2+\alpha_1^2+4\eta_H^2}}{\alpha_0\alpha_1}\tan(\omega_M)\frac{1}{\sqrt{2\pi}}\frac{1}{\eta_H}$$

$$A = \frac{1}{\eta_H^2}\frac{\alpha_0^2\alpha_1^2 + \left(\alpha_0^2+\alpha_1^2\right)\eta_H^2}{\alpha_0^2\alpha_1^2}$$

$$B = \frac{1}{\eta_H^2}\frac{\alpha_0^2 + 2\eta_H^2}{\alpha_0^2}\tan(\omega_M)$$

$$C = \frac{1}{\eta_H^2}\frac{\alpha_0^2 + 4\eta_H^2}{\alpha_0^2}\tan^2(\omega_M).$$

If the crystal mosaic is zero, then$\Delta\lambda$ and γ are perfectly correlated so that

$$P_{H0}(\gamma_1, x) = N_{H0}\delta(\gamma_1 - x\tan(\omega_M))\exp\left\{-\frac{1}{2}C_0x^2\right\} d\gamma_1 dx$$

where now

$$N_{H0} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\alpha_0^2+\alpha_1^2}}{\alpha_0\alpha_1}\tan(\omega_M)$$

$$C_0 = \frac{\alpha_0^2+\alpha_1^2}{\alpha_0^2\alpha_1^2}\tan^2(\omega_M).$$

If there is no crystal then just the collimation determines the angle spread which is independent of the wavelength distribution.

$$P_{H00} = \frac{1}{\sqrt{2\pi}\sigma_\gamma} \exp\left\{-\frac{1}{2}\gamma_1^2/\sigma_\gamma^2\right\} d\gamma_1 \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{1}{2}x^2/\sigma_x^2\right\} dx$$

$$\sigma_\gamma^2 = \frac{\alpha_0^2\alpha_1^2}{\alpha_0^2+\alpha_1^2}$$

If we integrate over all the deviation angles we get the uncorrelated wavelength distribution for the neutrons scattered from the crytstal, $x = \frac{\Delta\lambda}{\lambda}$

$$\int P_H(\gamma_1, x) d\gamma_1 = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{1}{2}x^2/\sigma_x^2\right\} dx$$

$$\sigma_x^2 = \left(C - B^2/A\right)^{-1/2} = \cot(\omega_M) \left(\frac{\alpha_0^2\alpha_1^2 + \left(\alpha_0^2+\alpha_1^2\right)\eta_H^2}{\alpha_0^2+\alpha_1^2+4\eta_H^2}\right)^{1/2}$$

The TPF for deviation angles out of the scattering plane is

$$P_V(\delta_0, \delta_1) = N_V \exp\left\{-\frac{1}{2}\left[\left(\frac{\delta_0}{\beta_0}\right)^2 + \left(\frac{\delta_1-\delta_0}{2\eta_V \sin(\omega_M)}\right)^2 + \left(\frac{\delta_1}{\beta_1}\right)^2\right]\right\} d\delta_0 d\delta_1$$

where δ_0 and $\delta_1$are the deviation angles before and after the crystal, β_0 and β_1 the corresponding vertical effective collimations and η_V the crystal mosaic in the out of scattering-plane direction. Integrating over δ_0 and normalizing gives

$$P_V(\delta_1) = \frac{1}{\sqrt{2\pi}\sigma_\delta} \exp\left\{-\frac{1}{2}\delta_1^2/\sigma_\delta^2\right\} d\delta_1$$

where

$$\sigma_\delta^2 = \frac{\beta_1^2\left[\beta_0^2 + (2\eta_V \sin(\omega_M))^2\right]}{\beta_1^2 + \left[\beta_0^2 + (2\eta_V \sin(\omega_M))^2\right]}$$

To remove the crystal just set $\eta_V \sin(\omega_M) = 0$.

Now the average transmission can be calculated. For this calculation use $\lambda = \lambda_M(1+x)$ in the expression for the transmission so that

$$t_\pm = t_E \exp\left(-\tau_{\pm M}(1+x)(1+\frac{1}{2}\gamma^2+\frac{1}{2}\delta^2)\right)$$

where $\tau_{\pm M} = \tau_{\pm 0}\frac{\lambda_M}{\lambda_0}$. λ_M is the average wavelength produced by the monochromator and $\lambda_0$is the reference wavelength at which the He-3 absorption cross-section in $\tau_{\pm 0}$ is evaluated. The expansion of the transmission up to second order in the deviations is just

$$t_{\pm} = t_{\pm 0\epsilon} \left\{ 1 - \tau_{\pm M}(x + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2) + \frac{1}{2}\tau_{\pm M}^2 x^2 \right\}$$

where $t_{\pm 0\epsilon} = t_E \exp\left(-\tau_{\pm M}\right)$ is the transmission for zero deviations. The average transmission requires the integrals

$$< t_{\pm} >= t_{\pm 0\epsilon} \int P_H(\gamma, x) P_V(\delta) \left\{ 1 - \tau_{\pm M}(x + \frac{1}{2}\gamma^2 + \frac{1}{2}\delta^2) + \frac{1}{2}\tau_{\pm M}^2 x^2 \right\} d\gamma dx\, d\delta$$

The perfect crystal case with a delta function produces

$$< t_{\pm} >= t_{\pm 0\epsilon} \left\{ 1 - \frac{1}{2}\tau_{\pm M} \left[\left(\tan^2(\omega_M) - \tau_{\pm M} \right) \sigma_\gamma^2 + \sigma_\delta^2 \right] \right\} = \hat{C}_{\pm 0} t_{\pm 0},$$

where $\sigma_\gamma^2 = 1/C_0$ and $\sigma_\delta^2 = 1/A_V$. Note that the sign can change for the in-plane part of the correction for τ_{+M} and τ_{-M}.

The more general case requires tedious integration. The δ^2 integral term is simply

$$\int P_H(\gamma, x) P_V(\delta) \delta^2 d\gamma dx\, d\delta = \sigma_\delta^2.$$

The γ^2term integrated over γ yields

$$\int P_H(\gamma, x) P_V(\delta) \gamma^2 d\gamma dx\, d\delta = N_H \sqrt{\frac{2\pi}{A}} \int dx\, \exp\left[-\frac{1}{2}\left(C - \frac{B^2}{A} \right) x^2 \right] \left[\frac{1}{A} + \left(\frac{B}{A} x \right)^2 \right]$$

which is

$$\int P_H(\gamma, x) P_V(\delta) \gamma^2 d\gamma dx\, d\delta = \frac{C}{AC - B^2} = \frac{\alpha_1^2 \left(\alpha_0^2 + 4\eta_H^2 \right)}{\alpha_1^2 + \alpha_0^2 + 4\eta_H^2} = \sigma_\gamma^2.$$

To remove the crystal set $\eta_H = 0$.

The integral of the linear x term can be shown to be zero. The integral of the x^2 term is

$$\int P_H(\gamma, x) P_V(\delta) x^2\, d\gamma dx\, d\delta = \frac{A}{AC - B^2} = \frac{\left(\alpha_1^2 + \alpha_0^2 \right) \eta_H^2 + \alpha_0^2 \alpha_1^2}{\alpha_1^2 + \alpha_0^2 + 4\eta_H^2} \cot^2(\omega_M) = \sigma_x^2 = \left(\frac{\sigma_\lambda}{\lambda_M} \right)^2.$$

To modify these expressions for an analyzer cell in 2-axis mode, fudge the cell d-spacing to be that of the monochromator, as well as the horizontal collimations, so that one gets roughly the wavelength spread due to the monochromator, but set $\eta_H = 0$ and $\eta_V = 0$. The sum of all terms yields

$$< t_{\pm} >= t_{\pm 0\epsilon} \left\{ 1 - \frac{1}{2}\tau_{\pm M} \left[\sigma_\gamma^2 + \sigma_\delta^2 \right] + \frac{1}{2}\tau_{\pm M}^2 \sigma_x^2 \right\} = \hat{C}_{\pm} t_{\pm 0}$$

This expression agrees roughly with the formalism layed out previously, as

$$t_{\pm 0\epsilon} \exp\left(-\frac{1}{2}\tau_{\pm M}\left[\sigma_\gamma^2+\sigma_\delta^2\right]+\frac{1}{2}\tau_{\pm M}^2\sigma_x^2\right) = t_{\pm 0} \exp\left(\frac{1}{2}\tau_{\pm M}^2\sigma_x^2\right)$$

which is what we would calculate if the width of the pathlength distribution is zero. For thermal triple-axis instruments $\tau_{\pm M}$ is never very large, while the angular divergences are relatively small.

Note that in this case of flat windows, the wavelength variation increases the transmission while the angular distribution decreases the transmission. The effects are largest for the non-preferred spin-state.

7.2 Effect of cell geometry on the path length

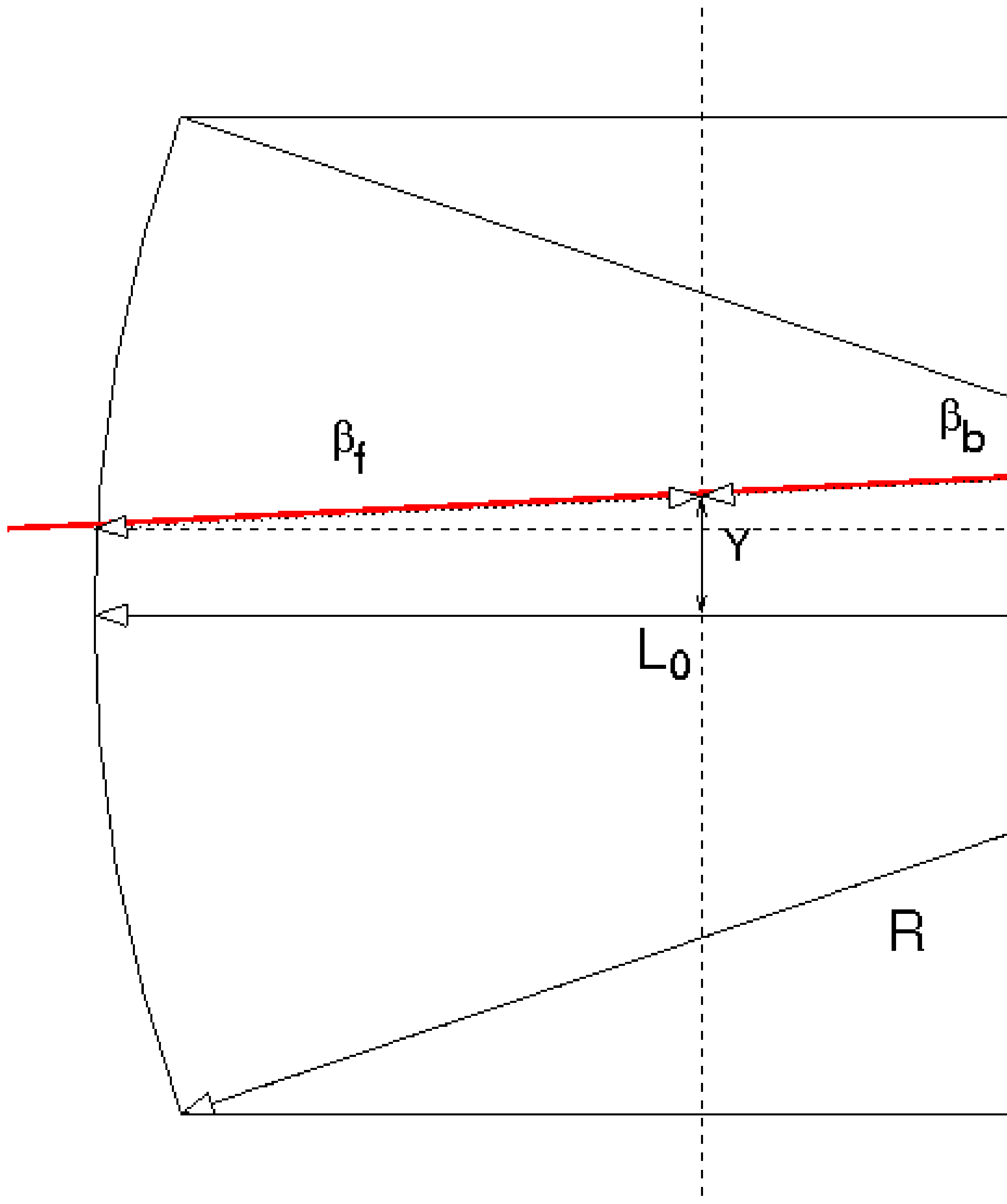
7.2.1 end-window geometry

There is a correction to the transmission from the varying path lengths due to the shape of the end windows. The end-window shape is characterized by its radius of curvature, R ($R \to \infty$ for flat windows). In order to calculate this, take the coordinate system origin at the center of the He-3 cell with y-axis up and the primary beam direction as the z-axis, so that in terms of the x(γ) and y(δ) deviation angles the neutron direction is

$$\hat{\mathbf{n}} = \cos(\delta)\left[\cos(\gamma)\,\hat{z} + \sin(\gamma)\,\hat{\mathbf{x}}\right] + \sin(\delta)\,\hat{y}.$$

In the $z = 0$ plane passing through the cell center, assume that the neutron passes through the point $\mathbf{P} = (X, Y, 0)$. Then the neutron path is along the line

$$\mathbf{r}_n = \left[X + \beta\cos(\delta)\sin(\gamma)\,, Y + \beta\sin(\delta)\,, \beta\cos(\delta)\cos(\gamma)\right].$$



βf
βb
Y
L0
R

If the primary neutron beam is displaced along the Y-axis, the same coordinate system can be used, but divergence angles will be associated with a non-zero Y-value. The expression for beam direction remains the same. Find the intersections of this line with the front and back spherical faces of the He-3 cell in order to calculate the path length. If $R > L_0$ is the radius of curvature for the spherical faces and L_0 is the straight through diameter of the cell, then points (x_f, y_f, z_f) on the front (beam entrance) face (where $z_f < 0$) satisfy

$$\left[\left(R - \frac{L_0}{2}\right) - z_f\right]^2 + x_f^2 + y_f^2 = R^2,$$

$$z_f^2 - 2\left(R - \frac{L_0}{2}\right) z_f + \left(R - \frac{L_0}{2}\right)^2 + x_f^2 + y_f^2 = R^2$$

$$z_f^2 - 2\left(R - \frac{L_0}{2}\right) z_f + x_f^2 + y_f^2 + \frac{1}{4}L_0^2 - RL_0 = 0$$

$$z_f = \left(R - \frac{L_0}{2}\right) - \left[\left(R - \frac{L_0}{2}\right)^2 + RL - x_f^2 - y_f^2 - \frac{1}{4}L_0^2\right]^{1/2}$$

$$z_f = \left(R - \frac{L_0}{2}\right) - R\left[1 - \left(x_f^2 + y_f^2\right)/R^2\right]^{1/2}$$

$$z_f = \left(R - \frac{L_0}{2}\right) - \left(R^2 - x_f^2 - y_f^2\right)^{1/2}$$

$$z_f = -\frac{L_0}{2} + R\left\{1 - \left[1 - (x_f^2 + y_f^2)/R^2\right]^{1/2}\right\}$$

Similarly, the back (beam exit) points (x_b, y_b, z_b) with $z_b > 0$ satisfy

$$\left[\left(R - \frac{L_0}{2}\right) + z_b\right]^2 + x_b^2 + y_b^2 = R^2$$

$$z_b = -\left(R - \frac{L_0}{2}\right) + \left(R^2 - x_b^2 - y_b^2\right)^{1/2}$$

$$z_b = +\frac{L_0}{2} - R\left\{1 - \left[1 - (x_b^2 + y_b^2)/R^2\right]^{1/2}\right\}$$

and if it can be assumed that $x_f^2 + y_f^2 \ll R^2$, an approximate expression for z_f is

$$z_f = -\frac{L_0}{2} + \frac{1}{2}\frac{x_f^2 + y_f^2}{R}.$$

Now use the expression for the neutron path to find the intersection point

$$x_f = X + \beta_f \cos(\delta)\sin(\gamma)$$

$$y_f = Y + \beta_f \sin(\delta)$$

$$z_f = \beta_f \cos(\delta)\cos(\gamma).$$

Exactly,

$$\left[\beta_f \cos(\delta)\cos(\gamma) - \left(R - \frac{L_0}{2}\right)\right]^2 + [X + \beta_f \cos(\delta)\sin(\gamma)]^2 + [Y + \beta_f \sin(\delta)]^2 = R^2$$

$$\left[\beta_b \cos(\delta)\cos(\gamma) + \left(R - \frac{L_0}{2}\right)\right]^2 + [X + \beta_b \cos(\delta)\sin(\gamma)]^2 + [Y + \beta_b \sin(\delta)]^2 = R^2$$

Cancel out the R^2 terms and find

$$\beta_f^2 + 2B_{fe}\beta_f + C_e = 0$$

$$\beta_b^2 + 2B_{be}\beta_b + C_e = 0$$

$$B_{fe} = -\left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma) + X\cos(\delta)\sin(\gamma) + Y\sin(\delta)$$

$$B_{be} = +\left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma) + X\cos(\delta)\sin(\gamma) + Y\sin(\delta)$$

$$C_e = -L_0\left(R - \frac{L_0}{4}\right) + \left(X^2 + Y^2\right)$$

The correct signs for the roots are determined by the known signs of β_{fe} and β_{be},

$$\beta_{fe} = -B_{fe} - \left(B_{fe}^2 - C_e\right)^{1/2}$$

$$\beta_{be} = -B_{be} + \left(B_{be}^2 - C_e\right)^{1/2}$$

so that

$$L_e(\gamma, \delta, X, Y) = \beta_{be} - \beta_{fe} = -2\left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma) + \left(B_{be}^2 - C_e\right)^{1/2} + \left(B_{fe}^2 - C_e\right)^{1/2}$$

Let $K = \left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma)$ and $A = X\cos(\delta)\sin(\gamma) + Y\sin(\delta)$. Then

$$L_e(\gamma, \delta, X, Y) = -2K + K\left[\left(\frac{A}{K} - 1\right)^2 - \frac{C_e}{K^2}\right]^{1/2} + K\left[\left(\frac{A}{K} + 1\right)^2 - \frac{C_e}{K^2}\right]^{1/2}$$

This becomes an expansion in $1/R$ by using a Taylor's series for the square-root terms.

$$L_e(\gamma, \delta, X, Y) = -2K + K\left[1 - \epsilon_1 + \epsilon_2\right]^{1/2} + K\left[1 + \epsilon_1 + \epsilon_2\right]^{1/2}$$

where

$$\epsilon_1 = \frac{2A}{K}$$

$$\epsilon_2 = \frac{A^2}{K^2} - \frac{C_e}{K^2}.$$

Now we can expand in terms of the small $1/R$ quantities ϵ_1 and ϵ_2.
Use

$$[1 + \epsilon]^{1/2} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 ...$$

First look at the solution for the case that $\delta = \gamma = 0$

$$B_{fe}(\delta = \gamma = 0) = -\left(R - \frac{L_0}{2}\right)$$

$$B_{be}(\delta = \gamma = 0) = +\left(R - \frac{L_0}{2}\right)$$

$$C_e(\delta = \gamma = 0) = -L_0\left(R - \frac{L_0}{4}\right) + \left(X^2 + Y^2\right)$$

$$B_{fe}^2 - C_e = B_{be}^2 - C_e = R^2 - (X^2 + Y^2)$$

$$\left(B_{fe}^2 - C_e\right)^{1/2} = \left(B_{be}^2 - C_e\right)^{1/2} = R\left[1 - (X^2 + Y^2)/R^2\right]^{1/2}$$

$$\beta_{fe} = -B_{fe} - \left(B_{fe}^2 - C_e\right)^{1/2} = \left(R - \frac{L_0}{2}\right) - R\left[1 - (X^2 + Y^2)/R^2\right]^{1/2}$$

$$\beta_{be} = -B_{be} + \left(B_{be}^2 - C_e\right)^{1/2} = -\left(R - \frac{L_0}{2}\right) + R\left[1 - (X^2 + Y^2)/R^2\right]^{1/2}$$

Approximating the square root

$$\beta_{fe}(\delta = \gamma = 0) = -\frac{L_0}{2} + \frac{1}{2}(X^2 + Y^2)/R$$

$$\beta_{be}(\delta = \gamma = 0) = +\frac{L_0}{2} - \frac{1}{2}(X^2 + Y^2)/R$$

$$L_e(0, 0, X, Y) = \beta_{be} - \beta_{fe} = L_0 - (X^2 + Y^2)/R$$

Also look at the case that $X = Y = 0$.

$$B_{fe}(X = Y = 0) = -\left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma)$$

$$B_{be}(X = Y = 0) = +\left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma)$$

$$C_e(X = Y = 0) = -L_0\left(R - \frac{L_0}{4}\right)$$

$$\left(B_{fe}^2 - C_e\right)^{1/2} = \left(B_{be}^2 - C_e\right)^{1/2} = \left\{\left(R - \frac{L_0}{2}\right)^2 \cos^2(\delta)\cos^2(\gamma) + L_0\left(R - \frac{L_0}{4}\right)\right\}^{1/2}$$

$$\beta_{fe}(X = Y = 0) = \left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma) - \left\{\left(R - \frac{L_0}{2}\right)^2 \cos^2(\delta)\cos^2(\gamma) + L_0\left(R - \frac{L_0}{4}\right)\right\}^{1/2}$$

$$\beta_{be}(X = Y = 0) = -\left(R - \frac{L_0}{2}\right)\cos(\delta)\cos(\gamma) + \left\{\left(R - \frac{L_0}{2}\right)^2 \cos^2(\delta)\cos^2(\gamma) + L_0\left(R - \frac{L_0}{4}\right)\right\}^{1/2}$$

$$L_e(\gamma,\delta,0,0) = \beta_{be} - \beta_{fe} = -2\left(R-\frac{L_0}{2}\right)\cos(\delta)\cos(\gamma) + 2\left\{\left(R-\frac{L_0}{2}\right)^2\cos^2(\delta)\cos^2(\gamma) + L_0\left(R-\frac{L_0}{4}\right)\right\}^{1/2}$$

Note that for $R = L_0/2$ this correctly gives $L_e(\gamma,\delta,0,0,R = L_0/2) = L_0$. For $R \to \infty$, approximate the square root,

$$L_e(\gamma,\delta,0,0) = \beta_{be} - \beta_{fe} = -2\left(R-\frac{L_0}{2}\right)\cos(\delta)\cos(\gamma)\left[1-\left\{1+L_0\left(R-\frac{L_0}{4}\right)/\left[\left(R-\frac{L_0}{2}\right)^2\cos^2(\delta)\cos^2(\gamma\right.\right.\right.$$

$$L_e(\gamma,\delta,0,0) = \beta_{be} - \beta_{fe} = L_0\left(R-\frac{L_0}{4}\right)/\left[\left(R-\frac{L_0}{2}\right)\cos(\delta)\cos(\gamma)\right]$$

$$L_e(\gamma,\delta,0,0,R\to\infty) = \beta_{be} - \beta_{fe} = L_0/\left[\cos(\delta)\cos(\gamma)\right]$$

The exact expression is unwieldy for doing averaging over divergence angles and or beam coordinates. Practically, the small angle approximation for the divergence angles can be used, and if we discard terms of order 3 or greater in the angles,

$$B_f = -\left(R-\frac{L_0}{2}\right)\left[1-\frac{1}{2}\left(\delta^2+\gamma^2\right)\right] + X\gamma + Y\delta$$

$$B_b = +\left(R-\frac{L_0}{2}\right)\left[1-\frac{1}{2}\left(\delta^2+\gamma^2\right)\right] + X\gamma + Y\delta$$

$$C = -L_0\left(R-\frac{L_0}{4}\right) + \left(X^2+Y^2\right)$$

$$B_f^2 = \left(R-\frac{L_0}{2}\right)^2\left[1-\left(\delta^2+\gamma^2\right)\right] - 2\left(R-\frac{L_0}{2}\right)(X\gamma+Y\delta) + \left(R-\frac{L_0}{2}\right)\left(\delta^2+\gamma^2\right)(X\gamma+Y\delta) + (X\gamma+Y\delta)^2$$

$$B_b^2 = \left(R-\frac{L_0}{2}\right)^2\left[1-\left(\delta^2+\gamma^2\right)\right] + 2\left(R-\frac{L_0}{2}\right)(X\gamma+Y\delta) - \left(R-\frac{L_0}{2}\right)\left(\delta^2+\gamma^2\right)(X\gamma+Y\delta) + (X\gamma+Y\delta)^2$$

Normally the last two terms should be smaller than the others and can be omitted, as they are quartic in small quantities. Note that the terme$_1$ $= \left(R-\frac{L_0}{2}\right)\left(\delta^2+\gamma^2\right)(X\gamma+Y\delta)$ which is cubic in the small divergence angles could be comparable to the last term, $e_2 = (X\gamma+Y\delta)^2$ becaus of R.

Approximating the square root and omitting e_1 and e_2,

$$\left(B_f^2 - C\right)^{1/2} = R\left\{1 - \frac{1}{2}\left(1 - \frac{L_0}{2R}\right)^2\left(\delta^2 + \gamma^2\right) - \left(1 - \frac{L_0}{2R}\right)\left(X\gamma + Y\delta\right)/R - \frac{1}{2}\left(X^2 + Y^2\right)/R^2\right\}$$

$$\left(B_b^2 - C\right)^{1/2} = R\left\{1 - \frac{1}{2}\left(1 - \frac{L_0}{2R}\right)^2\left(\delta^2 + \gamma^2\right) + \left(1 - \frac{L_0}{2R}\right)\left(X\gamma + Y\delta\right)/R - \frac{1}{2}\left(X^2 + Y^2\right)/R^2\right\}$$

$$\beta_f = \left(R - \frac{L_0}{2}\right)\left[1 - \frac{1}{2}\left(\delta^2 + \gamma^2\right)\right] + X\gamma + Y\delta - \left(B_f^2 - C\right)^{1/2}$$

$$\beta_b = -\left(R - \frac{L_0}{2}\right)\left[1 - \frac{1}{2}\left(\delta^2 + \gamma^2\right)\right] + X\gamma + Y\delta + \left(B_b^2 - C\right)^{1/2}$$

$$L(\gamma, \delta, X, Y) = \beta_b - \beta_f = -2\left(R - \frac{L_0}{2}\right)\left[1 - \frac{1}{2}\left(\delta^2 + \gamma^2\right)\right] + \left(B_b^2 - C\right)^{1/2} + \left(B_f^2 - C\right)^{1/2}$$

$$L(\gamma, \delta, X, Y) = \beta_b - \beta_f = L_0\left[1 + \frac{1}{2}\left(\delta^2 + \gamma^2\right)\left(1 - \frac{L_0}{2R}\right)\right] - \left(X^2 + Y^2\right)/R$$

$$L(\gamma, \delta, X, Y)/L_0 = \left[1 + \frac{1}{2}\left(\delta^2 + \gamma^2\right)\left(1 - \frac{L_0}{2R}\right)\right] - \left(X^2 + Y^2\right)/\left(L_0 R\right)$$

Note that $\delta^2+\gamma^2$ just gets multiplied by $1-\frac{L_0}{2R}$, so this is the curvature correction factor that multiplies $\sigma_\gamma^2 + \sigma_\delta^2$ in the expression for the averaged transmission. Thus, curvature weakens the increase in pathlength due to angular divergence, but also strengthens the decrease in pathlength due to beam cross-sectional area. The general expression for the transmission becomes

$$t_\pm = t_E \exp\left(-\tau_{\pm 0}\frac{\lambda}{\lambda_0}\frac{L}{L_0}\right)$$

$$= t_E \exp\left(-\tau_{\pm M}(1+x)\left[1 + \frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{L_0}{2R}\right)\right] + \tau_{\pm M}(1+x)\left(X^2 + Y^2\right)/\left(L_0 R\right)\right)$$

$$= t_E \exp\left(-\tau_{\pm M}\left(1 - \frac{X^2 + Y^2}{L_0 R}\right) - \tau_{\pm M}(1+x)\left[\frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{L_0}{2R}\right)\right] - \tau_{\pm M}x\left(1 - \frac{X^2 + Y^2}{L_0 R}\right)\right).$$

Keeping only terms up to second order this can be approximated as

$$t_\pm = t_E \exp\left(-\tau_{\pm M}\left(1 - \frac{X^2 + Y^2}{L_0 R}\right) - \tau_{\pm M}\left[\frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{L_0}{2R}\right)\right] - \tau_{\pm M}x\right)$$

$$= t_E \exp\left[-\tau_{\pm M}\left(1-\frac{X^2+Y^2}{L_0R}\right)\right]\left\{1-\tau_{\pm M}\left[\frac{1}{2}\left(\gamma^2+\delta^2\right)\left(1-\frac{L_0}{2R}\right)\right]-\tau_{\pm M}x+\frac{1}{2}\tau_{\pm M}^2x^2\right\}.$$

As before, the linear term in x will average to zero, so for the averaged transmission

$$< t_\pm >= t_E \exp\left[-\tau_{\pm M}\left(1-\frac{\left\langle X^2+Y^2\right\rangle}{L_0R}\right)\right]\left\{1-\tau_{\pm M}\left[\frac{1}{2}\left\langle\gamma^2+\delta^2\right\rangle\left(1-\frac{L_0}{2R}\right)\right]+\frac{1}{2}\tau_{\pm M}^2\left\langle x^2\right\rangle\right\},$$

and we have the previous expression for the average transmission, with now $\sigma_\gamma^2+\sigma_\delta^2$ scaled by $\left(1-\frac{L_0}{2R}\right)$ and $\tau_{\pm M}$ scaled by $1-\frac{\left\langle X^2+Y^2\right\rangle}{L_0R}$.

Again this fits into the formalism for averaging the transmission as

$$t_{\pm 0} = t_E \exp\left[-\tau_{\pm M}\left(1-\frac{\left\langle X^2+Y^2\right\rangle}{L_0R}+\frac{1}{2}\left\langle\gamma^2+\delta^2\right\rangle\left(1-\frac{L_0}{2R}\right)\right)\right]$$

$$C_\pm = \exp\left(\frac{1}{2}\tau_{\pm M}^2\left\langle x^2\right\rangle\right)$$

and the pathlength distribution width is assumed zero. A rough estimate for the pathlength distribution width is

$$\sigma_L^2 \cong \frac{1}{2}\left(\sigma_\gamma^4+\sigma_\delta^4\right)+2\left(\sigma_X^4+\sigma_Y^4\right)/(L_0R)^2$$

As promised, we have arranged the averaged transmission in the form

$$< t_\pm >= \tilde{C}_\pm t_E \exp\left(-\tilde{\tau}_{\pm M}\right)$$

with $\tilde{\tau}_{\pm M} = \tilde{\tau}_M\left(1 \mp P_{He3}\right)$ and $\tilde{C}_\pm \cong 1$.

To compute the unpolarized beam transmission requires $\langle C\rangle = (C_+ + C_-)/2$.

$$(C_+ + C_-)/2 = 1-\frac{1}{2}\left\langle\epsilon_L\right\rangle\left(\tau_{+m}+\tau_{-m}\right)+\frac{1}{4}\left(\left\langle\epsilon_\lambda^2\right\rangle+\left\langle\epsilon_L^2\right\rangle\right)\left(\tau_{+m}^2+\tau_{-m}^2\right)$$

$$\langle C\rangle = 1-\tau_m\left\langle\epsilon_L\right\rangle+\frac{1}{2}\tau_m^2\left(\left\langle\epsilon_\lambda^2\right\rangle+\left\langle\epsilon_L^2\right\rangle\right)\left(1+P_{He3}^2\right)$$

Soller type collimators with vertical blades, provide some degree of translational invariance across the beam in the scattering plane. This makes the divergence angles for the most part uncorrelated with the neutron cell-crossing point $(X, Y, 0)$ and the averages can be performed separately. This is no longer true when using radial collimators which are typically focussed to the sample position. Then γ and X are correlated. If the sample is small enough then we can correlate δ with Y as well. We can directly use the formulae in the following section

$$\gamma = (X - S_x)/\,|S_z|$$

$$\delta = (Y - S_y)/\,|S_z|$$

where $|S_z|$ is the distance from the sample to the analyzing He3 cell center, and S_x, S_y locate the neutron at the sample.

7.2.2 SANS flat area detector with end-window geometry

In the case of small-angle scattering onto an area detector, wavelength and pathlength deviations are uncorrelated, but the pathlength and deviation angles are very correlated. The neutron coordinate at the detector is relatively well defined, so that the best formulation is obtained by averaging over sample coordinates. Let $S_z < 0$ be the distance from the sample to the analyzing He3 cell center, let $D = D_z - S_z$ be the sample to detector distance (D_z is the He3 cell center to detector distance). Then if D_x, D_y locate the neutron on the area-detector, define $(D_x^2 + D_y^2)/D^2 = tan^2\theta$, and let S_x, S_y locate the neutron at the sample. That is, given the detector position, there are only two degrees of freedom to average over, the sample coordinates, S_x and S_y. We can use the following relations between the deviation angles and He3 cell midplane coordinates of the neutron, X and Y,

$$\gamma = (S_x - X)/S_z = (D_x - S_x)/D$$

$$\delta = (S_y - Y)/S_z = (D_y - S_y)/D$$

$$X = S_x - S_z(D_x - S_x)/D$$

$$Y = S_y - S_z(D_y - S_y)/D$$

Then we can series expand the approximated expression for the path-length in terms of the sample and detector coordinates.

$$L_s/L_0 = L(S_x, S_y, D_x, D_y)/L_0 = 1 + \left[\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{S_z^2}{RL_0}\right]\frac{D_x^2 + D_y^2}{D^2} + \left[\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{D_z^2}{RL_0}\right]\frac{S_x^2 + S_y^2}{D^2}$$

This expression assumes that the sample coordinates will be averaged and the sample is symmetric about the origin so that cross terms in S_x and S_y will average to zero. Otherwise add the cross term (from $X^2 + Y^2$ and $\gamma^2 + \delta^2$) so that total $L/L_0 = L_s/L_0 + L_{xy}/L_0$, with

$$L_{xy}(S_x, S_y, D_x, D_y)/L_0 = -2\left[\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{S_z D_z}{RL_0}\right]\frac{S_x D_x + S_y D_y}{D^2}$$

$$\langle L_{xy}(S_x, S_y, D_x, D_y)/L_0 \rangle = -2\,(c - sdz)\,\frac{\langle S_x\rangle \langle D_x\rangle + \langle S_y\rangle \langle D_y\rangle}{D^2}$$

If this asymmetry is not needed,

$$L_s/L_0 = L(S_x, S_y, D_x, D_y)/L_0 = 1 + (c - sz)\,\frac{D_x^2 + D_y^2}{D^2} + (c - dz)\,\frac{S_x^2 + S_y^2}{D^2}$$

where $c = \frac{1}{2}\left(1 - \frac{L_0}{2R}\right)$, $sz = \frac{S_z^2}{RL_0}$, $dz = \frac{D_z^2}{RL_0}$ and $sdz = \frac{S_z D_z}{RL_0}$. In all cases our approximation can be written as,

$$\langle L/L_0 \rangle = 1 + (c - sz)\,\frac{\langle D_x^2\rangle + \langle D_y^2\rangle}{D^2} + (c - dz)\,\frac{\langle S_x^2\rangle + \langle S_y^2\rangle}{D^2} - 2\,(c - sdz)\,\frac{\langle S_x\rangle \langle D_x\rangle + \langle S_y\rangle \langle D_y\rangle}{D^2}$$

Similarly we can break the variance of the normalized pathlength into contributions that apply when the sample (or detector) coordinates are zero, and contributions that apply when this is not the case. Note, however, that L_{xy} has a contribution to the variance in the zero-coordinate case.

$$\sigma_{Ls}^2 = \left\langle (L/L_0)^2 \right\rangle_s - \langle L/L_0\rangle_s^2 = (c-sz)^2\,\frac{\sigma_{Dx^2}^2 + \sigma_{Dy^2}^2}{D^4} + (c-dz)^2\,\frac{\sigma_{Sx^2}^2 + \sigma_{Sy^2}^2}{D^4} + 4(c-sdz)^2\,\frac{\sigma_{SxDx}^2 + \sigma_{SyDy}^2}{D^4}$$

The part of the variance that will contribute only when averaged sample and detector source and destination coordinates are non-zero is,

$$\sigma_{Lxy}^2 = \left\langle (L/L_0)^2 \right\rangle_{xy} - \langle L/L_0\rangle_{xy}^2 = -4(c-sz)(c-sdz)\,\frac{\langle S_x\rangle D_x^{321} + \langle S_y\rangle D_y^{321}}{D^4} + -4(c-dz)(c-sdz)\,\frac{\langle D_x\rangle S_x^{321} + \langle D}{D^4}$$

Here, $D_x^{321} = \langle D_x^3\rangle - \langle D_x^2\rangle \langle D_x\rangle$ and similarly for the other like terms. Note also that $\sigma_{Dx^2}^2 = \langle D_x^4\rangle - \langle D_x^2\rangle^2$ and $\sigma_{SxDx}^2 = \langle S_x^2\rangle \langle D_x^2\rangle - \langle S_x\rangle^2 \langle D_x\rangle^2$.

These are easy to calculate for the case that source and destination coordinates are uniformly distributed (rectangle distribution) with mean coordinate symbolized by μ and object size (coordinate range) symbolized by Δ. Then since the distributions for sample and detector are independent of one another,

$$\langle D_x\rangle = \mu_{Dx}$$

$$\sigma_{Dx}^2 = \Delta_{Dx}^2/12$$

$$\sigma_{Dx^2}^2 = \Delta_{Dx}^2\left(\Delta_{Dx}^2/20 + \mu_{Dx}^2/3\right)$$

$$\sigma_{SxDx}^2 = \Delta_{Dx}^2\left(\Delta_{Sx}^2/24 + \mu_{Sx}^2\right)/12 + \Delta_{Sx}^2\left(\Delta_{Dx}^2/24 + \mu_{Dx}^2\right)/12$$

$$D_x^{321} = \Delta_{Dx}\mu_{Dx}^2/3$$

For SANS the detector coordinates are relatively well defined (no averaging required), and we can average over the sample coordinates. A common case is the disc approximation with the radius of the disc as R_s and the sample cross-sectional area as $A_s = \pi R_s^2$, so that the average SANS path-length in terms of the scattering angle, θ, becomes, in the symmetric sample-at-He3-cell XY-origin,

$$\langle L(\theta, A_s)\rangle / L_0 = 1 + \left[\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{S_z^2}{RL_0}\right] tan^2\theta + \left[\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{D_z^2}{RL_0}\right] \frac{A_s}{2\pi D^2}$$

Note that in the SANS case $A_s/D^2/4\pi$ is typically much smaller than $\frac{1}{2}tan^2\theta$ ($S_r \ll D_r$). Dropping this term leaves

$$\langle L(\theta, A_s)\rangle / L_0 = 1 + \left[\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{S_z^2}{RL_0}\right] tan^2\theta - \frac{D_z^2}{RL_0}\frac{A_s}{2\pi D^2}$$

so for the averaged transmission (with $1+S_z/D = D_z/D$ approximated as unity for SANS case),

$$< t_\pm >= \exp\left[-\tau_{\pm M}\left(1 - \frac{\langle S_x^2 + S_y^2\rangle}{RL_0}\right)\right]\left\{1 - \tau_{\pm M}\left[\left(\frac{1}{2}\left(1 - \frac{L_0}{2R}\right) - \frac{S_z^2}{RL_0}\right) tan^2\theta\right] + \frac{1}{2}\tau_{\pm M}^2\left(\frac{\Delta\lambda}{\lambda}\right)^2\right\}.$$

This is different than the triple-axis case as the coordinate average is now over sample coordinates instead of the beam coordinates at the center of the He3 cell, and the angular divergence term is different.

Interestingly, it is possible in the approximation to cancel the $tan^2\theta$ or A_s dependence of the pathlength. For example, the $tan^2\theta$ dependence vanishes by choosing S_z such that

$$\frac{1}{2}L_0\left(R - \frac{L_0}{2}\right) = S_z^2$$

With $L_0 = 10cm$ and $R = 40cm$, S_z (sample to cell distance) would have to be about $13cm$ which is pretty small. Typically S_z is larger than this number so that the pathlength decreases with $tan^2\theta$. The averaged pathlength also decreases with the sample-area. For example, if the sample is a 1 cm disc with $R = 40cm$ then the reduction in path-length due to the sample area is just 0.012 cm. Take the example where $S_z = 50cm$. Then the correction is approximately

$$L_0\left(1 - 5\theta^2\right)$$

so that when θ reaches 5 degrees (0.1 radians) this becomes a 5% effect.

The dependence on sample cross-sectional area vanishes when

$$\frac{1}{2}L_0\left(R - \frac{L_0}{2}\right) = D_z^2$$

where $D - |S_z| = D_z$ is the He3-cell to detector distance. D_z is much larger than R or L_0 so that this condition is never approached.

Note that we have used $\sigma_L = 0$ in our formula, but $\tau_{\pm M}$ can become quite large for SANS experiments. We can estimate σ_L^2 just from the variance due to sample size, as the angle variation for a given detector pixel is quite small. Then

$$\sigma_L^2 = 2\left[\frac{1}{2}\left(1-\frac{L_0}{2R}\right)-\frac{D_z^2}{RL_0}\right]^2\frac{\sigma_{Sx^2}^2+\sigma_{Sy^2}^2}{D^4} \cong 2\frac{\Delta_{Sx}^4+\Delta_{Sy}^4}{20\left(RL_0\right)^2} \approx \frac{\left(A_s/2\pi\right)^2}{\left(RL_0\right)^2}$$

For a sample radius of $2cm$, $R = 25cm$ and $L_0 = 8cm$ find $\sigma_L^2 = 0.0008$. For SANS τ_-can reach 10-15 at long wavelengths so that this correction can become important.

7.2.3 Non-SANS PSD radial collimator end-window geometry

In the general non-SANS radial collimator case we cannot drop any of the terms. The radial collimator correlates pathlength with angular deviation just as in the SANS case, and the wavelength angle correlation depends on whether a crystal is used before the detector. One must calculate σ_x accordingly. In addition if the PSD is one-dimensional in the scattering plane, we need to average over the the detector Y direction in the pathlength expression. For example, take the distribution for neutrons along $D_y = D\beta$ as a Gaussian with some effective detector vertical-divergence standard deviation angle, σ_β (The δ averaging above was with D_y fixed). We need to average $(D_x^2 + D_y^2)/D^2 = tan^2\theta$ over this distribution.

$$\left\langle tan^2\theta\right\rangle = N_\beta \int \left(\left(D_x/D\right)^2+\beta^2\right)\exp\left(-\frac{1}{2}\beta^2/\sigma_\beta^2\right)d\beta$$

$$\gamma = a4det - a4He3$$

$$\left\langle tan^2\theta\right\rangle = \left\langle \left(D_x/D\right)^2\right\rangle + \sigma_\beta^2 = \left\langle tan^2\left(\gamma\right)\right\rangle + \sigma_\beta^2.$$

If there is only a single detector then D_x is averaged as well so that

$$\left\langle tan^2\theta\right\rangle = \sigma_\beta^2 + \sigma_\alpha^2.$$

These are the replacements for $tan^2\theta$ in $\left\langle L(\theta, A_s)\right\rangle/L_0$. We also need to do the average $\left\langle tan^2\left(\gamma\right)\right\rangle$ over detector acceptance angle.

$$\left\langle tan^2\left(\gamma\right)\right\rangle = N\int \exp-\frac{1}{2}\left(\frac{x}{\sigma_d}\right)^2 tan^2\left(\gamma+x\right)dx.$$

$$\left\langle tan^2\left(\gamma\right)\right\rangle \cong tan^2\left(\gamma\right)+\left(1+4tan^2\left(\gamma\right)+3tan^4\left(\gamma\right)\right)\sigma_d^2$$

$$L(S_x, S_y, \gamma, \beta)/L_0 = 1+\left[\frac{1}{2}\left(1-\frac{L_0}{2R}\right)-\frac{S_z^2}{RL_0}\right]\left(tan^2(\gamma)+\beta^2\right)+\left[\frac{1}{2}\left(1-\frac{L_0}{2R}\right)-\frac{D_z^2}{RL_0}\right]\frac{S_x^2+S_y^2}{D^2}$$

$$L(S_x, S_y, \gamma, \beta)/L_0 = 1+(c-sz)\left(tan^2(\gamma)+\beta^2\right)+(c-dz)\frac{S_x^2+S_y^2}{D^2}$$

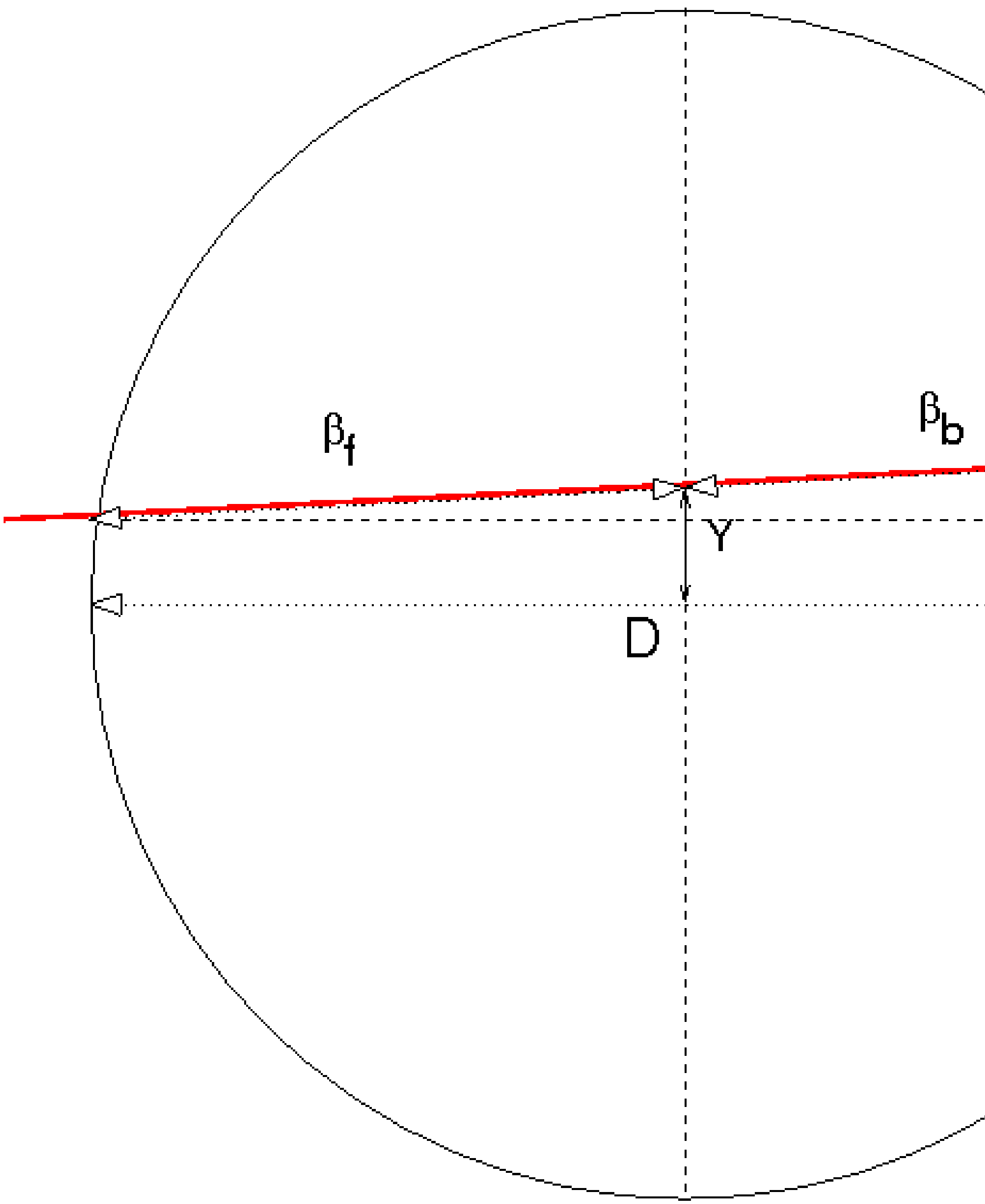
$$\langle L/L_0\rangle = 1+(c-sz)\left[tan^2(\gamma)+\left(1+4tan^2(\gamma)\right)\sigma_d^2+\sigma_\beta^2\right]+(c-dz)\frac{\sigma_{Sx}^2+\sigma_{Sy}^2}{D^2}$$

$$\left\langle (L/L_0)^2\right\rangle - \langle L/L_0\rangle^2 \cong \sigma_L^2 = 2(c-sz)^2\left[2tan^2(\gamma)\sec^4(\gamma)\,\sigma_d^2+\sigma_\beta^4\right]+2(c-dz)^2\frac{\sigma_{Sx}^4+\sigma_{Sy}^4}{D^4}$$

$$\left\langle (L/L_0)^2\right\rangle - \langle L/L_0\rangle^2 \cong \sigma_L^2 = 2(c-sz)^2\left[\sigma_\gamma^4+\sigma_\beta^4\right]+2(c-dz)^2\frac{\sigma_{Sx}^4+\sigma_{Sy}^4}{D^4}$$

7.2.4 cylinder geometry

In some cases He3-transmission filters are used with the neutrons passing through the cell cylinder walls instead of the end windows. The geometry is shown in the following diagram where D is the cylinder diameter and R is the radius. Recall that the primary beam direction is +z and +y is up out of the scattering plane.



β_f
β_b
Y
D

First look at the pathlength when the neutron is in a plane perpendicular to the cylinder axis so that the x-coordinate is fixed for the entire neutron path. Then the beam entrance and exit points satisfy

$$z_f^2 + y_f^2 = R^2,$$

$$z_b^2 + y_b^2 = R^2.$$

Now use the expression for the neutron path to find the intersection points. For the front face

$$x_f = X + \beta_f \cos(\delta) \sin(\gamma)$$

$$y_f = Y + \beta_f \sin(\delta)$$

$$z_f = \beta_f \cos(\delta) \cos(\gamma) .$$

But we are fixing $\gamma = 0$ so

$$x_f = X$$

$$y_f = Y + \beta_f \sin(\delta)$$

$$z_f = \beta_f \cos(\delta) .$$

The expressions for the intersection with the back wall are the same,

$$x_b = X$$

$$y_b = Y + \beta_b \sin(\delta)$$

$$z_b = \beta_b \cos(\delta) .$$

Then

$$Y^2 + 2Y\beta_f \sin(\delta) + \beta_f^2 = R^2$$

$$Y^2 + 2Y\beta_b \sin(\delta) + \beta_b^2 = R^2$$

$$\beta_f = -Y \sin(\delta) - \left[R^2 - Y^2 cos^2(\delta)\right]^{1/2}$$

$$\beta_b = -Y\sin(\delta) + \left[R^2 - Y^2 cos^2(\delta)\right]^{1/2}$$

and the pathlength is

$$L(Y,\delta) = \beta_b - \beta_f = 2R\left[1 - (Y/R)^2 cos^2(\delta)\right]^{1/2}.$$

These formula all can be obtained by setting $\gamma = 0$, $X = 0$ and $L_0 = 2R$ in the standard configuration. Since there is no Soller collimation in the vertical divergence, δ is correlated with with Y. Using $S_z < 0$ as the distance from cell center to the sample, $S_y$the neutron height coordinate at the sample, D_z the distance from the cell center to the detector, and D as the distance from sample to detector (no longer the cell diameter).

$$\delta = (Y - S_y)/|S_z| = (R/|S_z|)(Y/R) - S_y/|S_z| = (D_y - S_y)/D = (D_y - Y)/D_z.$$

An expansion in terms of S_y and D_y as in the SANS case is what we want to end up with since the two distribution functions should be approximately independent. The expansion in terms of Y and S_y also reveals some of the symmetry of the problem. Here, $|Y/R|$ could easily reach values of 0.3 so we make the expansion in terms of $z = Y/R$ fourth-order. On the other hand $y = S_y/|S_z|$ should be much less than 1 in magnitude and we should be able to use a second order expansion. The resulting power series for the reduced pathlength with $(R/|S_z|) = r$ is

$$L(z,y)/(2R) = 1 - \frac{1}{2}z^2 - \frac{1}{8}z^4 + \frac{1}{2}r^2z^4 - rz^3y + \left(\frac{1}{2}z^2 + \frac{1}{4}z^4 - r^2z^4\right)y^2.$$

$$L(z,y)/(2R) = 1 - \frac{1}{2}z^2 - \frac{1}{8}\left(1 - 4r^2\right)z^4 - rz^3y + \left[\frac{1}{2}z^2 + \frac{1}{4}\left(1 - 4r^2\right)z^4\right]y^2.$$

If $|z| \leq 0.5$ this series approximation is good to about 0.0005 of the pathlength. This is now easily averaged over a normalized height distribution in the He3 cell and the the distribution of neutrons from sample height. Using normal distributions for both with $\langle z^4 \rangle = 3\sigma_z^4$

$$\langle L(z,y)\rangle/(2R) = 1 - \frac{1}{2}\sigma_z^2 - \frac{3}{8}\left(1 - 4r^2\right)\sigma_z^4 + \left[\frac{1}{2}\sigma_z^2 + \frac{3}{4}\left(1 - 4r^2\right)\sigma_z^4\right]\frac{\sigma_{Sy}^2}{S_z^2}.$$

Note that the fourth-order terms drop out when $R/|S_z| = 1/2$. This is actually pretty close to the MACS conditions. This was the reason for doing the expansion in terms of Y and S_y.

Now the expansion in terms of S_y and D_y can be done using

$$\delta = \frac{D_y}{D} - \frac{S_y}{D} = dy - sy$$

$$\frac{Y}{R} = \frac{D_z}{R}\frac{S_y}{D} + \frac{|S_z|}{R}\frac{D_y}{D} = dz * sy + sz * dy$$

At MACS $2\max(Dy)/D$ is approximately the vertical divergence angle which is on the order of 10 degrees. Also at MACS the He3 cell placement is such that $|S_z|/R \cong 2$. The fourth order expansion, keeping only terms even in S_y and D_y since their distributions will be even as well, is

$$L(sy, dy)/(2R) = 1 - \frac{1}{2}sz^2dy^2 - \frac{1}{2}dz^2sy^2 + \left(\frac{1}{2}dz^2 + \frac{1}{2}sz^2 - 2dz * sz - \frac{3}{4}dz^2sz^2\right) dy^2sy^2.$$

For the MACS geometry using just the quadratic terms alone gets the path correct to about 0.0001. Then

$$\langle L(sz, dz)\rangle /(2R) \cong 1 - \frac{1}{2}sz^2\sigma_{dy}^2 - \frac{1}{2}dz^2\sigma_{sy}^2$$

$$\sigma_L^2 \cong \frac{1}{2}\left(sz^4\sigma_{dy}^4 + dz^4\sigma_{sy}^4\right)$$

MACS is a cold neutron triple-axis, so τ can be large enough that the σ_L^2 correction can become significant.

This result is indendent of X as long as the neutron stays away from the end windows, and the γ dependence can be added easily by noting that for fixed δ it is equivalent to the zero curvature end-window case. There the γ dependence of the pathlength just multiplies by $1/cos\,(\gamma - \gamma_0)$, where γ_0 is any tilt angle of the cylinder with respect to the perpendicular to the fiduciary beam direction. and in the small angle approximation we have

$$\frac{1}{cos\,(\gamma - \gamma_0)} = 1 + \epsilon_{L\gamma}$$

$$\gamma = a4detector - a4He3Center$$

If we average the a4 detector angle over the detector acceptance using the standard deviation angle for the detector collimation, σ_d,

$$\left\langle \frac{1}{cos\,(\gamma - \gamma_0)} \right\rangle = \frac{1 + \left[\frac{1}{2} + \tan^2(\gamma - \gamma_0)\right]\sigma_d^2}{cos\,(\gamma - \gamma_0)}$$

$$\langle L(sz, dz, \gamma)\rangle /(2R) \cong \left\{1 - \frac{1}{2}sz^2\sigma_{dy}^2 - \frac{1}{2}dz^2\sigma_{sy}^2\right\}\left\{1 + \left[\frac{1 + \left[\frac{1}{2} + \tan^2(\gamma - \gamma_0)\right]\sigma_d^2}{cos\,(\gamma - \gamma_0)} - 1\right]\right\}$$

$$\langle \epsilon_{L\gamma}\rangle = \left\langle \frac{1}{cos\,(\gamma - \gamma_0)} - 1 \right\rangle = \frac{1}{cos\,(\gamma - \gamma_0)} - 1 + \frac{1}{cos\,(\gamma - \gamma_0)}\left[\frac{1}{2} + \tan^2(\gamma - \gamma_0)\right]\sigma_d^2.$$

Making the small angle approximation for $\gamma - \gamma_0$
‘

$$\langle \epsilon_{L\gamma} \rangle \cong \frac{1}{2}(\gamma - \gamma_0)^2 + \left[1 + \frac{1}{2}(\gamma - \gamma_0)^2\right]\left[\frac{1}{2} + (\gamma - \gamma_0)^2\right]\sigma_d^2.$$

$$\langle \epsilon_{L\gamma} \rangle \cong \frac{1}{2}(\gamma - \gamma_0)^2 + \left[\frac{1}{2} + \frac{5}{4}(\gamma - \gamma_0)^2\right]\sigma_d^2 \cong \frac{1}{2}(\gamma - \gamma_0)^2 + \frac{1}{2}\sigma_d^2.$$

As usual, if we do the averages for Gaussian distributions just replace dy and sy by their standard deviations σ_{dy} and σ_{sy}. Remembering that it is the pathlength deviation average that goes into the transmission average, and using just the second order terms from the above expression,

$$\epsilon_L = -\frac{1}{2}\left(sz^2dy^2 + dz^2sy^2\right) + \frac{1}{cos\left(\gamma - \gamma_0\right)} - 1$$

$$\epsilon_L^2 = \frac{1}{4}\left(sz^4dy^4 + dz^4sy^4\right) + \left[\frac{1}{cos\left(\gamma - \gamma_0\right)} - 1\right]^2 - \left[\frac{1}{cos\left(\gamma - \gamma_0\right)} - 1\right]\left(sz^2dy^2 + dz^2sy^2\right)$$

$$\langle \epsilon_L \rangle = -\frac{1}{2}\left(sz^2\sigma_{dy}^2 + dz^2\sigma_{sy}^2\right) + \left\langle \frac{1}{cos\left(\gamma - \gamma_0\right)} - 1 \right\rangle$$

$$\langle \epsilon_L \rangle^2 = \frac{1}{4}\left(sz^4\sigma_{dy}^4 + dz^4\sigma_{sy}^4 + 2sz^2dz^2\sigma_{dy}^2\sigma_{sy}^2\right) + \left\langle \frac{1}{cos\left(\gamma - \gamma_0\right)} - 1 \right\rangle^2 - \left\langle \frac{1}{cos\left(\gamma - \gamma_0\right)} - 1 \right\rangle\left(sz^2\sigma_{dy}^2 + dz^2\sigma_{sy}^2\right)$$

$$\left\langle \epsilon_L^2 \right\rangle = \frac{3}{4}\left(sz^4\sigma_{dy}^4 + dz^4\sigma_{sy}^4\right) + \frac{1}{2}dz^2sz^2\sigma_{dy}^2\sigma_{sy}^2 + \left\langle \left[\frac{1}{cos\left(\gamma - \gamma_0\right)} - 1\right]^2 \right\rangle - \left\langle \frac{1}{cos\left(\gamma - \gamma_0\right)} - 1 \right\rangle\left(sz^2\sigma_{dy}^2 + dz^2\sigma_{sy}^2\right)$$

$$\left\langle \epsilon_L^2 \right\rangle - \langle \epsilon_L \rangle^2 = \frac{1}{2}\left(sz^4\sigma_{dy}^4 + dz^4\sigma_{sy}^4\right) + \frac{\tan^2\left(\gamma - \gamma_0\right)}{cos^2\left(\gamma - \gamma_0\right)}\sigma_d^2 \cong \sigma_{\epsilon L}^2$$

Currently this geometry is only being used with detectors along the x-direction. We can replace $1 + \frac{1}{2}\gamma^2$ by its Gaussian average, or if there are multiple detectors behind the He3 cell, use it to vary the pathlength for each detector.

$$t_{\pm} = t_E \exp\left(-\tau_{\pm 0}\frac{\lambda}{\lambda_0}\frac{L}{L_0}\right)$$

7.2.5 alternative derivation for end window geometry

The following alternative derivation produces a slightly different approximated form. This was the first approximation I used, but it turns out the approximation in a previous section is not only more accurate, it is also easier to average. The inclusion here is somewhat historical. If instead we start with the approximation

$$z_f = -\frac{L_0}{2} + \frac{1}{2}\frac{x_f^2 + y_f^2}{R}$$

the result of substitutions in the small angle approximation is a quadratic equation for the beam-path intersection length parameter, β_f, (which must be negative for the front face)

$$A\beta_f^2 + B_f\beta_f - C = 0,$$

where

$$A = \frac{1}{2R}\left[\cos^2(\delta)\sin^2(\gamma) + \sin^2(\delta)\right]$$

$$B_f = \frac{1}{R}\left[X\cos(\delta)\sin(\gamma) + Y\sin(\delta)\right] - \cos(\delta)\cos(\gamma)$$

$$C = \frac{L_0}{2} - \frac{X^2 + Y^2}{2R}.$$

For small beam divergence angles, γ and δ, $B_f^2 \cong 1$ and $|AC| \ll 1$, so that the solution for β_f can be approximated as

$$\beta_f = -C\left\{1 + \frac{X}{R}\gamma + \frac{Y}{R}\delta + \frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{C}{R}\right)\right\}.$$

The quadratic equation for β_b, the beam exit intersection path length parameter, has the same coefficients A and C, but there is a sign change in B_b

$$B_b = \frac{1}{R}\left[X\cos(\delta)\sin(\gamma) + Y\sin(\delta)\right] + \cos(\delta)\cos(\gamma),$$

so that

$$\beta_b = C\left\{1 - \frac{X}{R}\gamma - \frac{Y}{R}\delta + \frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{C}{R}\right)\right\}.$$

The total path length is then

$$L(\gamma, \delta, X, Y) = \beta_b - \beta_f = \left(L_0 - \frac{X^2 + Y^2}{R}\right)\left\{1 + \frac{1}{2}\left(\gamma^2 + \delta^2\right)\left(1 - \frac{C}{R}\right)\right\}.$$

Note that if $R \to \infty$ the expression for the path length in the flat wall case is recovered. The $X^2 + Y^2$ dependence can be handled by assuming that the probability distribution for beam divergence angles is independent of X^2+Y^2 (which should be true for small enough X^2+Y^2), and then replacing X^2+Y^2 by its average over the effective beam cross sectional area in the $x=0$ plane. For example, if the effective beam cross sectional area is a disc of radius r then $\left\langle X^2+Y^2\right\rangle = \left\langle \rho^2\right\rangle = \frac{1}{2}r^2$. The final result is that the previous expression for the transmission as a function of deviation angles and wavelength deviation, which was

$$t_{\pm} = t_E \exp\left(-\tau_{\pm M}(1+x)(1+\frac{1}{2}\gamma^2+\frac{1}{2}\delta^2)\right)$$

can be simply modified by scaling down $\tau_{\pm M}$

$$\tilde{\tau}_{\pm M} = \tau_{\pm M}\left(1-\frac{\left\langle \rho^2\right\rangle}{L_0 R}\right)$$

where L is the straight through path length of the cell. Also, the $\frac{1}{2}$ coefficients of γ^2 and $\delta^2$are scaled down

$$\frac{1}{2} \to \frac{1}{2}\left\{1-\frac{L_0}{2R}\left[1-\frac{\left\langle \rho^2\right\rangle}{L_0 R}\right]\right\} = \frac{1}{2}P.$$

Note that for a completely spherical cell ($R=L_0/2$) and a beam that must pass through the cell center ($\left\langle \rho^2\right\rangle = 0$) the dependence on angular deviation becomes zero, as it should. The scaling of γ^2 and δ^2translate directly into scaling of σ_γ^2 and σ_δ^2 in the results for the averaged transmission. Disregarding angle and wavelength deviations the basic transmission is modified to

$$\tilde{t}_{\pm 0} = t_E \exp\left[-\left(1-\frac{\left\langle \rho^2\right\rangle}{L_0 R}\right) n\sigma_0\frac{\lambda_M}{\lambda_0}L_0\left(1\mp P_{He}\right)\right] = t_E\exp\left(-\tilde{\tau}_{\pm M}\right).$$

The general form for the averaged transmission for all corrections is

$$< t_{\pm} >= \tilde{t}_{\pm 0}\left\{1-\frac{1}{2}\tilde{\tau}_{\pm M}P\left[\sigma_\gamma^2+\sigma_\delta^2\right]+\frac{1}{2}\tilde{\tau}_{\pm M}^2\sigma_x^2\right\} = \hat{C}_{\pm}\tilde{t}_{\pm 0}$$

Also consider the effect of higher order wavelength contamination of the neutron beam. In this case the wavelength probability distribution is a sum of probability distributions centered at each higher order wavelength, $\lambda_n = \lambda_1/n$, so that

$$P\left(\lambda\right) = \sum_{n=1} a_n P_n\left(\lambda_n\right),$$

where the sum of wavelength fractions is unity

$$\sum_{n=1} a_n = 1.$$

All the wavelength fractions are at the same settings for angles and angle distribution parameters so that the transmission correction factor, $\hat{C}_{\pm}$, should be approximately wavelength order independent. The averaged transmission factor is then

$$\langle t_{\pm} \rangle = \hat{C}_{\pm} t_E \sum_{n=1} a_n \exp\left(-\tau_{\pm n}\right),$$

where $\tau_{\pm n} = \tau_{\pm 1}\lambda_n/\lambda_1 = \tau_{\pm}/n$. Thus

$$\langle t_{\pm} \rangle = \hat{C}_{\pm} t_E \sum_{n=1} a_n \exp\left(-\frac{1}{n}\tau_{\pm}\right) = \widetilde{C}_{\pm} t_{\pm},$$

where the correction factor is now

$$\widetilde{C}_{\pm} = \hat{C}_{\pm}\left\{1 + \sum_{n=2} a_n K_{\pm n}\right\}$$

and

$$K_{\pm n} = \exp\left[\left(1 - \frac{1}{n}\right)\tau_{\pm}\right] - 1$$

For example, take $\tau_m = 1.8662$, $P_{He} = 0.7$ and the primary wavelength as 1.77 Angstroms. For the uncorrelated beam correction, using $\frac{\sigma_\lambda}{\lambda_m} = 0.05$, $\sigma_\alpha = 0.01$ and $\sigma_\beta = 0.04$, $C_+ = 1.0001$ and $C_- = 1.01$. For the correlated beam correction with $\cot(\theta_m) = 1$ and the same σ_α and σ_β values, $C_+ = 0.9998$ and $C_- = 1.0017$. The second order wavelength contamination factors (which still have to be multiplied by a_n) are $K_{+2} = 0.323$ and $K_{-2} = 3.885$. This means that the corrections to the transmission factors due to second order wavelength contamination can be significant (depending on the fraction a_2).

8 monitoring He-3 polarization and neutron polarization

If the transmission, t_{00}, through the unpolarized He-3 cell is measured ($P_{He} = 0$), then measurements of $t_0(P_{He})$ can be used to monitor the He-3 polarization, P_{He}, of the He-3 cell, assuming $\tilde{\tau}$ has been determined by a transmission measurement of the unpolarized cell. This is most conveniently done when there are no higher order wavelength contaminations, so that

$$r(P_{He}) = \frac{t_0(P_{He})}{t_{00}} = \cosh(\tilde{\tau}P_{He}) + \Delta\sinh(\tilde{\tau}P_{He}). \tag{26}$$

Neglecting the correction term in Δ, the coshfunction can be inverted to give

$$\tilde{\tau}P_{He} \cong x_0 = \ln\left(r + \sqrt{r^2 - 1}\right).$$

If the correction coefficient is known then

$$\tilde{\tau} P_{He} \cong x_0 - \Delta.$$

The outgoing neutron polarization, $-1 \leq P_n \leq 1$, after an incident unpolarized beam passes through the cell is

$$P_n = \frac{n_+ - n_-}{n_+ + n_-} = \tanh\left(\tilde{\tau}_M P_{He}\right) + \frac{\Delta}{\cosh^2(\tilde{\tau}_M P_{He})}. \tag{27}$$

As in the example above, using a 7 cm gas-thickness He-3 cell at 2 bars has $\tau_M = 1.8662$. With $P_{He} = 0.7$ He-3 polarization and $t_E = 0.86$, the cell transmits an uncorrected $t_0 = 0.2636$ of an incident unpolarized beam at 1.77 Angstroms and produces an outgoing beam that is $P_n = 0.8633$ polarized ($n_-/n_+ = 0.0733$). Making the corrections as in the example above, for the uncorrelated beam case, $t_0 = 0.2638$, and for the correlated beam case, $t_0 = 0.2637$. The corrections to the polarization for these two cases yield $P_n = 0.8621$ and $P_n = 0.8631$.

The best way to keep track of the polarization of the He-3 cells is to use beam monitors as shown in the diagram at the start of this document, and measure the transmissions as a function of time. If this is not possible, the remaining handle on the polarized beam performance is the flipping ratio, preferably measured with a non-spin-flip cross section. Recall that this flipping ratio is

$$R_{nsf} = \frac{t_+ + e_t t}{t_+ - e_{tF} t_-}$$

with $e_{tF} = e_t(2e_{FP,A} - 1)$. Now it is assumed that the correction factors for the He-3 transmission factors are unity. When the transport and flipping efficiencies are unity this simplifies to

$$R_{0,nsf} = \frac{\cosh(\tau_{M1} P_{He1} + \tau_{M2} P_{He2})}{\cosh(\tau_{M1} P_{He1} - \tau_{M2} P_{He2})}$$

and in terms of this ideal flipping ratio

$$R_{nsf} = R_{0,nsf} \frac{1 + e_t}{1 + e_{tF}} \left(1 + \epsilon_t / R_{0,nsf} - \epsilon_{tF} R_{0,nsf}\right)$$

where the transport loss is $\epsilon_t = (1 - e_t)/(1 + e_t)$ and transport-flipper loss is $\epsilon_{tF} = (1 - e_{tF})/(1 + e_{tF})$. If the cell parameters τ_{M1} and $\tau_{M2}$are known, as well as the cell He-3 polarizations (through transmission measurements and known time dependences) and beam efficiencies, then the calculated R_{nsf}can be compared to measured values.