Solutions of the Maxwell equations and photon wave functions*

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Abstract

Properties of six-component electromagnetic field solutions of a matrix form of the Maxwell equations, analogous to the four-component solutions of the Dirac equation, are described. It is shown that the six-component equation, including sources, is invariant under Lorentz transformations. Complete sets of eigenfunctions of the Hamiltonian for the electromagnetic fields, which may be interpreted as photon wave functions, are given both for plane waves and for angular-momentum eigenstates. Rotationally invariant projection operators are used to identify transverse or longitudinal electric and magnetic fields. For plane waves, the velocity transformed transverse wave functions are also transverse, and the velocity transformed longitudinal wave functions include both longitudinal and transverse components. A suitable sum over these eigenfunctions provides a Green function for the matrix Maxwell equation, which can be expressed in the same covariant form as the Green function for the Dirac equation. Radiation from a dipole source and from a Dirac atomic transition current are calculated to illustrate applications of the Maxwell Green function.

1. Introduction

For quantum mechanics to provide a complete description of nature, it is necessary to have a wave function for something as important as electromagnetic radiation or photons. This has become increasingly relevant as the number of experiments on single photon production and detection, motivated by interest in the fields of quantum computation and quantum cryptography, has grown rapidly over the past two decades [1]. The history of theoretical efforts to define photon wave functions dates back to the early days of quantum mechanics and is still unfolding. Overviews have been given in [2–4]. However, there is not yet a consensus on the form a photon wave function should take or the properties it should have. Further investigation of these questions is warranted, and possible answers are given in this paper.

Quantum electrodynamics (QED) accurately describes the interaction of radiation with free electrons and electrons bound in atoms, but as it is formulated in terms of an S matrix,

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asymptotic states, and Feynman diagrams, it does not readily lend itself to the description of the time evolution of radiation. In particular, interference effects or the space-time behavior of a photon wave packet would be more naturally described in the framework of wave mechanics with a wave function for a single photon.

There are a number of requirements that need to be be imposed on a formalism for a quantum mechanical description of photons. First, the predicted behavior of radiation should be consistent with the Maxwell equations. Time-dependent solutions of the Maxwell equations provide the basis for both classical electromagnetic theory and QED, and it can be expected that a photon wave function should also be based on solutions of the Maxwell equations. This means that the wave function is simultaneously the solution of both of the first-order Maxwell equations with time derivatives and not just a solution of a second-order scalar wave equation.

A second requirement is that the wave functions obey the quantum mechanical principle of linear superposition. Simply stated, this means that if two wave functions describe possible states of radiation, then a linear combination of these wave functions also describes a possible state. For example, a wave function for circularly polarized radiation can be written as a linear combination of two wave functions for linearly polarized radiation, all of which must be solutions of the same wave equation.

Another requirement is that the formalism be Lorentz invariant in order to properly describe the space-time behavior of radiation. An approximation scheme like the reduction of the Dirac equation to obtain the Schrödinger equation for electron velocities that are small compared to the speed of light is not an option for radiative photons.

Finally, it is necessary for the formalism to provide the tools for methods associated with quantum mechanics. This includes a wave equation with a Hamiltonian that describes the time development of states, wave functions that comprise a complete set of eigenfunctions of the Hamiltonian, normalizable states with a probability distribution that corresponds to the location of the photon, a law of conservation of probability, operators with expectation values for observables, and wave packets that realistically describe the propagation of photons in space and time.

To arrive at a wave equation that addresses these requirements, we examine an approach in which the four-component matrix Dirac equation for a spin one-half electron is adapted to a six-component form of the Maxwell equations for a spin-one photon. This version of the Maxwell equations is a direct extension of the Dirac equation for the electron in which two-by-two Pauli matrices are replaced by analogous three-by-three matrices. Since the quantum mechanical properties of the Dirac equation, Hamiltonian, and wave functions are well understood and tested experimentally, it is natural to consider the analogous Maxwell equation, Hamiltonian, and wave functions as a quantum mechanical description of photons.

There are fundamental differences between the Dirac equation and the matrix Maxwell equation, so the extension requires a detailed analysis. The most prominent difference is the fact that there is a possible source term in the Maxwell equation which has no analog for the Dirac equation [5]. Also, some properties of the three-by-three spin matrices differ from those of the Pauli matrices, even though they have the same commutation relations.

Linear operators that are representations of the inhomogeneous Lorentz group can replace the wave equation of a system for free electrons or transverse photons with no sources [6–8], but the source terms and longitudinal solutions of the Maxwell equation fall outside this framework. Taking the view that the Maxwell equation with a source is the most direct contact with experiment, our approach is to start from the matrix Maxwell equation with a source term and explicitly work out the Lorentz transformations of the solutions. It is shown that the six-component equation is invariant under Lorentz transformations, as it should be, but this is not self-evident, since the source term is essentially the three-vector current density.

Next, six-component solutions are constructed and shown to be complete sets of orthogonal coordinate-space eigenfunctions of the Maxwell Hamiltonian, parameterized by physical properties, such as linear momentum, angular momentum, and parity. These properties are associated with operators that commute with the Hamiltonian. Complete sets of both plane-wave solutions and angular-momentum eigenfunctions are given. Bilinear products of normalizable linear combinations of these functions provide expressions for the probability density and flux. The eigenfunctions are further classified according to whether they represent transverse or longitudinal states. These properties are associated with the electric and magnetic fields, with the result that under a velocity boost, the transformed transverse solutions are also transverse, unlike solutions corresponding to a transverse vector potential. Moreover, by summing over both transverse and longitudinal solutions, we obtain a covariant Green function for the Maxwell equation, which is of the same form as the Green function for the Dirac equation.

Solutions are obtained directly from the Maxwell equation, with no recourse to a vector potential. This avoids problems such as extra polarization components and ambiguities associated with gauge transformations [9–11]. Although integrals over a closed path of the potential may be observables, as discussed in [12], such integrals can be expressed in terms of the magnetic flux through the loop [13], so it is expected that the fields alone provide a complete description of electrodynamics. It is also clear that photon wave functions are closely aligned with electric and magnetic fields, and an approach that starts with classical electrodynamics expressed in terms of fields only provides a natural framework for the transition to the wave mechanics of photons. A possible advantage of using a vector potential is that it is the solution of a scalar wave equation, which has a well-known Green function. However, this advantage is offset by the fact that we provide a covariant Green function for the Maxwell equation.

This paper is organized as follows. In Sec. 2 the vector Maxwell equations and the Dirac equation are stated to define notation. The algebra of three-component spin matrices is reviewed in Sec. 3, where both a spherical basis, which is the direct extension of the Pauli matrices to three components, and a Cartesian basis with real components, are defined. The Maxwell equations are written in terms of the spherical spin matrices and combined into the Dirac equation form in Sec. 4. In Sec. 5, transverse and longitudinal projection operators are defined and used to separate the Maxwell equations and solutions into the corresponding disjoint sectors. Lorentz invariance is addressed in Sec. 6, where transformations of the coordinates and derivatives, transformations of the Maxwell equation, and transformations of the solutions are explicitly written. In Sec. 7, plane-wave solutions, which are eigenfunctions of the momentum operator as well as the Hamiltonian, are given for both transverse and longitudinal states, and the set of solutions is shown to be complete. The explicit action of Lorentz transformations on the plane-wave solutions is described. Properties of normalizable wave packets formed from the plane-wave solutions are illustrated. The angular-momentum operator and the corresponding eigenfunctions are given and shown to be complete in Sec. 8. In Sec. 9, the Maxwell Green function is written as an integral over the plane-wave solutions in a form analogous to the Dirac Green function. As examples of applications of the Maxwell Green function, formulas for radiation from a point dipole source and from a Dirac current source are derived in Sec. 10. A summary of the main points of the paper is in Sec. 11 and brief concluding remarks are made in Sec. 12.

The relation of the present study to earlier work is indicated in the sections where the particular topics are discussed.

2. Three-vector Maxwell equations and the Dirac equation

The Maxwell equations in vacuum, in the International System of Units (SI), are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0}, & (1)\\ \boldsymbol{\nabla}\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} &= \mu_0\boldsymbol{J}, & (2)\\ \boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t} &= 0, & (3)\\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0, & (4) \end{aligned}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, ρ and $\boldsymbol{J}$ are the charge and current densities, ϵ_0 and μ_0 are the electric and magnetic constants, and $c = (\epsilon_0\mu_0)^{-1/2}$ is the speed of light. The continuity equation

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \qquad (5)$$

follows from Eqs. (1) and (2).

The form of the Maxwell equations considered here is analogous to the Dirac equation for the electron. The Dirac wave function $\phi(x)$ is a four-component column matrix that is a function of the four-vector x. For a free electron, the Dirac equation is

$$(\,\mathrm{i}\,\hbar\gamma^\mu\partial_\mu - m_\mathrm{e}c)\,\phi(x) = 0, \qquad (6)$$

where $\hbar$ is the Planck constant divided by 2π, m_e is the mass of the electron, γ^μ, $\mu = 0,1,2,3$, are the 4×4 Dirac gamma matrices, given by

$$\gamma^0 = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}, \; i = 1,2,3, \qquad (7)$$

I and 0 are the 2×2 identity and zero matrices, σ^i, $i = 1, 2, 3$, are the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

and the derivatives ∂_μ are

$$\partial_0 = \frac{\partial}{\partial ct}; \quad \partial_i = \frac{\partial}{\partial x^i}, \ i = 1, 2, 3. \tag{9}$$

We take the metric tensor $g^{\mu\nu}$ to be

$$g^{00} = 1; \quad g^{ii} = -1, \ i = 1, 2, 3; \quad g^{\mu\nu} = 0, \ \mu \neq \nu. \tag{10}$$

In terms of the spin matrices, the derivative term in Eq. (6) can be written as

$$\gamma^\mu \partial_\mu \ = \ \begin{pmatrix} I \dfrac{\partial}{\partial ct} & \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ -\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & -I \dfrac{\partial}{\partial ct} \end{pmatrix}. \tag{11}$$

The Pauli spin matrices act on two-component spin matrices in the electron wave function. Oppenheimer has suggested that since the Maxwell equations involve three-vectors, three-component matrices should be considered for constructing a photon wave function [5]. Here we implement such an extension by replacing the Pauli spin matrices in the Dirac equation by the analogous 3×3 matrices described in the next section.

3. Three-component spin matrices

As is well known, three-vectors and operations among them are interchangeable with three-component matrices and matrix operations. In this section, formulas for these matrices relevant to subsequent work are given. Some of these formulas have been given in [14]. It is useful to define both Cartesian and spherical matrices to represent three-vectors.

The Cartesian matrix representing a vector $\boldsymbol{a}$ may be written as

$$\boldsymbol{a}_\mathrm{c} \ = \ \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \tag{12}$$

where $a^1, \ a^2, \ a^3$ are the rectangular components of the vector $\boldsymbol{a}$, and a spherical representation is denoted by

$$\boldsymbol{a}_\mathrm{s} \ = \ \boldsymbol{M}\boldsymbol{a}_\mathrm{c}, \tag{13}$$

where $\boldsymbol{M}$ is a 3×3 unitary matrix specified in the following. The dot product of two vectors is

$$\boldsymbol{a} \cdot \boldsymbol{b} \ = \ \boldsymbol{a}_\mathrm{c}^\dagger \boldsymbol{b}_\mathrm{c} = \boldsymbol{a}_\mathrm{s}^\dagger \boldsymbol{b}_\mathrm{s}, \tag{14}$$

where † denotes the combined operations of matrix transposition and complex conjugation.

Explicit Hermitian $\boldsymbol{\tau}$ matrices ($\boldsymbol{\tau}^\dagger = \boldsymbol{\tau}$), which are 3×3 versions of the Pauli matrices, are obtained by taking τ^3 to be diagonal

$$\tau^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{15}$$

and applying appropriate rotation matrices to obtain τ^1 and τ^2:

$$\tau^1 = \mathfrak{D}^{(1)}(\{0,\tfrac{\pi}{2},0\})\,\tau^3\,\mathfrak{D}^{(1)}(\{0,-\tfrac{\pi}{2},0\}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{16}$$

$$\tau^2 = \mathfrak{D}^{(1)}(\{0,0,\tfrac{\pi}{2}\})\,\tau^1\,\mathfrak{D}^{(1)}(\{0,0,-\tfrac{\pi}{2}\}) = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{17}$$

where $\mathfrak{D}^{(1)}(\{\alpha,\beta,\gamma\})$ is the $j=1$ representation of the rotation group, parameterized by the Euler angles α,β,γ [15]. In particular, $\mathfrak{D}^{(1)}(\{0,\frac{\pi}{2},0\})$ represents the rotation about the 2 axis by the angle $\pi/2$ and $\mathfrak{D}^{(1)}(\{0,0,\frac{\pi}{2}\})$ represents the rotation about the 3 axis by the angle $\pi/2$. The same rotations starting from σ^3, with the $j=\frac{1}{2}$ representation, reproduce σ^1 and σ^2. The $\boldsymbol{\tau}$ matrices are related by

$$\left[\tau^i, \tau^j\right] = \mathrm{i}\,\epsilon_{ijk}\,\tau^k, \tag{18}$$

where ϵ_{ijk} is the Levi-Civita symbol.[1]

The cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in terms of the scalar product of the tau matrices with the vector $\boldsymbol{a}$

$$\boldsymbol{\tau}\cdot\boldsymbol{a} = \tau^i a^i \tag{19}$$

acting on the spherical matrix for the vector $\boldsymbol{b}$ as

$$\boldsymbol{\tau}\cdot\boldsymbol{a}\;\boldsymbol{b}_{\mathrm{s}} = \mathrm{i}\,(\boldsymbol{a}\times\boldsymbol{b})_{\mathrm{s}}\,, \tag{20}$$

provided the matrix $\boldsymbol{M}$ in Eq. (13) is suitably chosen. To determine $\boldsymbol{M}$, we take the Cartesian definition

$$(\boldsymbol{a}\times\boldsymbol{b})^i = \epsilon_{ijk}a^j b^k \tag{21}$$

[1] The tau matrices defined by Oppenheimer [5] are $\boldsymbol{N}^\dagger\tau^i\boldsymbol{N}$, where $\boldsymbol{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. [The minus sign in τ^2 in Eq. (10) of that paper apparently is a typographical error, as indicated by inspection of Eq. (11).] The matrices defined by Majorana [16] are $\boldsymbol{M}^\dagger\tau^i\boldsymbol{M}$, where $\boldsymbol{M}$ is given in Eq. (23).

and write Eq. (20) as

$$\boldsymbol{\tau}\cdot\boldsymbol{a}\,\boldsymbol{M}\,\boldsymbol{b}_{\rm c} \;=\; {\rm i}\boldsymbol{M}(\boldsymbol{a}\times\boldsymbol{b})_{\rm c}. \tag{22}$$

Imposing the requirement that this equation be valid for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ fixes $\boldsymbol{M}$, up to a phase factor, to be

$$\boldsymbol{M} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & {\rm i} & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & {\rm i} & 0 \end{pmatrix}, \tag{23}$$

which yields

$$\boldsymbol{a}_{\rm s} \;=\; \begin{pmatrix} -\frac{1}{\sqrt{2}}(a^1 - {\rm i}\,a^2) \\ a^3 \\ \frac{1}{\sqrt{2}}(a^1 + {\rm i}\,a^2) \end{pmatrix}. \tag{24}$$

Consequences of Eq. (20) are

$$\boldsymbol{\tau}\cdot\boldsymbol{a}\;\boldsymbol{a}_{\rm s} \;=\; 0, \tag{25}$$

$$\boldsymbol{\tau}\cdot\boldsymbol{a}\;\boldsymbol{b}_{\rm s} + \boldsymbol{\tau}\cdot\boldsymbol{b}\;\boldsymbol{a}_{\rm s} \;=\; 0, \tag{26}$$

$$\boldsymbol{a}_{\rm s}^\dagger\,\boldsymbol{\tau}\cdot\boldsymbol{b}\;\boldsymbol{c}_{\rm s} \;=\; {\rm i}\,\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}), \tag{27}$$

$$\begin{aligned}(\boldsymbol{\tau}\cdot\boldsymbol{a}\,\boldsymbol{c}_{\rm s})\cdot(\boldsymbol{\tau}\cdot\boldsymbol{b}\,\boldsymbol{d}_{\rm s}) \;&=\; (\boldsymbol{a}\times\boldsymbol{c})\cdot(\boldsymbol{b}\times\boldsymbol{d}) \\ &=\; \boldsymbol{a}\cdot\boldsymbol{b}\;\boldsymbol{c}\cdot\boldsymbol{d} - \boldsymbol{a}\cdot\boldsymbol{d}\;\boldsymbol{c}\cdot\boldsymbol{b}.\end{aligned} \tag{28}$$

Equation (28) can be written as

$$\boldsymbol{c}_{\rm s}^\dagger(\boldsymbol{\tau}\cdot\boldsymbol{a})^\dagger\boldsymbol{\tau}\cdot\boldsymbol{b}\,\boldsymbol{d}_{\rm s} \;=\; \boldsymbol{c}_{\rm s}^\dagger(\boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{b}_{\rm s}\,\boldsymbol{a}_{\rm s}^\dagger)\boldsymbol{d}_{\rm s} \tag{29}$$

for any vectors $\boldsymbol{c}$ and $\boldsymbol{d}$, which yields the relation

$$(\boldsymbol{\tau}\cdot\boldsymbol{a})^\dagger\boldsymbol{\tau}\cdot\boldsymbol{b} \;=\; \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{b}_{\rm s}\boldsymbol{a}_{\rm s}^\dagger, \tag{30}$$

where it is understood that the first term on the right includes the 3×3 identity matrix as a factor and the second term is also a 3×3 matrix. If $\boldsymbol{a}_{\rm c}$ has real components, then

$$(\boldsymbol{\tau}\cdot\boldsymbol{a})^\dagger \;=\; \boldsymbol{\tau}\cdot\boldsymbol{a}, \tag{31}$$

$$(\boldsymbol{\tau}\cdot\boldsymbol{a})^3 \;=\; \boldsymbol{a}^2\;\boldsymbol{\tau}\cdot\boldsymbol{a}, \tag{32}$$

where $\boldsymbol{a}^2 = \boldsymbol{a}\cdot\boldsymbol{a}$ is the ordinary real vector scalar product.

Real Cartesian tau matrices $\tilde{\boldsymbol{\tau}}$ may be defined so that

$$\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{a}\;\boldsymbol{b}_{\rm c} \;=\; (\boldsymbol{a}\times\boldsymbol{b})_{\rm c}\,. \tag{33}$$

This relation follows from Eqs. (20) and (13) with the definition

$$\tilde{\tau}^i \;=\; -\mathrm{i}\boldsymbol{M}^\dagger \tau^i \boldsymbol{M}, \quad i = 1,2,3. \tag{34}$$

These matrices are antisymmetric $\tilde{\boldsymbol{\tau}}^\top = -\tilde{\boldsymbol{\tau}}$, where $\top$ denotes matrix transposition, in contrast to $\boldsymbol{\tau}^\dagger = \boldsymbol{\tau}$. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with real Cartesian components, we have

$$\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{a}\;\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{b} \;=\; \boldsymbol{b}_\mathrm{c}\boldsymbol{a}_\mathrm{c}^\top - \boldsymbol{a}\cdot\boldsymbol{b}, \tag{35}$$

$$(\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{a})^3 \;=\; -\boldsymbol{a}^2\,(\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{a}), \tag{36}$$

$$(\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{a})^{ij} \;=\; -\epsilon_{ijk}a^k. \tag{37}$$

The matrix

$$\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{a} \;=\; \begin{pmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{pmatrix} \tag{38}$$

has the form of the lower right portion of the electromagnetic field-strength tensor $F^{\mu\nu}$, as given in [17] for example.

4. Matrix Maxwell equation

In terms of the notation of the previous section, the matrix forms of the Maxwell equations in (2) and (3), for the source-free case ($\boldsymbol{J} = 0$), are

$$\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{B}_\mathrm{s} + \frac{1}{c}\frac{\partial\boldsymbol{E}_\mathrm{s}}{\partial ct} \;=\; 0, \tag{39}$$

$$\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{E}_\mathrm{s} - c\frac{\partial\boldsymbol{B}_\mathrm{s}}{\partial ct} \;=\; 0. \tag{40}$$

These equations may be written as two uncoupled equations

$$\left(\boldsymbol{I}\frac{\partial}{\partial ct} + \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{E}_\mathrm{s} + \mathrm{i}\,c\boldsymbol{B}_\mathrm{s}) = 0, \tag{41}$$

$$\left(\boldsymbol{I}\frac{\partial}{\partial ct} - \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{E}_\mathrm{s} - \mathrm{i}\,c\boldsymbol{B}_\mathrm{s}) = 0, \tag{42}$$

where $\boldsymbol{I}$ is the 3×3 identity matrix. In the Cartesian basis, Eqs. (41) and (42) are

$$\left(\boldsymbol{I}\frac{\partial}{\partial ct} + \mathrm{i}\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{E}_\mathrm{c} + \mathrm{i}\,c\boldsymbol{B}_\mathrm{c}) = 0, \tag{43}$$

$$\left(\boldsymbol{I}\frac{\partial}{\partial ct} - \mathrm{i}\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{E}_\mathrm{c} - \mathrm{i}\,c\boldsymbol{B}_\mathrm{c}) = 0. \tag{44}$$

In these expressions, it is evident that for real electric and magnetic fields, Eqs. (43) and (44) are complex conjugates of each other and reduce to a single complex equation. It was recognized in lectures by Riemann in the nineteenth century that this complex combination of $\boldsymbol{E}$ and $\boldsymbol{B}$ is a solution of a single equation [18]. This fact was also discussed in [19, 20] and is included in many works up to the present. Equations (43) and (44) may be interpreted as Maxwell equations for right- and left-circularly polarized radiation, analogous to the Weyl equations for right- and left-handed neutrino fields [8, 14].

However, in this paper, we consider the more restrictive case of complex electric and magnetic fields that are simultaneously solutions of both Eqs (43) and (44), or equivalently both Eqs. (39) and (40), for any polarization of radiation. The question of whether such solutions can be found is answered by their explicit construction in subsequent sections of the paper. To formulate this approach, we follow the Dirac equation and write

$$\begin{pmatrix} \boldsymbol{I}\,\dfrac{\partial}{\partial ct} & \boldsymbol{\tau}\cdot\boldsymbol{\nabla} \\ -\boldsymbol{\tau}\cdot\boldsymbol{\nabla} & -\boldsymbol{I}\,\dfrac{\partial}{\partial ct} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_{\rm s} \\ {\rm i}\,c\boldsymbol{B}_{\rm s} \end{pmatrix} = 0, \qquad (45)$$

which is a restatement of Eqs. (39) and (40) in the form of the Dirac equation for an electron wave function. It is a matrix equation with six components that may be viewed as a single equation equivalent to Eqs. (39) and (40) for any polarization of the fields. Any complex solution of Eq. (45) is a solution of both Eqs. (41) and (42). Similar wave functions have been discussed in [21–23]. It should be noted that this formulation is different from the six-component form considered by Oppenheimer in which the upper-three components and lower-three components represent opposite helicity states [5].

If we define 6×6 gamma matrices by

$$\gamma^0 = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} \boldsymbol{0} & \tau^i \\ -\tau^i & \boldsymbol{0} \end{pmatrix}, \ i = 1, 2, 3, \qquad (46)$$

where $\boldsymbol{0}$ is the 3×3 zero matrix, and write

$$\mathit{\Psi}(x) \ = \ \begin{pmatrix} \boldsymbol{E}_{\rm s}(x) \\ {\rm i}\,c\boldsymbol{B}_{\rm s}(x) \end{pmatrix}, \qquad (47)$$

then Eq. (45) takes the covariant Dirac equation form

$$\gamma^\mu \partial_\mu \mathit{\Psi}(x) \ = \ 0, \qquad (48)$$

which provides a concise expression for two of the Maxwell equations. We can also write this as

$$\overline{\mathit{\Psi}}(x)\overleftarrow{\partial}_\mu \gamma^\mu \ = \ 0, \qquad (49)$$

where $\overline{\mathit{\Psi}}(x) = \mathit{\Psi}^\dagger(x)\,\gamma^0$ and $\overleftarrow{\partial}_\mu$ denotes differentiation of the function to the left. Although these equations are simply algebraic rearrangements of the two Maxwell equations, the resemblance to the Dirac equation and wave function is suggestive of a form that photon wave functions might take.

It is of interest to note that for solutions of the Dirac equation for the hydrogen atom, the lower two components are small and approach zero in the nonrelativistic limit, *i.e.,* as the velocity of the bound electron approaches zero. Similarly, for local electromagnetic fields generated by moving charges, the magnetic field, given by the lower three components of $\mathit{\Psi}$, also approaches zero in the limit as the velocity of the charges approaches zero.

To take source currents into account, Eq. (2) is written as

$$\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{B}_{\mathrm{s}}+\frac{1}{c}\frac{\partial\boldsymbol{E}_{\mathrm{s}}}{\partial ct} \;=\; -\mu_0\boldsymbol{J}_{\mathrm{s}}, \tag{50}$$

and a source term $\mathit{\Xi}$ is defined to be

$$\mathit{\Xi}(x) \;=\; \left(\begin{array}{c} -\mu_0 c\boldsymbol{J}_{\mathrm{s}}(x) \\ \boldsymbol{0} \end{array}\right), \tag{51}$$

where $\boldsymbol{0}$ is a 3×1 matrix of zeros. This yields the expressions

$$\gamma^\mu\partial_\mu\mathit{\Psi}(x) \;=\; \mathit{\Xi}(x) \tag{52}$$

and

$$\overline{\mathit{\Psi}}(x)\overleftarrow{\partial}_\mu\gamma^\mu \;=\; \overline{\mathit{\Xi}}(x), \tag{53}$$

either of which is referred to as the Maxwell equation here. The source term in Eq. (52) or (53) represents a fundamental difference between the Dirac equation and the Maxwell equation, as mentioned in Sec. 1 [5].

In this framework, an energy-momentum density operator is

$$p^\mu \;=\; \frac{\epsilon_0}{2c}\,\gamma^\mu, \tag{54}$$

which gives

$$\overline{\mathit{\Psi}}\,cp^0\,\mathit{\Psi} \;=\; \frac{1}{2}\left(\epsilon_0|\boldsymbol{E}|^2+\frac{1}{\mu_0}|\boldsymbol{B}|^2\right)=u, \tag{55}$$

$$\begin{aligned}\overline{\mathit{\Psi}}\,\boldsymbol{p}\,\mathit{\Psi} \;&=\; \frac{\mathrm{i}\,\epsilon_0}{2}\left(\boldsymbol{E}_{\mathrm{s}}^\dagger\boldsymbol{\tau}\boldsymbol{B}_{\mathrm{s}}-\boldsymbol{B}_{\mathrm{s}}^\dagger\boldsymbol{\tau}\boldsymbol{E}_{\mathrm{s}}\right)\\ &=\; \frac{1}{c^2\mu_0}\,\mathrm{Re}\,\boldsymbol{E}\times\boldsymbol{B}^*=\boldsymbol{g}.\end{aligned} \tag{56}$$

Eqs. (52) and (53) imply that

$$\partial_\mu\overline{\mathit{\Psi}}(x)\gamma^\mu\,\mathit{\Psi}(x) \;=\; \overline{\mathit{\Xi}}(x)\,\mathit{\Psi}(x)+\overline{\mathit{\Psi}}(x)\,\mathit{\Xi}(x), \tag{57}$$

which is a complex form of the Poynting theorem [see Eq. (14)]

$$\frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{S}=-\mathrm{Re}\,\boldsymbol{E}\cdot\boldsymbol{J}, \tag{58}$$

where

$$\boldsymbol{S}=c^2\boldsymbol{g}, \tag{59}$$

which gives the conventional result if the fields and current are real [17].

5. Transverse and longitudinal fields

To make a Helmholtz decomposition of electromagnetic fields expressed in matrix form into transverse and longitudinal components, we define 3×3 matrix transverse and longitudinal Hermitian projection operators $\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a})$ and $\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a})$ to be

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a}) = \frac{(\boldsymbol{\tau}\cdot\boldsymbol{a})^{\dagger}(\boldsymbol{\tau}\cdot\boldsymbol{a})}{\boldsymbol{a}\cdot\boldsymbol{a}}, \tag{60}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a}) = \frac{\boldsymbol{a}_{\mathrm{s}}\boldsymbol{a}_{\mathrm{s}}^{\dagger}}{\boldsymbol{a}\cdot\boldsymbol{a}}. \tag{61}$$

Based on identities in Sec. 3, these operators have the following properties:

$$\left[\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a})\right]^{2} = \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a}), \tag{62}$$

$$\left[\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a})\right]^{2} = \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a}), \tag{63}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a}) + \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a}) = \boldsymbol{I}, \tag{64}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a})\,\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a}) = 0, \tag{65}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a})\,\boldsymbol{a}_{\mathrm{s}} = 0, \tag{66}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a})\,\boldsymbol{a}_{\mathrm{s}} = \boldsymbol{a}_{\mathrm{s}}. \tag{67}$$

Acting on the matrix of an arbitrary vector $\boldsymbol{b}$, the operators project the components perpendicular to and parallel to the argument $\boldsymbol{a}$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{a})\,\boldsymbol{b}_{\mathrm{s}} = \boldsymbol{b}_{\mathrm{s}} - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\,\boldsymbol{a}_{\mathrm{s}}, \tag{68}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{a})\,\boldsymbol{b}_{\mathrm{s}} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\,\boldsymbol{a}_{\mathrm{s}}. \tag{69}$$

In addition to these algebraic relations, usefulness of the projection operators arises from an extension to include differential and integral operations acting on coordinate-space functions. Formally, we write

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) = \frac{(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})^{2}}{\boldsymbol{\nabla}^{2}}, \tag{70}$$

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla}) = \frac{\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{\nabla}_{\mathrm{s}}^{\dagger}}{\boldsymbol{\nabla}^{2}}, \tag{71}$$

which takes into account that fact that $\boldsymbol{\nabla}$ has real Cartesian components (in the sense that they give real values when acting on a real function). The inverse Laplacian is defined by the relation

$$\frac{1}{\boldsymbol{\nabla}^{2}}\,f(\boldsymbol{x}) = -\frac{1}{4\pi}\int \mathrm{d}\,\boldsymbol{x}'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,f(\boldsymbol{x}'), \tag{72}$$

which yields

$$\boldsymbol{\nabla}^{2}\frac{1}{\boldsymbol{\nabla}^{2}}\,f(\boldsymbol{x}) = -\frac{1}{4\pi}\int \mathrm{d}\,\boldsymbol{x}'\,\boldsymbol{\nabla}^{2}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,f(\boldsymbol{x}') = f(\boldsymbol{x}), \tag{73}$$

based on

$$\boldsymbol{\nabla}^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = -4\pi\,\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{74}$$

where

$$\delta(\boldsymbol{x}-\boldsymbol{x}') = \delta(x^1-x'^1)\,\delta(x^2-x'^2)\,\delta(x^3-x'^3). \tag{75}$$

Equation (73) indicates that the Laplacian operator follows analogs of the rules of algebra in this context. For example, we have

$$\begin{aligned}\left[\boldsymbol{\Pi}_{\rm s}^{\rm T}(\boldsymbol{\nabla})\right]^2 &= \left[\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})^2}{\boldsymbol{\nabla}^2}\right]^2 = \frac{\boldsymbol{\nabla}^2\,(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})^2}{\boldsymbol{\nabla}^2\boldsymbol{\nabla}^2}\\ &= \frac{(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})^2}{\boldsymbol{\nabla}^2} = \boldsymbol{\Pi}_{\rm s}^{\rm T}(\boldsymbol{\nabla})\,,\end{aligned} \tag{76}$$

where either simply canceling the $\boldsymbol{\nabla}^2$ factors in the numerator and denominator or applying the definition in Eq. (72) for the operators acting on a suitable function gives the same result. Transverse and longitudinal components of the electric and magnetic fields and the current density are identified by writing

$$\boldsymbol{F}_{\rm s} = \boldsymbol{F}_{\rm s}^{\rm T} + \boldsymbol{F}_{\rm s}^{\rm L}, \tag{77}$$

where

$$\boldsymbol{F}_{\rm s}^{\rm T} = \boldsymbol{\Pi}_{\rm s}^{\rm T}(\boldsymbol{\nabla})\boldsymbol{F}_{\rm s}, \tag{78}$$

$$\boldsymbol{F}_{\rm s}^{\rm L} = \boldsymbol{\Pi}_{\rm s}^{\rm L}(\boldsymbol{\nabla})\boldsymbol{F}_{\rm s}, \tag{79}$$

and $\boldsymbol{F}_{\rm s}$ may be any of $\boldsymbol{E}_{\rm s}$, $\boldsymbol{B}_{\rm s}$, or $\boldsymbol{J}_{\rm s}$.

The separation of the Maxwell equations into two independent sets of equations which involve either transverse components or longitudinal components takes the following form. In terms of the spherical matrices, Eq. (1) is

$$\boldsymbol{\nabla}_{\rm s}^{\dagger}\boldsymbol{E}_{\rm s}^{\rm L} = \frac{\rho}{\epsilon_0}, \tag{80}$$

where the transverse component of the electric field is absent, because $\boldsymbol{\nabla}_{\rm s}^{\dagger}\boldsymbol{\Pi}_{\rm s}^{\rm T}(\boldsymbol{\nabla}) = 0$. Equations (1) and (80) are equivalent, in the sense that each can be derived from the other; Eq. (1) follows from Eq. (80) if the vanishing transverse component is added to the latter equation. In the separated form, it is evident that the equation neither contains information about or places any constraint on the transverse component $\boldsymbol{E}_{\rm s}^{\rm T}$. The transverse and longitudinal projection operators acting on Eq. (50), the matrix form of Eq. (2), yield

$$\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{B}_{\rm s}^{\rm T} + \frac{1}{c}\frac{\partial \boldsymbol{E}_{\rm s}^{\rm T}}{\partial ct} = -\mu_0\boldsymbol{J}_{\rm s}^{\rm T}, \tag{81}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}_{\rm s}^{\rm L}}{\partial ct} = -\mu_0\boldsymbol{J}_{\rm s}^{\rm L} \tag{82}$$

respectively, which take into account the commutation relation $[\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}), \boldsymbol{\tau}\cdot\boldsymbol{\nabla}] = 0$ and the fact that $\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla})\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla} = 0$. Together, these equations are equivalent to Eq. (50) which can be restored by writing the sum of Eq. (81) and Eq. (82) and adding the term that vanishes. Evidently, this pair of equations is independent of $\boldsymbol{B}_{\mathrm{s}}^{\mathrm{L}}$. Similarly, Eq. (3), or equivalently Eq. (40), can be written as the pair

$$\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{E}_{\mathrm{s}}^{\mathrm{T}} - c\frac{\partial \boldsymbol{B}_{\mathrm{s}}^{\mathrm{T}}}{\partial ct} = 0, \tag{83}$$

$$\frac{\partial \boldsymbol{B}_{\mathrm{s}}^{\mathrm{L}}}{\partial ct} = 0, \tag{84}$$

which are independent of $\boldsymbol{E}_{\mathrm{s}}^{\mathrm{L}}$. Equation (4) takes the form

$$\boldsymbol{\nabla}_{\mathrm{s}}^{\dagger}\boldsymbol{B}_{\mathrm{s}}^{\mathrm{L}} = 0, \tag{85}$$

independent of $\boldsymbol{B}_{\mathrm{s}}^{\mathrm{T}}$. The transverse and longitudinal equations comprise two independent sets.

Six-dimensional transverse and longitudinal projection operators are defined by

$$\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}) = \begin{pmatrix} \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) \end{pmatrix}, \tag{86}$$

$$\mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}) = \begin{pmatrix} \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla}) \end{pmatrix}, \tag{87}$$

where $\mathbf{0}$ is the 3×3 matrix of zeros, $\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}) + \mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}) = \mathcal{I}$, and $\mathcal{I}$ is the 6×6 identity matrix. The transverse equations are summarized by writing

$$\gamma^{\mu}\partial_{\mu}\mathit{\Psi}^{\mathrm{T}}(x) = \mathit{\Xi}^{\mathrm{T}}(x), \tag{88}$$

where

$$\mathit{\Psi}^{\mathrm{T}}(x) = \mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla})\,\mathit{\Psi}(x), \tag{89}$$

$$\mathit{\Xi}^{\mathrm{T}}(x) = \mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla})\,\mathit{\Xi}(x). \tag{90}$$

Equation (88) also follows directly from Eq. (52) and the fact that $\left[\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}), \gamma^{\mu}\partial_{\mu}\right] = 0$. The longitudinal equations are Eqs. (80), (82), (84), and (85), together with the continuity equation, Eq. (5), which can be expressed as

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}_{\mathrm{s}}^{\dagger}\boldsymbol{J}_{\mathrm{s}}^{\mathrm{L}} = 0. \tag{91}$$

Since the continuity equation follows from Eqs. (80) and (82), it is not necessary to include it in an independent set of equations; it is listed here only to show that it provides no restriction on $\boldsymbol{J}_{\mathrm{s}}^{\mathrm{T}}$. Equations (84) and (85) are eliminated from consideration by taking

$$\boldsymbol{B}_{\mathrm{s}}^{\mathrm{L}} = 0. \tag{92}$$

Constant fields are eliminated by the requirement that static fields vanish at infinite distances for finite source distributions. However, a constant magnetic field may be approximated by the field at the center of a current loop with a radius that is large compared to the extent of the region of interest. Such a steady-state current density is transverse, as shown by Eq. (82), and so the magnetic field, given by Eq. (81), is also transverse, which is consistent with Eq. (92). A complete set of equations, equivalent to the set of Maxwell equations, is provided by Eqs. (52), (80), and (92), and the transverse fields are completely described by Eq. (88).

6. Lorentz transformations

Lorentz transformations of the matrix Maxwell equation are examined here in order to confirm that this form of the Maxwell equations is Lorentz invariant. We adopt the convention that transformations apply to the physical system rather than to the observer's coordinates.

To represent four-vector coordinates, the Cartesian matrices are extended to include a time component $x^0 = ct$, so coordinate vectors take the form

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \boldsymbol{x}_{\rm c} \end{pmatrix}. \tag{93}$$

We employ the Cartesian basis for coordinate and momentum vectors and the spherical basis for fields and currents, with a few exceptions that will be apparent. The notation x represents either the four-coordinate argument of a function or a column matrix, depending on the context. It is sufficient for our purpose to consider only homogeneous Lorentz transformations and to consider rotations and velocity transformations separately. These transformations acting on four-vectors leave the scalar product

$$x \cdot x \;=\; x^\top g\, x = (ct)^2 - \boldsymbol{x}^2 \tag{94}$$

invariant, where g is the metric tensor given by

$$g = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{I} \end{pmatrix}. \tag{95}$$

A remark on notation is that a boldface $\mathbf{0}$ means either a 3×3, a 1×3, or a 3×1 rectangular array of zeros, as appropriate. We take the liberty of using an ordinary zero on the right-hand side of equations to mean whatever sort of zero matches the left-hand side.

6.1. Rotation of coordinates

Rotations are parameterized by a vector $\boldsymbol{u} = \theta\hat{\boldsymbol{u}}$, where $\hat{\boldsymbol{u}}$ is a unit vector in the direction of the axis of the rotation and θ is the angle of rotation. An infinitesimal rotation $\delta\theta\,\hat{\boldsymbol{u}}$ changes the point at position $\boldsymbol{x}$ to the point at position $\boldsymbol{x}'$, where

$$\boldsymbol{x}' \;=\; \boldsymbol{x} + \delta\theta\,\hat{\boldsymbol{u}} \times \boldsymbol{x} + \dots\;, \tag{96}$$

or

$$\boldsymbol{x}'_{\rm c} \;=\; (\boldsymbol{I} + \delta\theta\, \tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}})\, \boldsymbol{x}_{\rm c} + \ldots\ . \tag{97}$$

For a finite rotation, the operation is exponentiated to give

$$\boldsymbol{x}'_{\rm c} \;=\; {\rm e}^{\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{u}} \boldsymbol{x}_{\rm c} = \boldsymbol{R}_{\rm c}(\boldsymbol{u})\, \boldsymbol{x}_{\rm c}. \tag{98}$$

Expansion of the exponential function in powers of θ, taking into account the fact that $(\tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}})^3 = -\tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}}$, yields

$$\begin{aligned} \boldsymbol{R}_{\rm c}(\boldsymbol{u}) &= \boldsymbol{I} + (\tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}})^2 \,(1 - \cos\theta) + \tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}}\, \sin\theta \\ &= \hat{\boldsymbol{u}}_{\rm c} \hat{\boldsymbol{u}}_{\rm c}^{\top} - (\tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}})^2\, \cos\theta + \tilde{\boldsymbol{\tau}} \cdot \hat{\boldsymbol{u}}\, \sin\theta. \end{aligned} \tag{99}$$

Evidently, $\boldsymbol{R}_{\rm c}^{-1}(\boldsymbol{u}) = \boldsymbol{R}_{\rm c}(-\boldsymbol{u}) = \boldsymbol{R}_{\rm c}^{\top}(\boldsymbol{u})$. It is confirmed that this operator has the appropriate action on a vector by calculating

$$\boldsymbol{x}' \;=\; \hat{\boldsymbol{u}}\, \hat{\boldsymbol{u}} \cdot \boldsymbol{x} - \hat{\boldsymbol{u}} \times (\hat{\boldsymbol{u}} \times \boldsymbol{x}) \cos\theta + \hat{\boldsymbol{u}} \times \boldsymbol{x} \sin\theta. \tag{100}$$

We use the notation

$$\boldsymbol{x}' = \boldsymbol{R}(\boldsymbol{u})\boldsymbol{x} \tag{101}$$

to represent the transformation in Eq. (100).

Rotations of a four-vector only change the spatial coordinates and are written as

$$x' \;=\; R(\boldsymbol{u})\, x = \begin{pmatrix} ct \\ \boldsymbol{R}_{\rm c}(\boldsymbol{u})\, \boldsymbol{x}_{\rm c} \end{pmatrix}, \tag{102}$$

where

$$R(\boldsymbol{u}) \;=\; \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_{\rm c}(\boldsymbol{u}) \end{pmatrix}. \tag{103}$$

The scalar product $x \cdot x$ is invariant under rotations, since $\boldsymbol{x}^2$ is invariant.

The spatial coordinate rotation operator in the spherical basis, which follows from

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u}) \;=\; \boldsymbol{M}\boldsymbol{R}_{\rm c}(\boldsymbol{u})\boldsymbol{M}^{\dagger}, \tag{104}$$

is

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u}) \;=\; {\rm e}^{-{\rm i}\boldsymbol{\tau}\cdot\boldsymbol{u}} = \hat{\boldsymbol{u}}_{\rm s}\hat{\boldsymbol{u}}_{\rm s}^{\dagger} + (\boldsymbol{\tau} \cdot \hat{\boldsymbol{u}})^2\, \cos\theta - {\rm i}\,\boldsymbol{\tau} \cdot \hat{\boldsymbol{u}}\, \sin\theta, \tag{105}$$

and $\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u}) = \boldsymbol{R}_{\rm s}(-\boldsymbol{u}) = \boldsymbol{R}_{\rm s}^{\dagger}(\boldsymbol{u})$. Starting from the geometrical constraint that the rotated cross product of two vectors is the cross product of the rotated vectors, written as

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u})(\boldsymbol{a} \times \boldsymbol{b})_{\rm s} \;=\; (\boldsymbol{a}' \times \boldsymbol{b}')_{\rm s}, \tag{106}$$

we have

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{a}\,\boldsymbol{b}_{\rm s} \;=\; \boldsymbol{\tau}\cdot\boldsymbol{a}'\,\boldsymbol{b}'_{\rm s} = \boldsymbol{\tau}\cdot\boldsymbol{a}'\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{b}_{\rm s}. \tag{107}$$

Since this relation holds for any vector $\boldsymbol{b}$, it yields

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{a} \;=\; \boldsymbol{\tau}\cdot\boldsymbol{a}'\boldsymbol{R}_{\rm s}(\boldsymbol{u}) \tag{108}$$

and

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{a}\,\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u}) \;=\; \boldsymbol{\tau}\cdot\boldsymbol{a}'. \tag{109}$$

A direct calculation provides the same result.

The relation between the rotated and the unrotated gradient operators is given by

$$\nabla'^i \;=\; \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\,\frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial x'^i}\,\nabla^j, \tag{110}$$

where, from Eq. (98), we have

$$\frac{\partial x^j}{\partial x'^i} \;=\; \boldsymbol{R}_{{\rm c}\,ji}^{-1}(\boldsymbol{u}) = \boldsymbol{R}_{{\rm c}\,ij}(\boldsymbol{u}), \tag{111}$$

so that

$$\boldsymbol{\nabla}'_{\rm c} \;=\; \boldsymbol{R}_{\rm c}(\boldsymbol{u})\,\boldsymbol{\nabla}_{\rm c}, \tag{112}$$

and from Eq. (104),

$$\boldsymbol{\nabla}'_{\rm s} \;=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\nabla}_{\rm s}. \tag{113}$$

Since the spherical gradient operator transforms as a spherical vector, we also have

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u}) \;=\; \boldsymbol{\tau}\cdot\boldsymbol{\nabla}' \tag{114}$$

from Eq. (109). Equations (113) and (114) imply that transverse and longitudinal projection operators transform according to

$$\boldsymbol{\mathit{\Pi}}_{\rm s}^{\rm T}(\boldsymbol{\nabla}') \;=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\mathit{\Pi}}_{\rm s}^{\rm T}(\boldsymbol{\nabla})\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u}), \tag{115}$$
$$\boldsymbol{\mathit{\Pi}}_{\rm s}^{\rm L}(\boldsymbol{\nabla}') \;=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\mathit{\Pi}}_{\rm s}^{\rm L}(\boldsymbol{\nabla})\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u}). \tag{116}$$

The action of the inverse Laplacian in terms of the rotated coordinates is the same as it is for unrotated coordinates, which follows either because $\nabla'^2 = \nabla^2$ from Eq. (113) or by the definition in Eq. (72), taking into account the fact that the Jacobian for a rotation is unity.

6.2. Velocity transformation of coordinates

Velocity transformations are parameterized by a velocity vector $\boldsymbol{v} = c \tanh\zeta\, \hat{\boldsymbol{v}}$. If a space-time point is given an infinitesimal velocity boost of $\delta\zeta\, c\, \hat{\boldsymbol{v}}$, its spatial coordinate will change to

$$\boldsymbol{x}' = \boldsymbol{x} + \delta\zeta\, ct\, \hat{\boldsymbol{v}} + \dots\ , \tag{117}$$

and its time coordinate must transform in such a way that the scalar product is invariant. In particular, we require $x' \cdot x' = x \cdot x$, which yields

$$ct' = ct + \delta\zeta\, \hat{\boldsymbol{v}} \cdot \boldsymbol{x} + \dots\ . \tag{118}$$

The complete infinitesimal transformation is

$$\begin{pmatrix} ct' \\ \boldsymbol{x}'_{\mathrm{c}} \end{pmatrix} = \left[I + \delta\zeta \begin{pmatrix} 0 & \hat{\boldsymbol{v}}_{\mathrm{c}}^{\top} \\ \hat{\boldsymbol{v}}_{\mathrm{c}} & \boldsymbol{0} \end{pmatrix} \right] \begin{pmatrix} ct \\ \boldsymbol{x}_{\mathrm{c}} \end{pmatrix} + \dots\ . \tag{119}$$

This may be written in terms of a 4×4 matrix valued function of the velocity direction:

$$K(\hat{\boldsymbol{v}}) = \begin{pmatrix} 0 & \hat{\boldsymbol{v}}_{\mathrm{c}}^{\top} \\ \hat{\boldsymbol{v}}_{\mathrm{c}} & \boldsymbol{0} \end{pmatrix}, \tag{120}$$

for which

$$K^2(\hat{\boldsymbol{v}}) = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{v}}_{\mathrm{c}} \hat{\boldsymbol{v}}_{\mathrm{c}}^{\top} \end{pmatrix} \tag{121}$$

and $K^3(\hat{\boldsymbol{v}}) = K(\hat{\boldsymbol{v}})$. For a finite velocity, the transformation is exponentiated to give

$$x' = \mathrm{e}^{\zeta K(\hat{\boldsymbol{v}})}\, x = V(\boldsymbol{v})\, x. \tag{122}$$

Expansion in powers of ζ yields

$$\begin{aligned} V(\boldsymbol{v}) &= I + K^2(\hat{\boldsymbol{v}})(\cosh\zeta - 1) + K(\hat{\boldsymbol{v}}) \sinh\zeta \\ &= \begin{pmatrix} \cosh\zeta & \hat{\boldsymbol{v}}_{\mathrm{c}}^{\top} \sinh\zeta \\ \hat{\boldsymbol{v}}_{\mathrm{c}} \sinh\zeta & \boldsymbol{I} + \hat{\boldsymbol{v}}_{\mathrm{c}} \hat{\boldsymbol{v}}_{\mathrm{c}}^{\top} (\cosh\zeta - 1) \end{pmatrix}. \end{aligned} \tag{123}$$

The relations $V^{\top}(\boldsymbol{v}) = V(\boldsymbol{v})$ and $gV(\boldsymbol{v}) = V^{-1}(\boldsymbol{v})g$ confirm the invariance of the scalar product:

$$x' \cdot x' = x^{\top} V^{\top}(\boldsymbol{v})\, g\, V(\boldsymbol{v})\, x = x \cdot x. \tag{124}$$

The transformation yields

$$ct' = ct\, \cosh\zeta + \hat{\boldsymbol{v}} \cdot \boldsymbol{x} \sinh\zeta, \tag{125}$$

$$\boldsymbol{x}' = \boldsymbol{x} + \hat{\boldsymbol{v}}\, \hat{\boldsymbol{v}} \cdot \boldsymbol{x}(\cosh\zeta - 1) + ct\, \hat{\boldsymbol{v}}\, \sinh\zeta. \tag{126}$$

A point with $\boldsymbol{x} = 0$ has the boosted velocity

$$\frac{\boldsymbol{x}'}{t'} = c \tanh \zeta \, \hat{\boldsymbol{v}} = \boldsymbol{v}. \tag{127}$$

The spherical counterpart of the operator $V(\boldsymbol{v})$, in the velocity transformation

$$\begin{pmatrix} ct' \\ \boldsymbol{x}'_{\mathrm{s}} \end{pmatrix} = V_{\mathrm{s}}(\boldsymbol{v}) \begin{pmatrix} ct \\ \boldsymbol{x}_{\mathrm{s}} \end{pmatrix}, \tag{128}$$

is

$$\begin{aligned} V_{\mathrm{s}}(\boldsymbol{v}) &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M} \end{pmatrix} V(\boldsymbol{v}) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^\dagger \end{pmatrix} \\ &= \begin{pmatrix} \cosh \zeta & \hat{\boldsymbol{v}}_{\mathrm{s}}^\dagger \sinh \zeta \\ \hat{\boldsymbol{v}}_{\mathrm{s}} \sinh \zeta & \boldsymbol{I} + \hat{\boldsymbol{v}}_{\mathrm{s}} \hat{\boldsymbol{v}}_{\mathrm{s}}^\dagger (\cosh \zeta - 1) \end{pmatrix}. \end{aligned} \tag{129}$$

For the four-gradient operator, we have

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu, \tag{130}$$

and from Eq. (122), which can be written as

$$x = V^{-1}(\boldsymbol{v}) \, x' = V(-\boldsymbol{v}) \, x' \tag{131}$$

or

$$x^\nu = V_{\nu\mu}(-\boldsymbol{v}) \, x'^\mu, \tag{132}$$

we also have

$$\frac{\partial x^\nu}{\partial x'^\mu} = V_{\nu\mu}(-\boldsymbol{v}) = V_{\mu\nu}(-\boldsymbol{v}), \tag{133}$$

which yields

$$\partial'_\mu = V_{\mu\nu}(-\boldsymbol{v}) \, \partial_\nu. \tag{134}$$

If a Cartesian gradient operator is defined as

$$\partial_{\mathrm{c}} = \begin{pmatrix} \dfrac{\partial}{\partial ct} \\ -\boldsymbol{\nabla}_{\mathrm{c}} \end{pmatrix}, \tag{135}$$

then Eq. (134) gives

$$g \, \partial'_{\mathrm{c}} = V(-\boldsymbol{v}) \, g \, \partial_{\mathrm{c}} \tag{136}$$

or

$$\partial'_{\mathrm{c}} = V(\boldsymbol{v}) \, \partial_{\mathrm{c}}, \tag{137}$$

since $g \, V(-\boldsymbol{v}) \, g = V(\boldsymbol{v})$.

6.3. Parity and time reversal of coordinates

Lorentz transformations that leave the scalar product in Eq. (94) invariant include the parity transformation $P = g$, time reversal $T = -g$, and total inversion $PT = -I$ operations. These transformations have the following defining effects on the coordinate vectors:

$$Px = \begin{pmatrix} ct \\ -\boldsymbol{x}_{\rm c} \end{pmatrix}, \tag{138}$$

$$Tx = \begin{pmatrix} -ct \\ \boldsymbol{x}_{\rm c} \end{pmatrix}, \tag{139}$$

$$PTx = -x. \tag{140}$$

It is sufficient for the present purpose to consider only P and T. The coordinate derivatives transform as

$$P\partial_{\rm c} = \begin{pmatrix} \frac{\partial}{\partial ct} \\ \boldsymbol{\nabla}_{\rm c} \end{pmatrix}, \tag{141}$$

$$T\partial_{\rm c} = \begin{pmatrix} -\frac{\partial}{\partial ct} \\ -\boldsymbol{\nabla}_{\rm c} \end{pmatrix}. \tag{142}$$

Comparison of Eqs. (95) and (103) shows that parity transformations commute with rotations. On the other hand, for velocity transformations, the relation

$$PV(\boldsymbol{v}) = V(-\boldsymbol{v})P \tag{143}$$

applies as it should, because the space reflection of a point moving with a velocity $\boldsymbol{v}$ is a point at the reflected position moving with a velocity $-\boldsymbol{v}$. Similar conclusions follow for time-reversal transformations.

6.4. Rotation of $\mathit{\Psi}(x)$

The result of a rotation, parameterized by the vector $\boldsymbol{u}$, applied to the field $\mathit{\Psi}(x)$ in Eq. (52) is the field $\mathit{\Psi}'(x)$ given by

$$\mathit{\Psi}'(x) = \mathcal{R}(\boldsymbol{u})\,\mathit{\Psi}(R^{-1}(\boldsymbol{u})\,x), \tag{144}$$

where $\mathcal{R}(\boldsymbol{u})$ is a 6×6 matrix that gives the local transformation of the field $\mathit{\Psi}(x)$ at any point x. The inverse transformation of the argument on the right-hand-side takes into account the fact that the transformed field at the point x originated from the field at the point that is mapped into x by the transformation. Lorentz invariance is confirmed by showing that the transformed field satisfies the same equation as the original field. We expect the current to transform in the same way as $\mathit{\Psi}$ and write

$$\mathit{\Xi}'(x) = \mathcal{R}(\boldsymbol{u})\,\mathit{\Xi}(R^{-1}(\boldsymbol{u})\,x). \tag{145}$$

The objective is to show that

$$\gamma^\mu \partial_\mu \Psi'(x) \;=\; \Xi'(x), \tag{146}$$

for a suitable transformation $\mathcal{R}(\boldsymbol{u})$. In terms of the original field and source, Eq. (146) is given by

$$\gamma^\mu \partial_\mu \mathcal{R}(\boldsymbol{u})\, \Psi(R^{-1}(\boldsymbol{u})\, x) \;=\; \mathcal{R}(\boldsymbol{u})\, \Xi(R^{-1}(\boldsymbol{u})\, x) \tag{147}$$

or

$$\gamma^\mu \partial'_\mu \mathcal{R}(\boldsymbol{u})\, \Psi(x) \;=\; \mathcal{R}(\boldsymbol{u})\, \Xi(x), \tag{148}$$

where the variable x has been replaced by $x' = R(\boldsymbol{u})\, x$. Thus Eq. (146) will follow if

$$\mathcal{R}^{-1}(\boldsymbol{u}) \gamma^\mu \partial'_\mu \mathcal{R}(\boldsymbol{u}) \;=\; \gamma^\mu \partial_\mu. \tag{149}$$

We expect $\mathcal{R}(\boldsymbol{u})$ to be of the form

$$\mathcal{R}(\boldsymbol{u}) \;=\; \left(\begin{array}{cc} \boldsymbol{R}_{\rm s}(\boldsymbol{u}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_{\rm s}(\boldsymbol{u}) \end{array} \right), \tag{150}$$

which yields

$$\mathcal{R}^{-1}(\boldsymbol{u}) \gamma^\mu \partial'_\mu \mathcal{R}(\boldsymbol{u}) \;=\; \left(\begin{array}{cc} \boldsymbol{I}\, \frac{\partial}{\partial ct} & \boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u})\, \boldsymbol{\tau} \cdot \boldsymbol{\nabla}' \boldsymbol{R}_{\rm s}(\boldsymbol{u}) \\ -\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u})\, \boldsymbol{\tau} \cdot \boldsymbol{\nabla}' \boldsymbol{R}_{\rm s}(\boldsymbol{u}) & -\boldsymbol{I}\, \frac{\partial}{\partial ct} \end{array} \right), \tag{151}$$

so Eq. (149) follows from

$$\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u})\, \boldsymbol{\tau} \cdot \boldsymbol{\nabla}' \boldsymbol{R}_{\rm s}(\boldsymbol{u}) \;=\; \boldsymbol{\tau} \cdot \boldsymbol{\nabla}, \tag{152}$$

which, in turn, follows from Eq. (114). We conclude that, as expected, the solution and source terms, transformed according to Eqs. (144) and (145), where $\mathcal{R}(\boldsymbol{u})$ is given in Eq. (150), satisfy the same equation as the original solution and source terms. The six-dimensional rotation operator $\mathcal{R}(\boldsymbol{u})$ may be written as

$$\mathcal{R}(\boldsymbol{u}) \;=\; {\rm e}^{-{\rm i}\boldsymbol{\mathcal{S}}\cdot\boldsymbol{u}}, \tag{153}$$

where

$$\boldsymbol{\mathcal{S}} \;=\; \left(\begin{array}{cc} \boldsymbol{\tau} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\tau} \end{array} \right). \tag{154}$$

Equations (144), (145), and (150) correspond to the separate equations

$$\boldsymbol{E}'_{\rm s}(x) \;=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\, \boldsymbol{E}_{\rm s}(R^{-1}(\boldsymbol{u})\, x), \tag{155}$$

$$\boldsymbol{B}'_{\rm s}(x) \;=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\, \boldsymbol{B}_{\rm s}(R^{-1}(\boldsymbol{u})\, x), \tag{156}$$

$$\boldsymbol{J}'_{\rm s}(x) \;=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\, \boldsymbol{J}_{\rm s}(R^{-1}(\boldsymbol{u})\, x). \tag{157}$$

It can be confirmed that Eqs. (1) and (4) in spherical form are invariant under rotations. In particular, Eq. (1) for the rotated electric field and charge density $\rho'(x) = \rho(R^{-1}(\boldsymbol{u})x)$ is

$$\boldsymbol{\nabla}_{\rm s}^{\dagger}\,\boldsymbol{E}_{\rm s}'(x) \;=\; \frac{\rho'(x)}{\epsilon_0} \tag{158}$$

or

$$\boldsymbol{\nabla}_{\rm s}^{\dagger}\,\boldsymbol{R}_{\rm s}(\boldsymbol{u})\boldsymbol{E}_{\rm s}(R^{-1}(\boldsymbol{u})\,x) \;=\; \frac{\rho(R^{-1}(\boldsymbol{u})\,x)}{\epsilon_0}. \tag{159}$$

The substitution $x \rightarrow R(\boldsymbol{u})\,x$ gives

$$\boldsymbol{\nabla}_{\rm s}'^{\dagger}\,\boldsymbol{R}_{\rm s}(\boldsymbol{u})\boldsymbol{E}_{\rm s}(x) \;=\; \frac{\rho(x)}{\epsilon_0}, \tag{160}$$

and Eq. (113) yields

$$\boldsymbol{\nabla}_{\rm s}^{\dagger}\,\boldsymbol{E}_{\rm s}(x) \;=\; \frac{\rho(x)}{\epsilon_0}. \tag{161}$$

Hence, the transformed field and charge density satisfy Eq. (1) if the original field and charge density do. Similarly,

$$\boldsymbol{\nabla}_{\rm s}^{\dagger}\,\boldsymbol{B}_{\rm s}'(x) \;=\; \boldsymbol{\nabla}_{\rm s}^{\dagger}\,\boldsymbol{B}_{\rm s}(x) = 0. \tag{162}$$

Thus, all of the Maxwell equations in matrix form are invariant under rotations.

The separation into transverse and longitudinal components of the electric and magnetic fields is also invariant under rotations. This can be seen by considering the expression $\boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla})\,\boldsymbol{F}_{\rm s}(x)$, where $\boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla})$ is either $\boldsymbol{\Pi}_{\rm s}^{\rm T}(\boldsymbol{\nabla})$ or $\boldsymbol{\Pi}_{\rm s}^{\rm L}(\boldsymbol{\nabla})$ and $\boldsymbol{F}_{\rm s}(x)$ is any of $\boldsymbol{E}_{\rm s}(x)$, $\boldsymbol{B}_{\rm s}(x)$, or $\boldsymbol{J}_{\rm s}(x)$. We have

$$\boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla})\,\boldsymbol{F}_{\rm s}'(x) \;=\; \boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla})\,\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{F}_{\rm s}(R^{-1}(\boldsymbol{u})\,x) \tag{163}$$

or

$$\begin{aligned}\boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla}')\,\boldsymbol{F}_{\rm s}'(x') \;&=\; \boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla}')\,\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{F}_{\rm s}(x) \\ &=\; \boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\Pi}_{\rm s}(\boldsymbol{\nabla})\,\boldsymbol{F}_{\rm s}(x),\end{aligned} \tag{164}$$

where the last line follows from either Eq. (115) or Eq. (116). This means that if the original field is transverse or longitudinal, then the rotated field has the same character. These results extend directly to the six-dimensional projection operators $\mathit{\Pi}(\boldsymbol{\nabla})$, solution $\mathit{\Psi}(x)$, and source $\mathit{\Xi}(x)$.

6.5. Velocity transformation of $\Psi(x)$

The result of the velocity transformation, by a velocity $\boldsymbol{v}$, applied to the field $\Psi(x)$ in Eq. (52) is the function $\Psi'(x)$ given by

$$\Psi'(x) \;=\; \mathcal{V}(\boldsymbol{v})\,\Psi(V^{-1}(\boldsymbol{v})\,x), \tag{165}$$

where $\mathcal{V}(\boldsymbol{u})$ is a 6×6 matrix that gives the local transformation of the field $\Psi(x)$ at any point. The inverse transformation of the argument on the right-hand-side plays the same role as for rotations. Our objective is to establish the covariance of Eq. (52) by showing that if $\Psi(x)$ is a solution of that equation with a source $\Xi(x)$, then

$$\gamma^\mu\partial_\mu\,\Psi'(x) \;=\; \Xi'(x), \tag{166}$$

where $\Xi'(x)$ is a suitably transformed source term. Equation (166) can be written as

$$\gamma^\mu\partial_\mu\mathcal{V}(\boldsymbol{v})\,\Psi(V^{-1}(\boldsymbol{v})\,x) \;=\; \Xi'(x) \tag{167}$$

or

$$\gamma^\mu\partial'_\mu\mathcal{V}(\boldsymbol{v})\,\Psi(x) \;=\; \Xi'\left(V(\boldsymbol{v})\,x\right), \tag{168}$$

where the variable x has been replaced by $x' = V(\boldsymbol{v})\,x$. The 6×6 matrix $\mathcal{V}(\boldsymbol{v})$ is based on the conventional local velocity transformation of the electric and magnetic fields as discussed in Appendix A. Here we write it as

$$\mathcal{V}(\boldsymbol{v}) \;=\; \mathrm{e}^{\zeta\boldsymbol{\mathcal{K}}\cdot\hat{\boldsymbol{v}}}, \tag{169}$$

where

$$\boldsymbol{\mathcal{K}} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{\tau} \\ \boldsymbol{\tau} & \boldsymbol{0} \end{pmatrix}. \tag{170}$$

Expansion of the exponential function in Eq. (169) in powers of ζ yields

$$\begin{aligned}\mathcal{V}(\boldsymbol{v}) \;&=\; \mathcal{I} + (\boldsymbol{\mathcal{K}}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta - 1) + (\boldsymbol{\mathcal{K}}\cdot\hat{\boldsymbol{v}})\,\sinh\zeta \\ &=\; \begin{pmatrix} \boldsymbol{I} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta - 1) & \boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\sinh\zeta \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\sinh\zeta & \boldsymbol{I} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta - 1) \end{pmatrix},\end{aligned} \tag{171}$$

where $(\boldsymbol{\mathcal{K}}\cdot\hat{\boldsymbol{v}})^3 = \boldsymbol{\mathcal{K}}\cdot\hat{\boldsymbol{v}}$ is taken into account, so that

$$\mathcal{V}(\boldsymbol{v})\,\Psi \;=\; \begin{pmatrix} \boldsymbol{E}'_\mathrm{s} \\ \mathrm{i}\,c\boldsymbol{B}'_\mathrm{s} \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}_\mathrm{s} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\boldsymbol{E}_\mathrm{s}(\cosh\zeta - 1) + \mathrm{i}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,c\boldsymbol{B}_\mathrm{s}\sinh\zeta \\ \mathrm{i}\left[c\boldsymbol{B}_\mathrm{s} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2 c\boldsymbol{B}_\mathrm{s}(\cosh\zeta - 1) - \mathrm{i}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{E}_\mathrm{s}\sinh\zeta\,\right] \end{pmatrix}. \tag{172}$$

In Eq. (168), we have

$$\gamma^\mu\partial'_\mu \;=\; \begin{pmatrix} \boldsymbol{I}\,\dfrac{\partial}{\partial ct'} & \boldsymbol{\tau}\cdot\boldsymbol{\nabla}' \\ -\boldsymbol{\tau}\cdot\boldsymbol{\nabla}' & -\boldsymbol{I}\,\dfrac{\partial}{\partial ct'} \end{pmatrix}, \tag{173}$$

where, from Eqs. (135) and (137),

$$\frac{\partial}{\partial ct'} = \cosh\zeta\,\frac{\partial}{\partial ct} - \sinh\zeta\,\hat{\boldsymbol{v}}\cdot\boldsymbol{\nabla}, \tag{174}$$

$$\boldsymbol{\nabla}' = \boldsymbol{\nabla} + (\cosh\zeta - 1)\hat{\boldsymbol{v}}\,\hat{\boldsymbol{v}}\cdot\boldsymbol{\nabla} - \sinh\zeta\,\hat{\boldsymbol{v}}\,\frac{\partial}{\partial ct}. \tag{175}$$

Multiplication of Eq. (171) by Eq. (173) yields the identity

$$\begin{aligned}\gamma^\mu\partial'_\mu\,\mathcal{V}(\boldsymbol{v}) &= \begin{pmatrix} \boldsymbol{I} + \hat{\boldsymbol{v}}_{\rm s}\hat{\boldsymbol{v}}_{\rm s}^\dagger(\cosh\zeta - 1) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} + \hat{\boldsymbol{v}}_{\rm s}\hat{\boldsymbol{v}}_{\rm s}^\dagger(\cosh\zeta - 1)\end{pmatrix}\gamma^\mu\,\partial_\mu \\ &\quad + \begin{pmatrix} -\sinh\zeta\,\hat{\boldsymbol{v}}_{\rm s}\boldsymbol{\nabla}_{\rm s}^\dagger & \boldsymbol{0} \\ \boldsymbol{0} & \sinh\zeta\,\hat{\boldsymbol{v}}_{\rm s}\boldsymbol{\nabla}_{\rm s}^\dagger\end{pmatrix}\end{aligned} \tag{176}$$

and hence

$$\begin{aligned}\gamma^\mu\partial'_\mu\mathcal{V}(\boldsymbol{v})\,\Psi(x) &= \begin{pmatrix} \boldsymbol{I} + \hat{\boldsymbol{v}}_{\rm s}\hat{\boldsymbol{v}}_{\rm s}^\dagger(\cosh\zeta - 1) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} + \hat{\boldsymbol{v}}_{\rm s}\hat{\boldsymbol{v}}_{\rm s}^\dagger(\cosh\zeta - 1)\end{pmatrix}\Xi(x) \\ &\quad + \begin{pmatrix} -\sinh\zeta\,\hat{\boldsymbol{v}}_{\rm s}\,\boldsymbol{\nabla}\cdot\boldsymbol{E}(x) \\ \mathrm{i}\sinh\zeta\,\hat{\boldsymbol{v}}_{\rm s}\,c\,\boldsymbol{\nabla}\cdot\boldsymbol{B}(x)\end{pmatrix} \\ &= \begin{pmatrix} -\mu_0 c\left[\boldsymbol{J}_{\rm s}(x) + \hat{\boldsymbol{v}}_{\rm s}\hat{\boldsymbol{v}}\cdot\boldsymbol{J}(x)(\cosh\zeta - 1) + \sinh\zeta\,\hat{\boldsymbol{v}}_{\rm s}c\rho(x)\right] \\ \boldsymbol{0}\end{pmatrix} \\ &= \begin{pmatrix} -\mu_0 c\boldsymbol{J}'_{\rm s}(x) \\ \boldsymbol{0}\end{pmatrix},\end{aligned} \tag{177}$$

where $\boldsymbol{J}'_{\rm s}(x)$ is the velocity transformed three-vector source current, and

$$\Xi'(x) = \begin{pmatrix} -\mu_0 c\boldsymbol{J}'_{\rm s}(V^{-1}(\boldsymbol{v})\,x) \\ \boldsymbol{0}\end{pmatrix}. \tag{178}$$

The result in Eq. (177) takes into account the additional two Maxwell equations in (1) and (4), besides Eqs. (2) and (3) used to construct Eq. (52). It also requires the conventional result that the current

$$\begin{pmatrix} c\rho(x) \\ \boldsymbol{J}_{\rm s}(x)\end{pmatrix} \tag{179}$$

transforms as a four-vector under the velocity transformation given by Eq. (129). Equation (177) establishes the validity of Eq. (166), provided the transformed three-vector source current is the three-vector component of the transformed four-vector current.

The covariance of Eq. (52), even though the charge density does not appear in the source term, is linked to the fact that the current density satisfies the continuity equation. Since the continuity equation follows from the Maxwell equations, it cannot be expected that consistent solutions may be found for an arbitrary four-vector current density. However, for valid sources, information about the charge density may be obtained from the three-vector current density and the continuity equation. For example, if the electric field is specified at a particular time, then the charge density at that time is known from Eq. (1) and may be determined at any other time from knowledge of the three-vector current density by use of the continuity equation. Thus the time evolution of the electromagnetic fields can be described relativistically with no reference to the charge density. The Maxwell Green function, which provides the solutions of Eq. (52), is discussed in Sec. 9.

Since $\mathcal{V}^\dagger\gamma^0\mathcal{V} = \gamma^0$ and $\mathcal{V}^\dagger\gamma^0\eta\mathcal{V} = \gamma^0\eta$, where

$$\eta = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix}, \tag{180}$$

the invariance of the quantities

$$\overline{\Psi}\,\Psi = |\boldsymbol{E}|^2 - c^2|\boldsymbol{B}|^2, \tag{181}$$

$$\overline{\Psi}\eta\,\Psi = 2\,\mathrm{i}\,c\,\mathrm{Re}\,\boldsymbol{E}\cdot\boldsymbol{B} \tag{182}$$

is evident.

6.6. Parity and time-reversal transformations of $\Psi(x)$

We expect the fields $\Psi(x)$ to transform under a parity change according to

$$\Psi'(x) \;=\; \mathcal{P}\,\Psi(\,P^{-1}\,x), \tag{183}$$

where $\mathcal{P}$ is a 6×6 matrix, and we assume that the current transforms in the same way, so that

$$\Xi'(x) \;=\; \mathcal{P}\,\Xi(\,P^{-1}\,x). \tag{184}$$

We can obtain

$$\gamma^\mu\partial_\mu\Psi'(x) \;=\; \Xi'(x), \tag{185}$$

by finding a suitable matrix $\mathcal{P}$. In terms of the original field and source, Eq. (185) is given by

$$\gamma^\mu\partial_\mu\mathcal{P}\,\Psi(P^{-1}x) \;=\; \mathcal{P}\,\Xi(P^{-1}x) \tag{186}$$

or

$$\gamma^\mu\partial'_\mu\mathcal{P}\,\Psi(x) \;=\; \mathcal{P}\,\Xi(x), \tag{187}$$

where ∂'_μ is the parity transformed derivative given by Eq. (141). Equation (185) will follow if

$$\mathcal{P}^{-1}\gamma^\mu\partial'_\mu\mathcal{P} \;=\; \gamma^\mu\partial_\mu, \tag{188}$$

which corresponds to

$$\mathcal{P}^{-1}\gamma^0\mathcal{P} = \gamma^0, \tag{189}$$

$$\mathcal{P}^{-1}\gamma^i\mathcal{P} = -\gamma^i, \quad i = 1,2,3. \tag{190}$$

Solutions of these equations are provided by

$$\mathcal{P} = \pm\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{pmatrix}. \tag{191}$$

The minus sign corresponds to the conventional choice of how classical electric and magnetic fields transform under a parity change, that is, the current and electric fields change sign and the magnetic fields do not [17].

To examine time-reversal invariance, we first consider $\Psi(x)$ as a field which is real in the Cartesian basis. In this case, the conventional use of an anti-unitary operator for time reversal is unnecessary, and the same can be expected to be true for fields expressed in the spherical basis. We write

$$\Psi'(x) = \mathcal{T}\,\Psi(T^{-1}x) \tag{192}$$

and

$$\Xi'(x) = -\mathcal{T}\,\Xi(T^{-1}x), \tag{193}$$

where $\mathcal{T}$ is a suitable 6×6 matrix. The minus sign for the current provides the result that the electric field does not change sign under time reversal and the current does. The objective is to find a matrix $\mathcal{T}$ such that

$$\gamma^\mu\partial_\mu\Psi'(x) = \Xi'(x) \tag{194}$$

or

$$\gamma^\mu\partial_\mu\mathcal{T}\,\Psi(T^{-1}x) = -\mathcal{T}\,\Xi(T^{-1}x), \tag{195}$$

and so

$$\gamma^\mu\partial'_\mu\mathcal{T}\,\Psi(x) = -\mathcal{T}\,\Xi(x), \tag{196}$$

where ∂'_μ is given by Eq. (142). Such a matrix satisfies

$$\mathcal{T}^{-1}\gamma^\mu\partial'_\mu\mathcal{T} = -\gamma^\mu\partial_\mu \tag{197}$$

or

$$\mathcal{T}^{-1}\gamma^0\mathcal{T} = \gamma^0, \tag{198}$$

$$\mathcal{T}^{-1}\gamma^i\mathcal{T} = -\gamma^i, \quad i = 1,2,3, \tag{199}$$

with a solution provided by

$$\mathcal{T} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{pmatrix}. \tag{200}$$

The matrices $\mathcal{P}$ and $\mathcal{T}$ commute with the rotation matrix, as they should, and we have the result that the matrix form of the Maxwell equations is invariant under parity and time-reversal transformations.

For quantum-mechanical time reversal, the time-reversal operator is anti-unitary and includes complex conjugation, or Hermitian conjugation in the case of a matrix solution, which has the effect of interchanging initial and final states. For an example where such an interchange corresponds to observable consequences in QED, see [24, 25]. We thus write

$$\Psi'(x) = \mathfrak{T}\Psi(T^{-1}x) = \Psi^\dagger(T^{-1}x)\,\mathcal{U}^{-1}, \tag{201}$$

$$\Xi'(x) = -\mathfrak{T}\Xi(T^{-1}x) = -\Xi^\dagger(T^{-1}x)\,\mathcal{U}^{-1}, \tag{202}$$

where $\mathfrak{T} = \mathcal{C}\,\mathcal{U}$ is the product of the Hermitian conjugation operator $\mathcal{C}$, which has the action $\mathcal{C}\,\Psi(x) = \Psi^\dagger(x)$ and a unitary matrix $\mathcal{U}$. The objective is to find a $\mathcal{U}$ such that

$$\Psi'(x)\gamma^0\overleftarrow{\partial}_\mu\gamma^\mu = \Xi'(x)\gamma^0 \tag{203}$$

if $\Psi(x)$ is a solution of Eq. (53). Equation (203) can be written as

$$\Psi^\dagger(x)\,\mathcal{U}^{-1}\gamma^0\overleftarrow{\partial'}_\mu\gamma^\mu = -\Xi^\dagger(x)\,\mathcal{U}^{-1}\gamma^0, \tag{204}$$

where ∂'_μ is given by Eq. (142). Equation (204) follows from Eq. (53) provided

$$\gamma^0\,\mathcal{U}^{-1}\gamma^0\overleftarrow{\partial'}_\mu\gamma^\mu\gamma^0\,\mathcal{U}\gamma^0 = -\overleftarrow{\partial}_\mu\gamma^\mu, \tag{205}$$

which has as a solution

$$\mathcal{U} = \mathcal{T} \tag{206}$$

and

$$\mathfrak{T} = \mathcal{C}\,\mathcal{T}. \tag{207}$$

7. Plane-wave eigenfunctions

Following the analogy with the Dirac equation, the Hamiltonian for the Maxwell equation is

$$\mathcal{H} = c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} = -\mathrm{i}\,\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}, \tag{208}$$

where $\alpha^i = \gamma^0\,\gamma^i$, and the wave functions for the photon may be identified as the complete set of eigenfunctions of $\mathcal{H}$. The solutions considered here are coordinate-space plane waves characterized by a wave vector $\boldsymbol{k}$ and a polarization vector $\hat{\boldsymbol{\epsilon}}_\lambda$; both positive- and negative-energy solutions, as well as zero-energy solutions are included to form a complete set. These solutions are also eigenfunctions of the momentum operator

$$\boldsymbol{\mathcal{P}} = \mathcal{I}\boldsymbol{p} = -\mathrm{i}\,\hbar\mathcal{I}\,\boldsymbol{\nabla}, \tag{209}$$

which commutes with $\mathcal{H}$.

The plane-wave solutions are not normalizable, because their modulus squared is independent of $\boldsymbol{x}$ and the integral over all space does not exist. As a result, the solutions include an arbitrary multiplicative factor, that could be a function of $\boldsymbol{k}$. Here a factor is chosen to provide the simple result in Eq. (225).

7.1. Transverse plane-wave photons

We first consider transverse photons, *i.,e.,* photons for which the electric and magnetic fields are perpendicular to the wave vector. The polarization vector is a unit vector proportional to the electric or magnetic fields, represented by a three component, possibly complex, vector in the spherical basis. As such, the polarization vector does not transform as the spatial component of a four-vector under velocity transformations.

Two polarization vectors, both in the plane perpendicular to $\hat{\boldsymbol{k}}$, are denoted by

$$\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\,; \qquad \lambda = 1, 2. \tag{210}$$

They have the orthonormality properties

$$\hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_2}(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}_{\lambda_1}(\hat{\boldsymbol{k}}) = \delta_{\lambda_2,\lambda_1}, \tag{211}$$

$$\hat{\boldsymbol{k}}^\dagger_{\mathrm{s}}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) = 0, \tag{212}$$

and the completeness property

$$\sum_{\lambda=1}^{2}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_\lambda(\hat{\boldsymbol{k}}) = \boldsymbol{I} - \hat{\boldsymbol{k}}_{\mathrm{s}}\,\hat{\boldsymbol{k}}^\dagger_{\mathrm{s}} = (\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2 = \boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\hat{\boldsymbol{k}}). \tag{213}$$

From Eq. (212), we also have

$$(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) = \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}). \tag{214}$$

The polarization vectors can represent linear polarization, circular polarization, or any combination by a suitable choice of $\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})$. For example, for $\boldsymbol{k}$ in the $\hat{\boldsymbol{e}}^3$ direction, linear polarization vectors in the $\hat{\boldsymbol{e}}^1$ and $\hat{\boldsymbol{e}}^2$ directions are

$$\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}; \quad \hat{\boldsymbol{\epsilon}}_2(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}} \\ 0 \\ \frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix}, \tag{215}$$

according to Eq. (24). Similarly, circular polarization vectors are (see Sec. 8.2)

$$\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \hat{\boldsymbol{\epsilon}}_2(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{216}$$

These polarization vectors can be transformed to the vectors corresponding to any direction of $\boldsymbol{k}$ with the rotation operator in Eq. (105). (See also Sec. 7.5.)

Positive (+) and negative (−) energy transverse photon wave functions are given by

$$\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \frac{1}{\sqrt{2(2\pi)^3}} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \end{pmatrix} \mathrm{e}^{\pm \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{217}$$

Although constructed geometrically to be transverse by the choice of polarization vectors, these wave functions are also transverse in the sense defined in Sec. 5. We have (see Appendix B for more detail)

$$\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla})\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \mathit{\Pi}^{\mathrm{T}}(\hat{\boldsymbol{k}})\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}), \tag{218}$$

$$\mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla})\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \mathit{\Pi}^{\mathrm{L}}(\hat{\boldsymbol{k}})\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = 0, \tag{219}$$

where $\mathit{\Pi}^{\mathrm{T}}$ and $\mathit{\Pi}^{\mathrm{L}}$ are defined in Eqs. (86) and (87). The wave functions in Eq. (217) are eigenfunctions of the Hamiltonian in Eq. (208) with eigenvalues $\pm\hbar c\,|\boldsymbol{k}|$. In particular,

$$\mathcal{H}\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \pm\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{k}\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}), \tag{220}$$

and

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{\tau}\cdot\boldsymbol{k} \\ \boldsymbol{\tau}\cdot\boldsymbol{k} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \end{pmatrix} = |\boldsymbol{k}| \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \end{pmatrix}, \tag{221}$$

so that

$$\mathcal{H}\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \pm\hbar c\,|\boldsymbol{k}|\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}). \tag{222}$$

Also,

$$\boldsymbol{\mathcal{P}}\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = \pm\hbar\,\boldsymbol{k}\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}). \tag{223}$$

The wave functions have the expected property

$$\overline{\psi}^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x})\,\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) = 0, \tag{224}$$

since the electric and magnetic field strengths are equal for a transverse photon. Normalization and orthogonality relations are

$$\int \mathrm{d}\boldsymbol{x}\ \psi^{(\pm)\dagger}_{\boldsymbol{k}_2,\lambda_2}(\boldsymbol{x})\,\psi^{(\pm)}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}) = \delta_{\lambda_2\lambda_1}\delta(\boldsymbol{k}_2-\boldsymbol{k}_1), \tag{225}$$

$$\int \mathrm{d}\boldsymbol{x}\ \psi^{(\pm)\dagger}_{\boldsymbol{k}_2,\lambda_2}(\boldsymbol{x})\,\psi^{(\mp)}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}) = 0. \tag{226}$$

The latter relation follows from a cancellation of terms between the upper-three and lower-three components of the wave function:

$$\begin{aligned}\int \mathrm{d}\boldsymbol{x}\; \psi^{(\pm)\dagger}_{\boldsymbol{k}_2,\lambda_2}(\boldsymbol{x})\,\psi^{(\mp)}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}) &= \frac{1}{2(2\pi)^3}\int \mathrm{d}\boldsymbol{x}\;\hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_2}(\hat{\boldsymbol{k}}_2)\left[\boldsymbol{I}+\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_2\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_1\right]\hat{\boldsymbol{\epsilon}}_{\lambda_1}(\hat{\boldsymbol{k}}_1)\,\mathrm{e}^{\mp\mathrm{i}(\boldsymbol{k}_2+\boldsymbol{k}_1)\cdot\boldsymbol{x}} \\ &= \frac{1}{2}\,\hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_2}(\hat{\boldsymbol{k}}_2)\left[\boldsymbol{I}+\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_2\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_1\right]\hat{\boldsymbol{\epsilon}}_{\lambda_1}(\hat{\boldsymbol{k}}_1)\delta(\boldsymbol{k}_2+\boldsymbol{k}_1)=0. \qquad (227)\end{aligned}$$

The transverse wave functions constitute a complete set of such functions. The completeness is established by writing

$$\begin{aligned}&\sum_{\lambda=1}^{2}\int \mathrm{d}\boldsymbol{k}\;\psi^{(\pm)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_2)\psi^{(\pm)\dagger}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_1)\\ &=\sum_{\lambda=1}^{2}\int \mathrm{d}\boldsymbol{k}\;\frac{1}{2(2\pi)^3}\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_\lambda(\hat{\boldsymbol{k}}) & \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_\lambda(\hat{\boldsymbol{k}})\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_\lambda(\hat{\boldsymbol{k}}) & \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_\lambda(\hat{\boldsymbol{k}})\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}\\ &=\int \mathrm{d}\boldsymbol{k}\;\frac{1}{2(2\pi)^3}\begin{pmatrix}(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2 & \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}} & (\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}. \qquad (228)\end{aligned}$$

To evaluate the integrals for the sum over positive and negative energy solutions, we use κ to represent either a plus sign or a minus sign and write

$$\sum_{\kappa\to\pm}\int \mathrm{d}\boldsymbol{k}\;(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\,\mathrm{e}^{\kappa\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} = 2(2\pi)^3\,\boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1) \qquad (229)$$

and

$$\sum_{\kappa\to\pm}\int \mathrm{d}\boldsymbol{k}\;\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\mathrm{e}^{\kappa\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} = 0, \qquad (230)$$

which yields the transverse completeness relation

$$\sum_{\kappa\to\pm}\sum_{\lambda=1}^{2}\int \mathrm{d}\boldsymbol{k}\;\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_2)\psi^{(\kappa)\dagger}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_1) = \mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1). \qquad (231)$$

7.2. Longitudinal plane-wave photons

The transverse wave functions alone do not provide a complete description of electromagnetic fields. For example, the field of a point charge q at rest at the origin, given by

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\,\frac{\boldsymbol{x}_{\mathrm{s}}}{|\boldsymbol{x}|^3} = -\frac{q}{4\pi\epsilon_0}\boldsymbol{\nabla}_{\mathrm{s}}\,\frac{1}{|\boldsymbol{x}|}, \qquad (232)$$

is purely longitudinal, because

$$\boldsymbol{\mathit{\Pi}}^{\mathrm{L}}_{\mathrm{s}}(\boldsymbol{\nabla})\,\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{x}) = \boldsymbol{E}_{\mathrm{s}}(\boldsymbol{x}), \qquad (233)$$

$$\boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\boldsymbol{\nabla})\,\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{x}) = 0. \qquad (234)$$

The longitudinal photons are represented by a third polarization state, labeled $\lambda = 0$, with the polarization vector taken to be

$$\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) = \hat{\boldsymbol{k}}_{\rm s}. \tag{235}$$

If $\boldsymbol{k}$ is in the $\hat{\boldsymbol{e}}^3$ direction, the longitudinal polarization vector is (up to a phase factor)

$$\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{236}$$

Longitudinal wave functions are

$$\psi^{(+)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^3}} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \\ \boldsymbol{0} \end{pmatrix} {\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{237}$$

or

$$\psi^{(-)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^3}} \begin{pmatrix} \boldsymbol{0} \\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \end{pmatrix} {\rm e}^{-{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{238}$$

This polarization state has the property that

$$\mathit{\Pi}^{\rm L}(\boldsymbol{\nabla})\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \mathit{\Pi}^{\rm L}(\hat{\boldsymbol{k}})\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}), \tag{239}$$

$$\mathit{\Pi}^{\rm T}(\boldsymbol{\nabla})\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \mathit{\Pi}^{\rm T}(\hat{\boldsymbol{k}})\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = 0. \tag{240}$$

The wave function has an energy eigenvalue of zero,

$$\mathcal{H}\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \pm\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{k}\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = 0, \tag{241}$$

because $\boldsymbol{\tau}\cdot\boldsymbol{k}\,\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) = 0$. However,

$$\boldsymbol{\mathcal{P}}\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}) = \pm\hbar\,\boldsymbol{k}\,\psi^{(\pm)}_{\boldsymbol{k},0}(\boldsymbol{x}). \tag{242}$$

Normalization and orthogonality of the $\lambda = 0$ wave functions, as well as the transverse wave functions, are given by Eqs. (225) and (226), where λ_1 and λ_2 may take on any of the values 0, 1, or 2.

Completeness relations are given by

$$\int {\rm d}\boldsymbol{k}\;\psi^{(+)}_{\boldsymbol{k},0}(\boldsymbol{x}_2)\psi^{(+)\dagger}_{\boldsymbol{k},0}(\boldsymbol{x}_1) = \int {\rm d}\boldsymbol{k}\;\frac{1}{(2\pi)^3} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_0(\hat{\boldsymbol{k}}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} {\rm e}^{{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} \tag{243}$$

and

$$\int {\rm d}\boldsymbol{k}\;\psi^{(-)}_{\boldsymbol{k},0}(\boldsymbol{x}_2)\psi^{(-)\dagger}_{\boldsymbol{k},0}(\boldsymbol{x}_1) = \int {\rm d}\boldsymbol{k}\;\frac{1}{(2\pi)^3} \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}^\dagger_0(\hat{\boldsymbol{k}}) \end{pmatrix} {\rm e}^{-{\rm i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}, \tag{244}$$

where

$$\int \mathrm{d}\boldsymbol{k}\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}_0^\dagger(\hat{\boldsymbol{k}})\mathrm{e}^{\pm\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)} = (2\pi)^3\,\boldsymbol{\mathit{\Pi}}_\mathrm{s}^\mathrm{L}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1), \tag{245}$$

which yields

$$\sum_{\kappa\to\pm}\int \mathrm{d}\boldsymbol{k}\ \psi_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x}_2)\psi_{\boldsymbol{k},0}^{(\kappa)\dagger}(\boldsymbol{x}_1) = \mathit{\Pi}^\mathrm{L}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1). \tag{246}$$

For the example of a point charge at the origin, we have

$$\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}) \;=\; \frac{q}{4\pi\epsilon_0|\boldsymbol{x}|^3}\begin{pmatrix}\boldsymbol{x}_\mathrm{s}\\ \boldsymbol{0}\end{pmatrix}, \tag{247}$$

which may be written as (see Appendix C for some detail)

$$\begin{aligned}\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}) &= \mathit{\Pi}^\mathrm{L}(\boldsymbol{\nabla})\,\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}_1\mathit{\Pi}^\mathrm{L}(\boldsymbol{\nabla})\,\delta(\boldsymbol{x}-\boldsymbol{x}_1)\,\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}_1)\\ &= \sum_{\kappa\to\pm}\int \mathrm{d}\boldsymbol{k}\ \psi_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x})\int \mathrm{d}\boldsymbol{x}_1\,\psi_{\boldsymbol{k},0}^{(\kappa)\dagger}(\boldsymbol{x}_1)\,\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}_1)\\ &= -\frac{\mathrm{i}\,q}{\sqrt{(2\pi)^3}\,\epsilon_0}\int \mathrm{d}\boldsymbol{k}\,\frac{1}{|\boldsymbol{k}|}\,\psi_{\boldsymbol{k},0}^{(+)}(\boldsymbol{x}).\end{aligned} \tag{248}$$

7.3. Full orthogonality and completeness of the plane wave solutions

The full orthogonality relations are

$$\int \mathrm{d}\boldsymbol{x}\ \psi_{\boldsymbol{k}_2,\lambda_2}^{(\kappa_2)\dagger}(\boldsymbol{x})\,\psi_{\boldsymbol{k}_1,\lambda_1}^{(\kappa_1)}(\boldsymbol{x}) \;=\; \delta_{\kappa_2\kappa_1}\delta_{\lambda_2\lambda_1}\delta(\boldsymbol{k}_2-\boldsymbol{k}_1), \tag{249}$$

where the factor $\delta_{\kappa_2\kappa_1}$ is 1 if κ_2 and κ_1 represent the same sign and is 0 for opposite signs, and λ_2 and λ_1 may be any of 0,1,2. The combined result of the transverse and longitudinal completeness relations, Eqs. (231) and (246), is

$$\sum_{\kappa\to\pm}\sum_{\lambda=0}^{2}\int \mathrm{d}\boldsymbol{k}\ \psi_{\boldsymbol{k},\lambda}^{(\kappa)}(\boldsymbol{x}_2)\psi_{\boldsymbol{k},\lambda}^{(\kappa)\dagger}(\boldsymbol{x}_1) = \mathcal{I}\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1), \tag{250}$$

where $\mathit{\Pi}^\mathrm{T}(\boldsymbol{\nabla}) + \mathit{\Pi}^\mathrm{L}(\boldsymbol{\nabla}) = \mathcal{I}$.

7.4. Time dependence of the wave functions

The time dependence of the photon wave functions is given by[2]

$$\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(x) \;=\; \psi_{\boldsymbol{k},\lambda}^{(\kappa)}(\boldsymbol{x})\,\mathrm{e}^{-\kappa\mathrm{i}\omega t}, \tag{251}$$

[2]The notation $f(\boldsymbol{x}) = f(x)\big|_{t=0}$ is employed throughout this paper.

where ω is determined by the equation

$$\gamma^\mu \partial_\mu \, \psi^{(\kappa)}_{\boldsymbol{k},\lambda}(x) \;=\; \gamma^0 \left(\mathcal{I} \, \frac{\partial}{\partial ct} + \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} \right) \psi^{(\kappa)}_{\boldsymbol{k},\lambda}(x) = 0. \tag{252}$$

For transverse photons

$$\omega \;=\; c|\boldsymbol{k}|, \tag{253}$$

and for longitudinal photons

$$\omega \;=\; 0. \tag{254}$$

The complete exponential factor is thus

$$\mathrm{e}^{-\kappa \mathrm{i} k \cdot x}, \tag{255}$$

where

$$k \;=\; \begin{pmatrix} |\boldsymbol{k}| \\ \boldsymbol{k}_\mathrm{c} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \boldsymbol{k}_\mathrm{c} \end{pmatrix}, \tag{256}$$

depending on whether the photon is transverse or longitudinal. This corresponds to the eigenvalue equation

$$\mathcal{H} \, \psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) \;=\; \kappa \hbar \omega \, \psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}) \tag{257}$$

and a time dependence given by

$$\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(x) \;=\; \mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar} \psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}). \tag{258}$$

It is also of interest to consider the effect of a hypothetical photon mass m_γ on the longitudinal photon wave function. Such a modification, with an infinitesimal mass, resolves an ambiguity in the construction of the Green function as discussed in Sec. 9. Following the form of the Dirac equation in Eq. (6), we have

$$(\mathrm{i}\, \hbar \gamma^\mu \partial_\mu - m_\gamma c) \, \psi^{(\kappa)}_{\boldsymbol{k},0}(x) = 0 \tag{259}$$

or

$$\left(\frac{\hbar \kappa \omega}{c} \gamma^0 - m_\gamma c \, \mathcal{I} \right) \psi^{(\kappa)}_{\boldsymbol{k},0}(x) = 0, \tag{260}$$

which yields

$$\hbar \omega \;=\; m_\gamma c^2, \tag{261}$$

since

$$\gamma^0 \psi^{(\kappa)}_{\boldsymbol{k},0}(x) \;=\; \kappa \, \psi^{(\kappa)}_{\boldsymbol{k},0}(x). \tag{262}$$

7.5. Rotation of the wave functions

The rotations of the wave functions follow from the discussion of Sec. 6.4, with an additional consideration of the vector $\boldsymbol{k}$. On physical grounds, a rotation parameterized by the vector $\boldsymbol{u}$ of the state of a photon means rotation of the vector $\boldsymbol{k}$ into the vector $\boldsymbol{k}'$, according to

$$\boldsymbol{k}' = \boldsymbol{R}(\boldsymbol{u})\boldsymbol{k}, \tag{263}$$

where $\boldsymbol{R}(\boldsymbol{u})$ is defined by Eq. (101). Similarly, the polarization vector is transformed by the spherical rotation operator

$$\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}') = \boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \tag{264}$$

and

$$\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}') = \boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\boldsymbol{R}_{\rm s}^{-1}(\boldsymbol{u})\boldsymbol{R}_{\rm s}(\boldsymbol{u})\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}), \tag{265}$$

where the rotation angle θ, axis direction $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{k}}'$ are related by

$$\hat{\boldsymbol{u}}\sin\theta = \hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}'. \tag{266}$$

For applications, the rotation operator can be expressed as a function of $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}}'$ rather than $\boldsymbol{u}$. From

$$\boldsymbol{R}_{\rm s}(\boldsymbol{u}) = \boldsymbol{I} - (\boldsymbol{\tau}\cdot\hat{\boldsymbol{u}})^2\,(1-\cos\theta) - {\rm i}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{u}}\,\sin\theta, \tag{267}$$

one has for transverse polarization, $\lambda = 1, 2,$[3]

$$\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}') = \frac{(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}')^2 + \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}}{1+\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}), \tag{268}$$

$$\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}') = \frac{(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}')^2 + \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}}{1+\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}), \tag{269}$$

and for longitudinal polarization, $\lambda = 0$,

$$\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}') = \hat{\boldsymbol{k}}_{\rm s}'\hat{\boldsymbol{k}}_{\rm s}^\dagger\,\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}). \tag{270}$$

We thus have for a rotation of the wave functions characterized by the vector $\boldsymbol{u}$

$$\psi^{(\kappa)}_{\boldsymbol{k}',\lambda}(x) = \mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(R^{-1}(\boldsymbol{u})\,x), \tag{271}$$

[3] The identity $\boldsymbol{\tau}\cdot\boldsymbol{k}\times\boldsymbol{k}' = {\rm i}(\boldsymbol{k}_{\rm s}'\,\boldsymbol{k}_{\rm s}^\dagger - \boldsymbol{k}_{\rm s}\,\boldsymbol{k}_{\rm s}'^\dagger)$ is useful here.

where $\boldsymbol{k}'\cdot\boldsymbol{x} = \boldsymbol{k}\cdot\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}$ in the exponent of the wave function. This yields the result

$$\gamma^\mu\partial_\mu\psi^{(\kappa)}_{\boldsymbol{k}',\lambda}(x) \quad = \quad 0, \tag{272}$$

according to the discussion in Sec. 6.4.

The expected transformation in Eq. (271) can be confirmed by an explicit calculation based on the completeness of the wave functions. For a rotation of a transverse wave function, we write

$$\begin{aligned}\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(R^{-1}(\boldsymbol{u})\,x) &= \int \mathrm{d}\boldsymbol{x}_1\,\delta(\boldsymbol{x}-\boldsymbol{x}_1)\,\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}_1)\,\mathrm{e}^{-\kappa\mathrm{i}\omega t}\\ &= \sum_{\kappa_1\to\pm}\sum_{\lambda_1=1}^{2}\int\mathrm{d}\boldsymbol{k}_1\int\mathrm{d}\boldsymbol{x}_1\;\psi^{(\kappa_1)}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x})\psi^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}_1)\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}_1)\,\mathrm{e}^{-\kappa\mathrm{i}\omega t},\end{aligned} \tag{273}$$

where $\lambda = 1$ or 2, and from Eq. (164) and the subsequent remarks, it follows that the rotated wave function is also transverse, so λ_1 is restricted to 1 or 2. The evaluation requires the matrix element

$$\begin{aligned}&\int\mathrm{d}\boldsymbol{x}_1\;\psi^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}_1)\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}_1)\\ &\quad= \frac{1}{2(2\pi)^3}\int\mathrm{d}\boldsymbol{x}_1\left[\hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_1}(\hat{\boldsymbol{k}}_1)\,\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{u})\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) + \hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_1}(\hat{\boldsymbol{k}}_1)\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_1\,\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\right]\\ &\qquad\qquad\times\mathrm{e}^{-\kappa_1\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{\kappa\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}^{-1}(\boldsymbol{u})\boldsymbol{x}_1}\\ &\quad= \frac{1}{2}\,\delta_{\kappa_1\kappa}\left[\hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_1}(\hat{\boldsymbol{k}}')\,\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{u})\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) + \hat{\boldsymbol{\epsilon}}^\dagger_{\lambda_1}(\hat{\boldsymbol{k}}')\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{u})\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\right]\delta(\boldsymbol{k}_1-\boldsymbol{k}')\\ &\quad= \delta_{\kappa_1\kappa}\,\delta_{\lambda_1\lambda}\,\delta(\boldsymbol{k}_1-\boldsymbol{k}').\end{aligned} \tag{274}$$

In the exponent in Eq. (274), we have $\boldsymbol{k}\cdot\boldsymbol{R}^{-1}(\boldsymbol{u})\boldsymbol{x}_1 = \boldsymbol{R}(\boldsymbol{u})\boldsymbol{k}\cdot\boldsymbol{x}_1 = \boldsymbol{k}'\cdot\boldsymbol{x}_1$. The factor $\delta_{\kappa_1\kappa}$ results from the requirement that $\boldsymbol{k}'\to\boldsymbol{k}$ as the rotation angle $\theta\to 0$, *i.e.*, that $\boldsymbol{k}$ does not change sign for an infinitesimal rotation. The factor $\delta_{\lambda_1\lambda}$ follows from Eq. (264) and the discussion that follows it, together with the orthonormality of the polarization vectors. Substitution of Eq. (274) into (273) yields

$$\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(R^{-1}(\boldsymbol{u})\,x) \quad = \quad \psi^{(\kappa)}_{\boldsymbol{k}',\lambda}(x), \tag{275}$$

in accord with the general arguments leading to Eq. (271).

For a rotation of a longitudinal wave function, we have

$$\begin{aligned}\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},0}(R^{-1}(\boldsymbol{u})\,x) &= \int\mathrm{d}\boldsymbol{x}_1\,\delta(\boldsymbol{x}-\boldsymbol{x}_1)\,\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},0}(\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}_1)\\ &= \sum_{\kappa_1\to\pm}\int\mathrm{d}\boldsymbol{k}_1\int\mathrm{d}\boldsymbol{x}_1\;\psi^{(\kappa_1)}_{\boldsymbol{k}_1,0}(\boldsymbol{x})\psi^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,0}(\boldsymbol{x}_1)\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},0}(\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}_1)\end{aligned} \tag{276}$$

where only longitudinal wave functions contribute, and the evaluation requires the matrix element

$$\begin{aligned}\int \mathrm{d}\boldsymbol{x}_1\ \psi^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,0}(\boldsymbol{x}_1)\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},0}(\boldsymbol{R}^{-1}(\boldsymbol{u})\,\boldsymbol{x}_1)&\\ =\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1\delta_{\kappa_1\kappa}\,\hat{\boldsymbol{\epsilon}}^\dagger_0(\hat{\boldsymbol{k}}_1)\,\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{u})\,\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\,\mathrm{e}^{-\kappa_1\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{\kappa\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}^{-1}(\boldsymbol{u})\boldsymbol{x}_1}&\\ =\delta_{\kappa_1\kappa}\,\hat{\boldsymbol{\epsilon}}^\dagger_0(\hat{\boldsymbol{k}}')\,\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{u})\,\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\,\delta(\boldsymbol{k}_1-\boldsymbol{k}')&\\ =\delta_{\kappa_1\kappa}\,\delta(\boldsymbol{k}_1-\boldsymbol{k}').&\qquad(277)\end{aligned}$$

Substitution of Eq. (277) into (276) yields

$$\mathcal{R}(\boldsymbol{u})\,\psi^{(\kappa)}_{\boldsymbol{k},0}(R^{-1}(\boldsymbol{u})\,x)\ =\ \psi^{(\kappa)}_{\boldsymbol{k}',0}(x),\qquad(278)$$

in agreement with Eq. (271).

7.6. Velocity transformation of the wave functions

7.6.1. Transverse wave functions

Under the velocity transformation of a transverse photon by a velocity $\boldsymbol{v}$, the four-vector

$$k\ =\ \begin{pmatrix}|\boldsymbol{k}|\\ \boldsymbol{k}_{\mathrm{c}}\end{pmatrix}\qquad(279)$$

is transformed to

$$k'\ =\ V(\boldsymbol{v})\,k=\begin{pmatrix}|\boldsymbol{k}|\left(\cosh\zeta+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}\sinh\zeta\right)\\ \boldsymbol{k}_{\mathrm{c}}+|\boldsymbol{k}|\hat{\boldsymbol{v}}_{\mathrm{c}}\left[\sinh\zeta+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}(\cosh\zeta-1)\right]\end{pmatrix},\qquad(280)$$

and the wave function is expected to transform according to

$$\xi\,\psi'^{(\kappa)}_{\boldsymbol{k}',\lambda}(x)\ =\ \mathcal{V}(\boldsymbol{v})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(V^{-1}(\boldsymbol{v})\,x).\qquad(281)$$

The prime on the transformed function indicates that it is a function of modified polarization vectors, which in general are not simply rotated vectors corresponding to $\hat{\boldsymbol{k}}\to\hat{\boldsymbol{k}}'$. The transformed transverse wave function is proportional to

$$\begin{aligned}\mathcal{V}(\boldsymbol{v})\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\end{pmatrix}\ &=\ \begin{pmatrix}\left\{\boldsymbol{I}+(\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta-1)+\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \sinh\zeta\right\}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\\ \left\{\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\sinh\zeta+[\boldsymbol{I}+(\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta-1)]\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\right\}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\end{pmatrix}\\ &=\ \xi\begin{pmatrix}\hat{\boldsymbol{\epsilon}}'_\lambda(\hat{\boldsymbol{k}}')\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\ \hat{\boldsymbol{\epsilon}}'_\lambda(\hat{\boldsymbol{k}}')\end{pmatrix},\qquad(282)\end{aligned}$$

where the fact that transformed wave function can be written in the form given at the right-hand end is based on the three identities:

$$\left|\left\{\boldsymbol{I}+(\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta-1)+\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \sinh\zeta\right\}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\right|\ =\ \cosh\zeta+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}\sinh\zeta,\qquad(283)$$

$$\boldsymbol{k}_{\rm s}'^{\dagger} \left\{ \boldsymbol{I} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2 (\cosh\zeta - 1) + \boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\sinh\zeta \right\} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \;=\; 0, \tag{284}$$

$$\begin{aligned}\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\sinh\zeta + \left[\boldsymbol{I} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2 (\cosh\zeta - 1)\right] \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}} \\ = \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}' \left\{ \boldsymbol{I} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2 (\cosh\zeta - 1) + \boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\sinh\zeta \right\},\end{aligned} \tag{285}$$

where

$$\hat{\boldsymbol{k}}' \;=\; \frac{\hat{\boldsymbol{k}} + \hat{\boldsymbol{v}}\Big[\sinh\zeta + \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}(\cosh\zeta - 1)\Big]}{\cosh\zeta + \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}\sinh\zeta}. \tag{286}$$

Equation (283) determines the scalar multiplicative factor in Eq. (281) to be

$$\xi \;=\; \cosh\zeta + \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}\sinh\zeta. \tag{287}$$

According to Eq. (284), the transformed polarization vector is in the plane perpendicular to $\boldsymbol{k}'$, that is, the transformed transverse polarization vector is also transverse. This is in contrast to the vector potential, which does not maintain transversality under a velocity transformation, so the Coulomb, or radiation, gauge condition is not preserved. The difference is just the fact that the vector potential transforms as a vector, so that the angle between the space component of the vector potential and the space component of the wave vector is not necessarily preserved under a velocity transformation, whereas the polarization vector transforms as the electric field, *i.e.,* as a component of a second-rank tensor, and Eq. (284) shows that for this case the transversality is preserved. Equation (285) shows that the lower components of the transformed wave function, as given in Eq. (282), can be written as $\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'$ times the upper components in the same expression. This together with the relation $k'\cdot x = k\cdot V^{-1}(\boldsymbol{v})\,x$ in the exponent of the wave function insures that the transformed wave function is a solution of the source-free Maxwell equation. It also follows from Eq. (282) that if $\lambda_2 \neq \lambda_1$, then

$$\hat{\boldsymbol{\epsilon}}_{\lambda_2}'^{\dagger}(\hat{\boldsymbol{k}}')\,\hat{\boldsymbol{\epsilon}}_{\lambda_1}'(\hat{\boldsymbol{k}}') = 0, \tag{288}$$

so the orthogonality of the polarization vectors is preserved by the velocity transformation.

Equation (281) can be be obtained by an explicit calculation based on the completeness of the wave functions, as for rotations. A velocity transformation of a transverse wave function is given by

$$\begin{aligned}\mathcal{V}(\boldsymbol{v})\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(V^{-1}(\boldsymbol{v})\,x) \\ = \int \mathrm{d}\boldsymbol{x}_1\,\delta(\boldsymbol{x}-\boldsymbol{x}_1)\,\mathcal{V}(\boldsymbol{v})\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(V^{-1}(\boldsymbol{v})\,x_1) \\ = \sum_{\kappa_1\to\pm}\sum_{\lambda_1=0}^{2}\int \mathrm{d}\boldsymbol{k}_1 \int \mathrm{d}\boldsymbol{x}_1\;\psi_{\boldsymbol{k}_1,\lambda_1}'^{(\kappa_1)}(\boldsymbol{x})\psi_{\boldsymbol{k}_1,\lambda_1}'^{(\kappa_1)\dagger}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(V^{-1}(\boldsymbol{v})\,x_1),\end{aligned} \tag{289}$$

where $\lambda = 1$ or 2, $t_1 = t$, and the primes on the wave functions indicate that the velocity transformed polarization vectors provide the basis vectors for $\lambda_1 = 1, 2$. For $\lambda_1 = 0$,

$$\begin{aligned}
&\int \mathrm{d}\boldsymbol{x}_1 \; \psi'^{(+)\dagger}_{\boldsymbol{k}_1,0}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(V^{-1}(\boldsymbol{v})\,x_1) \\
&\quad = \frac{1}{\sqrt{2}(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \hat{\boldsymbol{\epsilon}}'^{\dagger}_0(\hat{\boldsymbol{k}}_1) & \boldsymbol{0} \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\;\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \end{array}\right) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{-\kappa \mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1} \\
&\quad = \frac{1}{\sqrt{2}}\,\delta_{+\kappa}\,\xi\,\hat{\boldsymbol{\epsilon}}'^{\dagger}_0(\hat{\boldsymbol{k}}')\,\hat{\boldsymbol{\epsilon}}'_\lambda(\hat{\boldsymbol{k}}')\,\delta(\boldsymbol{k}_1-\boldsymbol{k}')\,\mathrm{e}^{-\mathrm{i}k'^0ct} = 0,
\end{aligned} \tag{290}$$

$$\begin{aligned}
&\int \mathrm{d}\boldsymbol{x}_1 \; \psi'^{(-)\dagger}_{\boldsymbol{k}_1,0}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(V^{-1}(\boldsymbol{v})\,x_1) \\
&\quad = \frac{1}{\sqrt{2}(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \boldsymbol{0} & \hat{\boldsymbol{\epsilon}}'^{\dagger}_0(\hat{\boldsymbol{k}}_1) \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\;\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \end{array}\right) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{-\kappa \mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1} \\
&\quad = \frac{1}{\sqrt{2}}\,\delta_{-\kappa}\,\xi\,\hat{\boldsymbol{\epsilon}}'^{\dagger}_0(\hat{\boldsymbol{k}}')\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\hat{\boldsymbol{\epsilon}}'_\lambda(\hat{\boldsymbol{k}}')\,\delta(\boldsymbol{k}_1-\boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}k'^0ct} = 0.
\end{aligned} \tag{291}$$

In the exponent in Eqs. (290) and (291), $k\cdot V^{-1}(\boldsymbol{v})x_1 = V(\boldsymbol{v})k\cdot x_1 = k'\cdot x_1$. As for rotations, the factors $\delta_{+\kappa}$ and $\delta_{-\kappa}$ result from the requirement that $k' \to k$ as the velocity $|\boldsymbol{v}| \to 0$, and the last equalities follow from Eqs. (284) and (285). For $\lambda_1 = 1$ or 2, we have

$$\begin{aligned}
&\int \mathrm{d}\boldsymbol{x}_1 \; \psi'^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(V^{-1}(\boldsymbol{v})\,x_1) \\
&\quad = \frac{1}{2(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \hat{\boldsymbol{\epsilon}}'^{\dagger}_{\lambda_1}(\hat{\boldsymbol{k}}_1) & \hat{\boldsymbol{\epsilon}}'^{\dagger}_{\lambda_1}(\hat{\boldsymbol{k}}_1)\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_1 \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\;\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \end{array}\right) \\
&\qquad \times\mathrm{e}^{-\kappa_1\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{-\kappa \mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1} \\
&\quad = \delta_{\kappa_1\kappa}\,\delta_{\lambda_1\lambda}\,\xi\,\delta(\boldsymbol{k}_1-\boldsymbol{k}')\,\mathrm{e}^{-\kappa\mathrm{i}k'^0ct}.
\end{aligned} \tag{292}$$

Substitution of Eq. (292) into (289) yields

$$\mathcal{V}(\boldsymbol{v})\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(V^{-1}(\boldsymbol{v})\,x) \;=\; \xi\,\psi'^{(\kappa)}_{\boldsymbol{k}',\lambda}(x), \tag{293}$$

where

$$\xi\,\hat{\boldsymbol{\epsilon}}'_\lambda(\hat{\boldsymbol{k}}') \;=\; \left\{\boldsymbol{I} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\,(\cosh\zeta - 1) + \boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\sinh\zeta\right\}\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}), \tag{294}$$

in accord with Eq. (281).

7.6.2. Longitudinal wave functions

For a longitudinal solution, the four-vector k in the invariant phase factor $k\cdot x$, given by

$$k \;=\; \left(\begin{array}{c} 0 \\ \boldsymbol{k}_{\mathrm{c}} \end{array}\right) \tag{295}$$

from Eqs. (254)-(256), has a zero time component. However, the transformed phase, given by $k \cdot V^{-1}(\boldsymbol{v})\, x = V(\boldsymbol{v})k \cdot x = k' \cdot x$, where

$$k' \;=\; V(\boldsymbol{v})\, k = \begin{pmatrix} \hat{\boldsymbol{v}} \cdot \boldsymbol{k} \sinh\zeta \\ \boldsymbol{k}_{\rm c} + \hat{\boldsymbol{v}}_{\rm c}\, \hat{\boldsymbol{v}} \cdot \boldsymbol{k}(\cosh\zeta - 1) \end{pmatrix}, \tag{296}$$

does have time dependence. Thus the wave function is expected to transform according to

$$\sum_{\kappa'\to\pm} \sum_{\lambda'=0}^{3} \xi^{\kappa'\kappa}_{\lambda'0}(t)\, \psi'^{(\kappa')}_{\boldsymbol{k}',\lambda'}(x) \;=\; \mathcal{V}(\boldsymbol{v})\, \psi^{(\kappa)}_{\boldsymbol{k},0}(V^{-1}(\boldsymbol{v})\, x), \tag{297}$$

where the coefficients include the extra time dependence introduced by the velocity transformation. The sum over polarization states is necessary, because unlike the case of the transverse wave function, the transformed longitudinal wave function is mixture of longitudinal and transverse components. This is expected on physical grounds, since moving charges cause radiative atomic transitions. On the other hand, the transformed space-like wave vector does not the match the wave vector of either the longitudinal or transverse basis functions. Since the solutions are classified according to their three-wave-vector, there is a residual time dependence in the expansion that is included in the transformation coefficients.

To be explicit, for $\lambda \neq 0$, we consider specific polarization vectors. Let $\hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}')$ be a linear polarization vector in the plane of $\hat{\boldsymbol{k}}'$ and $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{\epsilon}}'_2(\hat{\boldsymbol{k}}')$ be a linear polarization vector perpendicular to $\hat{\boldsymbol{k}}'$ and $\hat{\boldsymbol{v}}$. These conditions, together with $\boldsymbol{k}'^{\dagger}_{\rm s}\, \hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') = 0$, yield (up to phase factors)

$$\hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') \;=\; \frac{\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}}'\, \hat{\boldsymbol{k}}'_{\rm s} - \hat{\boldsymbol{v}}_{\rm s}}{\sqrt{1 - (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}}')^2}}, \tag{298}$$

$$\hat{\boldsymbol{\epsilon}}'_2(\hat{\boldsymbol{k}}') \;=\; \frac{\boldsymbol{\tau} \cdot \hat{\boldsymbol{v}}\, \hat{\boldsymbol{k}}'_{\rm s}}{\sqrt{1 - (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}}')^2}}, \tag{299}$$

where

$$\hat{\boldsymbol{\epsilon}}'_2(\hat{\boldsymbol{k}}') \;=\; \boldsymbol{\tau} \cdot \hat{\boldsymbol{k}}'\, \hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}'), \tag{300}$$

$$\hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') \;=\; \boldsymbol{\tau} \cdot \hat{\boldsymbol{k}}'\, \hat{\boldsymbol{\epsilon}}'_2(\hat{\boldsymbol{k}}'). \tag{301}$$

The transformed upper- and lower-component longitudinal wave functions follow from the expressions (for $t = 0$ and $\boldsymbol{x} = 0$)

$$\mathcal{V}(\boldsymbol{v}) \begin{pmatrix} \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \\ \boldsymbol{0} \end{pmatrix} \;=\; \begin{pmatrix} [\boldsymbol{I} + (\boldsymbol{\tau} \cdot \hat{\boldsymbol{v}})^2 (\cosh\zeta - 1)]\, \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau} \cdot \hat{\boldsymbol{v}} \sinh\zeta\, \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \end{pmatrix} = \xi^{++}_{00}(0) \begin{pmatrix} \hat{\boldsymbol{\epsilon}}'_0(\hat{\boldsymbol{k}}') \\ \boldsymbol{0} \end{pmatrix}$$

$$+ \frac{\xi^{++}_{10}(0)}{\sqrt{2}} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') \\ \boldsymbol{\tau} \cdot \hat{\boldsymbol{k}}'\, \hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') \end{pmatrix} + \frac{\xi^{-+}_{10}(0)}{\sqrt{2}} \begin{pmatrix} \hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') \\ -\boldsymbol{\tau} \cdot \hat{\boldsymbol{k}}'\, \hat{\boldsymbol{\epsilon}}'_1(\hat{\boldsymbol{k}}') \end{pmatrix}, \tag{302}$$

and

$$\mathcal{V}(\boldsymbol{v})\begin{pmatrix}\boldsymbol{0}\\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\end{pmatrix} = \begin{pmatrix}\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\sinh\zeta\,\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\\ [\boldsymbol{I}+(\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2(\cosh\zeta-1)]\,\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\end{pmatrix} = \xi_{00}^{--}(0)\begin{pmatrix}\boldsymbol{0}\\ \hat{\boldsymbol{\epsilon}}_0'(\hat{\boldsymbol{k}}')\end{pmatrix}$$
$$+\frac{\xi_{20}^{--}(0)}{\sqrt{2}}\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_2'(\hat{\boldsymbol{k}}')\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\hat{\boldsymbol{\epsilon}}_2'(\hat{\boldsymbol{k}}')\end{pmatrix}+\frac{\xi_{20}^{+-}(0)}{\sqrt{2}}\begin{pmatrix}-\hat{\boldsymbol{\epsilon}}_2'(\hat{\boldsymbol{k}}')\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'\,\hat{\boldsymbol{\epsilon}}_2'(\hat{\boldsymbol{k}}')\end{pmatrix}, \tag{303}$$

based on the relations

$$\hat{\boldsymbol{k}} = \frac{\hat{\boldsymbol{k}}'-\hat{\boldsymbol{v}}\,\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\,(1-\operatorname{sech}\zeta)}{\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2\tanh^2\zeta}}, \tag{304}$$

$$\hat{\boldsymbol{k}}_{\rm s}'\hat{\boldsymbol{k}}_{\rm s}'^{\dagger}\left[\boldsymbol{I}+(\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2(\cosh\zeta-1)\right]\hat{\boldsymbol{k}}_{\rm s} = \cosh\zeta\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2\tanh^2\zeta}\;\hat{\boldsymbol{k}}_{\rm s}', \tag{305}$$

$$\hat{\boldsymbol{k}}_{\rm s}'\hat{\boldsymbol{k}}_{\rm s}'^{\dagger}\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\sinh\zeta\,\hat{\boldsymbol{k}}_{\rm s} = 0, \tag{306}$$

$$(\boldsymbol{I}-\hat{\boldsymbol{k}}_{\rm s}'\hat{\boldsymbol{k}}_{\rm s}'^{\dagger})\left[\boldsymbol{I}+(\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2(\cosh\zeta-1)\right]\hat{\boldsymbol{k}}_{\rm s} = \frac{\sinh\zeta\tanh\zeta\,\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\hat{\boldsymbol{k}}_{\rm s}'-\hat{\boldsymbol{v}}_{\rm s})}{\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2\tanh^2\zeta}}, \tag{307}$$

$$(\boldsymbol{I}-\hat{\boldsymbol{k}}_{\rm s}'\hat{\boldsymbol{k}}_{\rm s}'^{\dagger})\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\sinh\zeta\,\hat{\boldsymbol{k}}_{\rm s} = \frac{\sinh\zeta\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}'(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\hat{\boldsymbol{k}}_{\rm s}'-\hat{\boldsymbol{v}}_{\rm s})}{\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2\tanh^2\zeta}}, \tag{308}$$

with coefficients given by

$$\xi_{00}^{++}(0)=\xi_{00}^{--}(0) = \cosh\zeta\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2\tanh^2\zeta}\,, \tag{309}$$

$$\xi_{10}^{++}(0)=\xi_{20}^{--}(0) = \frac{\sinh\zeta}{\sqrt{2}}\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2}\sqrt{\frac{1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\tanh\zeta}{1-\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\tanh\zeta}}\,, \tag{310}$$

$$\xi_{10}^{-+}(0)=\xi_{20}^{+-}(0) = -\frac{\sinh\zeta}{\sqrt{2}}\sqrt{1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}')^2}\sqrt{\frac{1-\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\tanh\zeta}{1+\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}'\tanh\zeta}}\,. \tag{311}$$

Terms that do not appear in Eq. (302) or (303) make no contribution

$$\xi_{00}^{+-}(0)=\xi_{00}^{-+}(0)=\xi_{10}^{--}(0)=\xi_{10}^{+-}(0)=\xi_{20}^{++}(0)=\xi_{20}^{-+}(0)=0. \tag{312}$$

An explicit calculation of Eq. (297) is made by evaluating

$$\mathcal{V}(\boldsymbol{v})\,\psi_{\boldsymbol{k},0}^{(\kappa)}(V^{-1}(\boldsymbol{v})\,x) = \int \mathrm{d}\boldsymbol{x}_1\,\delta(\boldsymbol{x}-\boldsymbol{x}_1)\,\mathcal{V}(\boldsymbol{v})\,\psi_{\boldsymbol{k},0}^{(\kappa)}(V^{-1}(\boldsymbol{v})\,x_1)$$
$$= \sum_{\kappa_1\to\pm}\sum_{\lambda_1=0}^{2}\int \mathrm{d}\boldsymbol{k}_1\int \mathrm{d}\boldsymbol{x}_1\;\psi_{\boldsymbol{k}_1,\lambda_1}'^{(\kappa_1)}(\boldsymbol{x})\psi_{\boldsymbol{k}_1,\lambda_1}'^{(\kappa_1)\dagger}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\,\psi_{\boldsymbol{k},0}^{(\kappa)}(V^{-1}(\boldsymbol{v})\,x_1). \tag{313}$$

For $\lambda_1 = 0$,

$$\int \mathrm{d}\boldsymbol{x}_1\, \psi'^{(+)\dagger}_{\boldsymbol{k}_1,0}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\, \psi^{(+)}_{\boldsymbol{k},0}(V^{-1}(\boldsymbol{v})\, x_1)$$

$$= \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \hat{\boldsymbol{\epsilon}}'^{\dagger}_0(\hat{\boldsymbol{k}}_1) & \boldsymbol{0} \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \\ \boldsymbol{0} \end{array}\right) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{-\mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1}$$

$$= \xi^{++}_{00}(0)\, \delta(\boldsymbol{k}_1 - \boldsymbol{k}')\, \mathrm{e}^{-\mathrm{i}k'^0 ct}, \tag{314}$$

$$\int \mathrm{d}\boldsymbol{x}_1\, \psi'^{(-)\dagger}_{\boldsymbol{k}_1,0}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\, \psi^{(-)}_{\boldsymbol{k},0}(V^{-1}(\boldsymbol{v})\, x_1)$$

$$= \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \boldsymbol{0} & \hat{\boldsymbol{\epsilon}}'^{\dagger}_0(\hat{\boldsymbol{k}}_1) \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \boldsymbol{0} \\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \end{array}\right) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{\mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1}$$

$$= \xi^{--}_{00}(0)\, \delta(\boldsymbol{k}_1 - \boldsymbol{k}')\, \mathrm{e}^{\mathrm{i}k'^0 ct}, \tag{315}$$

where

$$\hat{\boldsymbol{k}}' \;=\; \frac{\hat{\boldsymbol{k}} + \hat{\boldsymbol{v}}\,\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}}(\cosh\zeta - 1)}{\sqrt{1 + (\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{k}})^2\sinh^2\zeta}}, \tag{316}$$

and k'^0 is the time component associated with the transformed space-like vector k

$$k'^0 \;=\; \hat{\boldsymbol{v}}\cdot\boldsymbol{k}\,\sinh\zeta = \hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\,\tanh\zeta. \tag{317}$$

For $\lambda_1 = 1$ or 2,

$$\int \mathrm{d}\boldsymbol{x}_1\, \psi'^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\, \psi^{(+)}_{\boldsymbol{k},0}(V^{-1}(\boldsymbol{v})\, x_1)$$

$$= \frac{1}{\sqrt{2}(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \hat{\boldsymbol{\epsilon}}'^{\dagger}_{\lambda_1}(\hat{\boldsymbol{k}}_1) & \hat{\boldsymbol{\epsilon}}'^{\dagger}_{\lambda_1}(\hat{\boldsymbol{k}}_1)\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_1 \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \\ \boldsymbol{0} \end{array}\right)$$
$$\times \mathrm{e}^{-\kappa_1\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{-\mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1}$$

$$= \delta_{\kappa_1+}\delta_{\lambda_1 1}\,\xi^{++}_{10}(0)\,\delta(\boldsymbol{k}_1 - \boldsymbol{k}')\,\mathrm{e}^{-\mathrm{i}k'^0 ct} + \delta_{\kappa_1-}\delta_{\lambda_1 1}\,\xi^{-+}_{10}(0)\,\delta(\boldsymbol{k}_1 + \boldsymbol{k}')\,\mathrm{e}^{-\mathrm{i}k'^0 ct}, \tag{318}$$

$$\int \mathrm{d}\boldsymbol{x}_1\, \psi'^{(\kappa_1)\dagger}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}_1)\mathcal{V}(\boldsymbol{v})\, \psi^{(-)}_{\boldsymbol{k},0}(V^{-1}(\boldsymbol{v})\, x_1)$$

$$= \frac{1}{\sqrt{2}(2\pi)^3}\int \mathrm{d}\boldsymbol{x}_1 \left(\begin{array}{cc} \hat{\boldsymbol{\epsilon}}'^{\dagger}_{\lambda_1}(\hat{\boldsymbol{k}}_1) & \hat{\boldsymbol{\epsilon}}'^{\dagger}_{\lambda_1}(\hat{\boldsymbol{k}}_1)\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_1 \end{array}\right) \mathcal{V}(\boldsymbol{v}) \left(\begin{array}{c} \boldsymbol{0} \\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}}) \end{array}\right)$$
$$\times \mathrm{e}^{-\kappa_1\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\mathrm{e}^{\mathrm{i}k\cdot V^{-1}(\boldsymbol{v})x_1}$$

$$= \delta_{\kappa_1-}\delta_{\lambda_1 2}\,\xi^{--}_{20}(0)\,\delta(\boldsymbol{k}_1 - \boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}k'^0 ct} + \delta_{\kappa_1+}\delta_{\lambda_1 2}\,\xi^{+-}_{20}(0)\,\delta(\boldsymbol{k}_1 + \boldsymbol{k}')\,\mathrm{e}^{\mathrm{i}k'^0 ct}. \tag{319}$$

Here both signs of κ_1 are included, because there is no continuity condition on the transverse solutions, which are absent in the limit of small velocity transformations. These results yield Eq. (297) with the non-zero coefficients given by

$$\xi_{00}^{++}(t) = \xi_{00}^{++}(0)\,\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\tanh\zeta\,ct}, \tag{320}$$
$$\xi_{00}^{--}(t) = \xi_{00}^{--}(0)\,\mathrm{e}^{\mathrm{i}\hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\tanh\zeta\,ct}, \tag{321}$$
$$\xi_{10}^{++}(t) = \xi_{10}^{++}(0)\,\mathrm{e}^{\mathrm{i}(|\boldsymbol{k}'|-\hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\tanh\zeta)\,ct}, \tag{322}$$
$$\xi_{10}^{-+}(t) = \xi_{10}^{-+}(0)\,\mathrm{e}^{-\mathrm{i}(|\boldsymbol{k}'|+\hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\tanh\zeta)\,ct}, \tag{323}$$
$$\xi_{20}^{+-}(t) = \xi_{20}^{+-}(0)\,\mathrm{e}^{\mathrm{i}(|\boldsymbol{k}'|+\hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\tanh\zeta)\,ct}, \tag{324}$$
$$\xi_{20}^{--}(t) = \xi_{20}^{--}(0)\,\mathrm{e}^{-\mathrm{i}(|\boldsymbol{k}'|-\hat{\boldsymbol{v}}\cdot\boldsymbol{k}'\tanh\zeta)\,ct}. \tag{325}$$

These transformed longitudinal solutions can be used, for example, to describe the fields of a moving charge by means of the expansion in Eq. (248).

7.7. Standing-wave parity eigenfunctions

The parity operator $\mathfrak{P}$ changes the sign of the coordinates and includes multiplication by the matrix $\mathcal{P} = -\gamma^0$, so that the transformed wave function is also a solution of the Maxwell equations, as discussed in Sec. 6.6. We thus have

$$\mathfrak{P}\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(\boldsymbol{x}) = -\gamma^0\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(-\boldsymbol{x}). \tag{326}$$

With this definition, the parity operator commutes with the Hamiltonian in Eq. (208)

$$\mathfrak{P}\mathcal{H} = \mathcal{H}\mathfrak{P}, \tag{327}$$

so we may identify eigenstates of both parity and energy. Since

$$\mathfrak{P}^2\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(\boldsymbol{x}) = \psi_{\boldsymbol{k},\lambda}^{(\kappa)}(\boldsymbol{x}), \tag{328}$$

the parity eigenvalues are ±1.

Transverse parity and energy eigenstates are

$$\psi_{\boldsymbol{k},\lambda}^{(\kappa,+)}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^3}}\begin{pmatrix} \kappa\mathrm{i}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\sin\boldsymbol{k}\cdot\boldsymbol{x} \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\cos\boldsymbol{k}\cdot\boldsymbol{x}\end{pmatrix}, \tag{329}$$

$$\psi_{\boldsymbol{k},\lambda}^{(\kappa,-)}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^3}}\begin{pmatrix} \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\cos\boldsymbol{k}\cdot\boldsymbol{x} \\ \kappa\mathrm{i}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\sin\boldsymbol{k}\cdot\boldsymbol{x}\end{pmatrix}, \tag{330}$$

where

$$\mathfrak{P}\psi_{\boldsymbol{k},\lambda}^{(\kappa,\pm)}(\boldsymbol{x}) = \pm\psi_{\boldsymbol{k},\lambda}^{(\kappa,\pm)}(\boldsymbol{x}), \tag{331}$$
$$\mathcal{H}\psi_{\boldsymbol{k},\lambda}^{(\kappa,\pm)}(\boldsymbol{x}) = \kappa\hbar c|\boldsymbol{k}|\psi_{\boldsymbol{k},\lambda}^{(\kappa,\pm)}(\boldsymbol{x}), \tag{332}$$

and $\lambda = 1,\ 2$. These states are linear combinations of plane-wave states that form standing plane waves. Orthogonality relations are calculated with the aid of the integrals

$$\int \mathrm{d}\boldsymbol{x}\ \cos\boldsymbol{k}_2\cdot\boldsymbol{x}\ \cos\boldsymbol{k}_1\cdot\boldsymbol{x} \quad = \quad 4\pi^3\left[\delta(\boldsymbol{k}_2-\boldsymbol{k}_1)+\delta(\boldsymbol{k}_2+\boldsymbol{k}_1)\right], \tag{333}$$

$$\int \mathrm{d}\boldsymbol{x}\ \sin\boldsymbol{k}_2\cdot\boldsymbol{x}\ \sin\boldsymbol{k}_1\cdot\boldsymbol{x} \quad = \quad 4\pi^3\left[\delta(\boldsymbol{k}_2-\boldsymbol{k}_1)-\delta(\boldsymbol{k}_2+\boldsymbol{k}_1)\right], \tag{334}$$

$$\int \mathrm{d}\boldsymbol{x}\ \cos\boldsymbol{k}_2\cdot\boldsymbol{x}\ \sin\boldsymbol{k}_1\cdot\boldsymbol{x} \quad = \quad 0, \tag{335}$$

which show that the states that differ only by the sign of $\boldsymbol{k}$ are not orthogonal. One of the overlapping states may be removed by including only states with $\boldsymbol{k}$ such that $\boldsymbol{k}\cdot\boldsymbol{k}_0 > 0$, where $\boldsymbol{k}_0$ is a fixed direction in space. With this restriction on $\boldsymbol{k}_2$ and $\boldsymbol{k}_1$, the orthonormality of the states is given by

$$\int \mathrm{d}\boldsymbol{x}\,\psi^{(\kappa_2,\pi_2)\dagger}_{\boldsymbol{k}_2,\lambda_2}(\boldsymbol{x})\,\psi^{(\kappa_1,\pi_1)}_{\boldsymbol{k}_1,\lambda_1}(\boldsymbol{x}) = \delta_{\kappa_2\kappa_1}\delta_{\pi_2\pi_1}\delta_{\lambda_2\lambda_1}\delta(\boldsymbol{k}_2-\boldsymbol{k}_1). \tag{336}$$

Completeness of these eigenfunctions, including the restriction on $\boldsymbol{k}$ provided by a factor $\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0)$, where the theta function is defined as

$$\theta(x) \quad = \quad \begin{cases} 1 & \text{for } x > 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases}, \tag{337}$$

follows from

$$\begin{aligned}
&\sum_{\kappa,\pi\to\pm}\sum_{\lambda=1}^{2}\int \mathrm{d}\boldsymbol{k}\,\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0)\,\psi^{(\kappa,\pi)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_2)\,\psi^{(\kappa,\pi)\dagger}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_1) \\
&\quad = \frac{2}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\,\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0)\mathit{\Pi}^{\mathrm{T}}(\hat{\boldsymbol{k}})(\cos\boldsymbol{k}\cdot\boldsymbol{x}_2\ \cos\boldsymbol{k}\cdot\boldsymbol{x}_1+\sin\boldsymbol{k}\cdot\boldsymbol{x}_2\ \sin\boldsymbol{k}\cdot\boldsymbol{x}_1) \\
&\quad = \mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1).
\end{aligned} \tag{338}$$

Longitudinal parity and energy eigenstates are given by

$$\psi^{(+,+)}_{\boldsymbol{k},0}(\boldsymbol{x}) \quad = \quad \frac{1}{\sqrt{4\pi^3}}\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\sin\boldsymbol{k}\cdot\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}, \tag{339}$$

$$\psi^{(+,-)}_{\boldsymbol{k},0}(\boldsymbol{x}) \quad = \quad \frac{1}{\sqrt{4\pi^3}}\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\cos\boldsymbol{k}\cdot\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}, \tag{340}$$

$$\psi^{(-,+)}_{\boldsymbol{k},0}(\boldsymbol{x}) \quad = \quad \frac{1}{\sqrt{4\pi^3}}\begin{pmatrix}\boldsymbol{0}\\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\cos\boldsymbol{k}\cdot\boldsymbol{x}\end{pmatrix}, \tag{341}$$

$$\psi^{(-,-)}_{\boldsymbol{k},0}(\boldsymbol{x}) \quad = \quad \frac{1}{\sqrt{4\pi^3}}\begin{pmatrix}\boldsymbol{0}\\ \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{k}})\sin\boldsymbol{k}\cdot\boldsymbol{x}\end{pmatrix}, \tag{342}$$

where

$$\mathfrak{P}\psi_{\boldsymbol{k},0}^{(\kappa,\pm)}(\boldsymbol{x}) = \pm\psi_{\boldsymbol{k},0}^{(\kappa,\pm)}(\boldsymbol{x}), \tag{343}$$

$$\mathcal{H}\psi_{\boldsymbol{k},0}^{(\kappa,\pm)}(\boldsymbol{x}) = 0. \tag{344}$$

As for the transverse eigenfunctions, there is overlap between states with opposite signs of $\boldsymbol{k}$, so the same condition on $\boldsymbol{k}$ may be applied here, that is $\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0) > 0$, to eliminate the redundancy. With this condition on $\boldsymbol{k}_2$ and $\boldsymbol{k}_1$, the orthonormality relation is

$$\int \mathrm{d}\boldsymbol{x}\, \psi_{\boldsymbol{k}_2,0}^{(\kappa_2,\pi_2)\dagger}(\boldsymbol{x})\, \psi_{\boldsymbol{k}_1,0}^{(\kappa_1,\pi_1)}(\boldsymbol{x}) = \delta_{\kappa_2\kappa_1}\delta_{\pi_2\pi_1}\delta(\boldsymbol{k}_2 - \boldsymbol{k}_1). \tag{345}$$

In addition, the longitudinal parity eigenfunctions are orthogonal to the transverse parity eigenfunctions. Completeness of the longitudinal parity eigenfunctions is given by

$$\begin{aligned}&\sum_{\kappa,\pi\to\pm}\int \mathrm{d}\boldsymbol{k}\,\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0)\,\psi_{\boldsymbol{k},0}^{(\kappa,\pi)}(\boldsymbol{x}_2)\,\psi_{\boldsymbol{k},0}^{(\kappa,\pi)\dagger}(\boldsymbol{x}_1)\\ &\quad = \frac{2}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\,\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0)\mathit{\Pi}^{\mathrm{L}}(\hat{\boldsymbol{k}})(\cos\boldsymbol{k}\cdot\boldsymbol{x}_2\,\cos\boldsymbol{k}\cdot\boldsymbol{x}_1 + \sin\boldsymbol{k}\cdot\boldsymbol{x}_2\,\sin\boldsymbol{k}\cdot\boldsymbol{x}_1)\\ &\quad = \mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2 - \boldsymbol{x}_1).\end{aligned} \tag{346}$$

The combined completeness relation is thus

$$\sum_{\kappa,\pi\to\pm}\sum_{\lambda=0}^{2}\int \mathrm{d}\boldsymbol{k}\,\theta(\boldsymbol{k}\cdot\boldsymbol{k}_0)\,\psi_{\boldsymbol{k},\lambda}^{(\kappa,\pi)}(\boldsymbol{x}_2)\,\psi_{\boldsymbol{k},\lambda}^{(\kappa,\pi)\dagger}(\boldsymbol{x}_1) = \mathcal{I}\delta(\boldsymbol{x}_2 - \boldsymbol{x}_1). \tag{347}$$

7.8. Wave packets

The plane wave solutions considered in this section are not normalizable as ordinary functions. Rather, integrals over products of solutions should be interpreted in the sense of distributions or generalized functions, like the delta function [26]. That is, they provide a well-defined value for an integral when they are included in the integrand together with a suitable weight, or test function. However, the plane waves can provide the basis for an expansion of a normalizable wave packet as a sum and integral over a complete set of solutions of the Maxwell equation. If $f_\lambda^{(\kappa)}(\boldsymbol{k})$ is a suitable function, we write

$$\mathit{\Psi}_f(x) = \sum_{\kappa\,\lambda}\int \mathrm{d}\boldsymbol{k}\, f_\lambda^{(\kappa)}(\boldsymbol{k})\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(x), \tag{348}$$

and $\mathit{\Psi}_f$ is a solution of the Maxwell equation

$$\gamma^\mu\partial_\mu\,\mathit{\Psi}_f(x) = \sum_{\kappa\,\lambda}\int \mathrm{d}\boldsymbol{k}\, f_\lambda^{(\kappa)}(\boldsymbol{k})\,\gamma^\mu\partial_\mu\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)}(x) = 0. \tag{349}$$

Further, $\mathit{\Psi}_f$ will be normalized if $f_\lambda^{(\kappa)}$ is, because

$$\int \mathrm{d}\boldsymbol{x}\ \mathit{\Psi}_f^\dagger(x)\,\mathit{\Psi}_f(x) = \sum_{\kappa\,\lambda}\sum_{\kappa'\,\lambda'}\int \mathrm{d}\boldsymbol{k}\int \mathrm{d}\boldsymbol{k}'\, f_\lambda^{(\kappa)*}(\boldsymbol{k})\int \mathrm{d}\boldsymbol{x}\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)\dagger}(x)\,\psi_{\boldsymbol{k}',\lambda'}^{(\kappa')}(x) f_{\lambda'}^{(\kappa')}(\boldsymbol{k}')$$
$$= \sum_{\kappa\,\lambda}\int \mathrm{d}\boldsymbol{k}\,\left|f_\lambda^{(\kappa)}(\boldsymbol{k})\right|^2 = 1. \qquad (350)$$

In view of the orthonormality (in the generalized sense) of the plane-wave solutions, Eq. (348) may be inverted to give

$$f_\lambda^{(\kappa)}(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{x}\,\psi_{\boldsymbol{k},\lambda}^{(\kappa)\dagger}(\boldsymbol{x})\,\mathit{\Psi}_f(\boldsymbol{x}), \qquad (351)$$

where we have specified $\mathit{\Psi}_f(\boldsymbol{x}) = \mathit{\Psi}_f(x)|_{t=0}$, and the time dependence of the wave function is given by Eq. (348).

An example is a normalized photon wave packet which at $t = 0$ has (approximately) a wave vector $\boldsymbol{k}_0$, a transverse polarization vector $\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0)$, and a Gaussian envelope of length a and width b centered at $\boldsymbol{x} = 0$:

$$\mathit{\Psi}_f(\boldsymbol{x}) = \frac{1}{a^{\frac{1}{2}}b}\left(\frac{2}{\pi^3}\right)^{\frac{1}{4}}\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0)\\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_0\ \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0)\end{pmatrix}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_0\cdot\boldsymbol{x}}\,\mathrm{e}^{-\left(\boldsymbol{x}_\parallel^2/a^2+\boldsymbol{x}_\perp^2/b^2\right)}, \qquad (352)$$

where $\boldsymbol{x}_\parallel = \boldsymbol{x}\cdot\hat{\boldsymbol{k}}_0\,\hat{\boldsymbol{k}}_0$ and $\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel$. For a and b large compared to $|\boldsymbol{k}_0|^{-1}$, the packet has a functional form that resembles a positive-energy transverse plane wave. From Eq. (351), we have

$$f_\lambda^{(\kappa)}(\boldsymbol{k}) = \frac{a^{\frac{1}{2}}b}{2}\left(\frac{2}{\pi^3}\right)^{\frac{1}{4}} F_\lambda^{(\kappa)}(\hat{\boldsymbol{k}})\ \mathrm{e}^{-\left[\left(\boldsymbol{k}_0-\kappa\boldsymbol{k}_\parallel\right)^2 a^2/4+\boldsymbol{k}_\perp^2 b^2/4\right]}, \qquad (353)$$

where $\boldsymbol{k}_\parallel = \boldsymbol{k}\cdot\hat{\boldsymbol{k}}_0\,\hat{\boldsymbol{k}}_0$, $\boldsymbol{k}_\perp = \boldsymbol{k} - \boldsymbol{k}_\parallel$, for $\lambda = 1, 2$,

$$F_\lambda^{(\kappa)}(\hat{\boldsymbol{k}}) = \frac{1}{2}\,\hat{\boldsymbol{\epsilon}}_\lambda^\dagger(\hat{\boldsymbol{k}})\left(\boldsymbol{I} + \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_0\right)\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0), \qquad (354)$$

and

$$F_0^{(+)}(\hat{\boldsymbol{k}}) = \frac{1}{\sqrt{2}}\,\hat{\boldsymbol{\epsilon}}_0^\dagger(\hat{\boldsymbol{k}})\,\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0), \qquad (355)$$

$$F_0^{(-)}(\hat{\boldsymbol{k}}) = \frac{1}{\sqrt{2}}\,\hat{\boldsymbol{\epsilon}}_0^\dagger(\hat{\boldsymbol{k}})\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_0\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0). \qquad (356)$$

The wave packet has small longitudinal components, because $F_0^{(\kappa)}(\hat{\boldsymbol{k}})$ is not necessarily zero unless $\hat{\boldsymbol{k}} = \pm\hat{\boldsymbol{k}}_0$. It has negative-energy components, but they are also suppressed, particularly as $a, b \to \infty$, because for $\kappa \to -$, the exponential factors in Eq. (353) favor $\boldsymbol{k} =$

$-\boldsymbol{k}_0$, and for $\lambda = 1, 2$, $F_\lambda^{(\kappa)}(-\hat{\boldsymbol{k}}_0) = 0$ as compared to $F_\lambda^{(\kappa)}(\hat{\boldsymbol{k}}_0) = \delta_{\lambda 1}$. Thus a cancellation between the upper-three and lower-three components of the wave function suppresses the contribution of negative-energy eigenstates to the wave packet.

The expectation value of the Hamiltonian $\mathcal{H}$, Eq. (208), is

$$\langle \mathit{\Psi}_f \,|\, \mathcal{H} \,|\, \mathit{\Psi}_f \rangle = -\mathrm{i}\,\hbar c \int \mathrm{d}\boldsymbol{x}\, \mathit{\Psi}_f^\dagger(\boldsymbol{x})\, \boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\, \mathit{\Psi}_f(\boldsymbol{x}) = \hbar c |\boldsymbol{k}_0| = \hbar\omega_0, \tag{357}$$

and the expectation value of the momentum $\boldsymbol{\mathcal{P}}$, Eq. (209), is

$$\langle \mathit{\Psi}_f \,|\, \boldsymbol{\mathcal{P}} \,|\, \mathit{\Psi}_f \rangle = \hbar \boldsymbol{k}_0. \tag{358}$$

The initial probability density $Q(\boldsymbol{x})$ is

$$Q(\boldsymbol{x}) = \mathit{\Psi}_f^\dagger(\boldsymbol{x})\, \mathit{\Psi}_f(\boldsymbol{x}) = \frac{2}{ab^2}\left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \mathrm{e}^{-2\left(\boldsymbol{x}_\parallel^2/a^2 + \boldsymbol{x}_\perp^2/b^2\right)}, \tag{359}$$

with

$$\int \mathrm{d}\boldsymbol{x}\, Q(\boldsymbol{x}) = 1, \tag{360}$$

and the initial energy density $E(\boldsymbol{x})$ is

$$E(\boldsymbol{x}) = \mathit{\Psi}_f^\dagger(\boldsymbol{x})\, \mathcal{H}\, \mathit{\Psi}_f(\boldsymbol{x}) = \hbar\omega_0 Q(\boldsymbol{x}) - \frac{\mathrm{i}}{2}\,\hbar c \hat{\boldsymbol{k}}_0 \cdot \boldsymbol{\nabla} Q(\boldsymbol{x}). \tag{361}$$

The real part of the energy density is proportional to the probability density for the photon, and the imaginary term, which vanishes upon integration to arrive at the expectation value in Eq. (357), reflects the change in the initial probability density at the point $\boldsymbol{x}$ due to the motion of the wave packet. At a fixed point in the path of the wave packet, the probability density increases as the packet approaches and decreases after the maximum of the wave packet has passed by. The time-dependent probability density is

$$Q(x) = \mathit{\Psi}_f^\dagger(x)\, \mathit{\Psi}_f(x) = \left| \mathrm{e}^{-\mathrm{i}\,\mathcal{H}\,t/\hbar}\, \mathit{\Psi}_f(\boldsymbol{x}) \right|^2, \tag{362}$$

and the change at $t = 0$ is

$$\begin{aligned} \left.\frac{\partial Q(x)}{\partial t}\right|_{t=0} &= -\frac{\mathrm{i}}{\hbar}\, \mathit{\Psi}_f^\dagger(\boldsymbol{x}) \left(\mathcal{H} - \overleftarrow{\mathcal{H}}^\dagger \right) \mathit{\Psi}_f(\boldsymbol{x}) \\ &= \frac{2}{\hbar}\, \mathrm{Im}\, \mathit{\Psi}_f^\dagger(\boldsymbol{x}) \mathcal{H}\, \mathit{\Psi}_f(\boldsymbol{x}) = -c \hat{\boldsymbol{k}}_0 \cdot \boldsymbol{\nabla} Q(\boldsymbol{x}). \end{aligned} \tag{363}$$

The gradient produces a vector that points toward the maximum of the wave packet, so that on the forward side of the packet $\hat{\boldsymbol{k}}_0 \cdot \boldsymbol{\nabla} Q(\boldsymbol{x})$ is negative and the probability density is increasing, as expected. Eq. (363) also shows that the wave packet is initially moving with velocity c in the direction of $\hat{\boldsymbol{k}}_0$.

The time dependence of the wave packet, Eq. (352) at $t = 0$, is given exactly by Eq. (348). However, approximations may be made in order to obtain a more transparent expression. In view of the exponential factors in Eq. (353), the assumption that $a, b \gg |\boldsymbol{k}_0|^{-1}$ implies $\boldsymbol{k} \approx \kappa \boldsymbol{k}_0$, and

$$F_\lambda^{(\kappa)}(\hat{\boldsymbol{k}}) \approx F_\lambda^{(\kappa)}(\kappa \hat{\boldsymbol{k}}_0) = \delta_{\lambda 1}\delta_{\kappa +}, \tag{364}$$

so that from Eq. (348), $\Psi_f \to \Psi_f'$, where

$$\Psi_f'(x) = \frac{a^{\frac{1}{2}} b}{8\pi^{\frac{3}{2}}}\left(\frac{2}{\pi^3}\right)^{\frac{1}{4}} \int \mathrm{d}\boldsymbol{k} \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\ \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}) \end{array}\right) \mathrm{e}^{-\mathrm{i}(|\boldsymbol{k}|ct - \boldsymbol{k}\cdot\boldsymbol{x})}\, \mathrm{e}^{-\left[\left(\boldsymbol{k}_0 - \boldsymbol{k}_\parallel\right)^2 a^2/4 + \boldsymbol{k}_\perp^2 b^2/4\right]}. \tag{365}$$

This is a normalized positive energy wave function with polarization $\hat{\boldsymbol{\epsilon}}_1$ that is an exact solution of the Maxwell equation $\gamma^\mu \partial_\mu \Psi_f'(x) = 0$. Further simplifications are the replacements $\hat{\boldsymbol{k}} \to \hat{\boldsymbol{k}}_0$ in the polarization vector matrix and $|\boldsymbol{k}| \to \boldsymbol{k}\cdot\hat{\boldsymbol{k}}_0$ in the exponent, which yield $\Psi_f' \to \Psi_f''$, with

$$\begin{aligned} \Psi_f''(x) &= \frac{a^{\frac{1}{2}} b}{8\pi^{\frac{3}{2}}}\left(\frac{2}{\pi^3}\right)^{\frac{1}{4}} \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_0\ \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0) \end{array}\right) \int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{-\mathrm{i}\left(\boldsymbol{k}\cdot\hat{\boldsymbol{k}}_0 ct - \boldsymbol{k}\cdot\boldsymbol{x}\right)}\, \mathrm{e}^{-\left[\left(\boldsymbol{k}_0 - \boldsymbol{k}_\parallel\right)^2 a^2/4 + \boldsymbol{k}_\perp^2 b^2/4\right]} \\ &= \frac{1}{a^{\frac{1}{2}} b}\left(\frac{2}{\pi^3}\right)^{\frac{1}{4}} \left(\begin{array}{c} \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}_0\ \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0) \end{array}\right) \mathrm{e}^{-\mathrm{i}(\omega_0 t - \boldsymbol{k}_0\cdot\boldsymbol{x})}\, \mathrm{e}^{-\left[\left(ct - \hat{\boldsymbol{k}}_0\cdot\boldsymbol{x}\right)^2/a^2 + \boldsymbol{x}_\perp^2/b^2\right]}, \end{aligned} \tag{366}$$

which is an approximate wave function with a normalized Gaussian probability distribution

$$Q''(x) = \frac{2}{ab^2}\left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} \mathrm{e}^{-2\left[\left(ct - \hat{\boldsymbol{k}}_0\cdot\boldsymbol{x}\right)^2/a^2 + \boldsymbol{x}_\perp^2/b^2\right]}, \tag{367}$$

that moves with velocity c in the $\hat{\boldsymbol{k}}_0$ direction.

7.9. Conservation of probability

The formulation of the Poynting theorem in Sec. 4 can be reinterpreted here to demonstrate conservation of probability. We define the probability density four-vector to be

$$q^\mu(x) = \overline{\Psi}(x)\gamma^\mu \Psi(x), \tag{368}$$

where in the previous section $Q(x) = q^0(x)$. For the source-free case, $\Xi(x) = 0$, Eq. (57) is

$$\partial_\mu \overline{\Psi}(x)\gamma^\mu \Psi(x) = 0, \tag{369}$$

which is the statement of conservation of probability

$$\frac{\partial}{\partial t} q^0(x) + c\,\boldsymbol{\nabla}\cdot\boldsymbol{q}(x) = 0. \tag{370}$$

Applying this relation to plane-wave states gives consistent, although trivial, results. We have

$$q^0(x) \;=\; \psi^{(\kappa)\dagger}_{\boldsymbol{k},\lambda}(x)\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(x) = (2\pi)^{-3}, \tag{371}$$

which reflects the fact that the probability distribution for a plane wave is uniform throughout space and not normalizable. For transverse plane waves, $\lambda = 1, 2$, the probability flux vector is

$$\boldsymbol{q}(x) \;=\; \psi^{(\kappa)\dagger}_{\boldsymbol{k},\lambda}(x)\,\boldsymbol{\alpha}\,\psi^{(\kappa)}_{\boldsymbol{k},\lambda}(x) = (2\pi)^{-3}\hat{\boldsymbol{k}}, \tag{372}$$

and for longitudinal plane waves $\boldsymbol{q}(x) = 0$.

For the wave packet in Eq. (352), at $t = 0$

$$\boldsymbol{q}(x) \;=\; \hat{\boldsymbol{k}}_0\,q^0(x), \tag{373}$$

so Eq. (363) is essentially the conservation of probability equation evaluated at $t = 0$. Conservation of probability is valid for any solution of the homogeneous Maxwell equation from the definition of the probability density operator. However, it also happens to be valid for the wave packet represented by Ψ''_f, which is not an exact solution of the wave equation. The wave equation is not satisfied by Ψ''_f, because there is a small extra term resulting from the gradient operator acting on the perpendicular coordinate $\boldsymbol{x}_\perp$. On the other hand, $\boldsymbol{q}(x)$ for Ψ''_f is proportional to $\hat{\boldsymbol{k}}_0$ and $\hat{\boldsymbol{k}}_0 \cdot \boldsymbol{\nabla}\,\boldsymbol{x}_\perp = 0$, so there is no corresponding extra term in $\boldsymbol{\nabla}\cdot\boldsymbol{q}(x)$.

8. Angular momentum eigenfunctions

Radiation emitted in atomic transitions is characterized by its angular momentum and parity. In this section, wave functions that are eigenfunctions of energy, angular momentum, and parity are given; they are also classified according to whether they are transverse or longitudinal. The three-component angular-momentum matrices given here are to some extent parallels of the three-vector functions of [27].

8.1. Angular momentum

The spatial angular-momentum operator is given by

$$\boldsymbol{L} \;=\; \boldsymbol{x}\times\boldsymbol{p} = -\mathrm{i}\,\hbar\,\boldsymbol{x}\times\boldsymbol{\nabla}, \tag{374}$$

and following the example of the Dirac equation, the total angular momentum is given as a 3×3 matrix by [5]

$$\boldsymbol{J} \;=\; \boldsymbol{L} + \hbar\boldsymbol{\tau}, \tag{375}$$

where it is understood that the first term on the right side is a 3×3 matrix with $\boldsymbol{L}$ for diagonal elements and zeros for the rest.[4] (In order to adhere to convention, we denote

[4] In some works, where electric and magnetic fields or the vector potential are three-vector valued fields, the spin operator is represented by a cross product. For example, Corben and Schwinger [27] write $J_z\boldsymbol{\Phi} = L_z\boldsymbol{\Phi} + \mathrm{i}\,\boldsymbol{e}_z\times\boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ is a vector potential, and in Edmonds [28], spin is represented symbolically as $\mathrm{i}\,\hat{\boldsymbol{e}}\times$.

both the current three-vector and the angular-momentum matrix by $\boldsymbol{J}$. In either case, the meaning should be clear from the context.) The extension to a 6×6 matrix is

$$\boldsymbol{\mathcal{J}} = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} \end{pmatrix} = \boldsymbol{x} \times \boldsymbol{\mathcal{P}} + \hbar \boldsymbol{\mathcal{S}} \tag{376}$$

and we have

$$[\mathcal{H}, \boldsymbol{\mathcal{J}}] = 0, \tag{377}$$

so eigenfunctions of both energy and angular momentum may be constructed. The vanishing of the commutator follows from the relations

$$[\boldsymbol{\tau} \cdot \boldsymbol{\nabla}, \boldsymbol{L}] = -\mathrm{i}\hbar \boldsymbol{\tau} \times \boldsymbol{\nabla}, \tag{378}$$

$$[\boldsymbol{\tau} \cdot \boldsymbol{\nabla}, \boldsymbol{\tau}] = \mathrm{i} \boldsymbol{\tau} \times \boldsymbol{\nabla}, \tag{379}$$

$$[\boldsymbol{\tau} \cdot \boldsymbol{\nabla}, \boldsymbol{J}] = 0. \tag{380}$$

It is of interest to see that $\boldsymbol{\mathcal{J}}$ commutes with $\mathcal{H}$ only for the (relative) combination of $\boldsymbol{L}$ and $\boldsymbol{\tau}$ given in Eq. (375). To obtain eigenstates of the square of the total angular momentum $\boldsymbol{\mathcal{J}}^2$ and the third component of angular momentum $\mathcal{J}^3$, given by

$$\boldsymbol{\mathcal{J}}^2 = \begin{pmatrix} \boldsymbol{J}^2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}^2 \end{pmatrix} \tag{381}$$

and

$$\mathcal{J}^3 = \begin{pmatrix} J^3 & \boldsymbol{0} \\ \boldsymbol{0} & J^3 \end{pmatrix}, \tag{382}$$

we construct matrix spherical harmonics that are analogous to conventional vector spherical harmonics and are three-component extensions of the Dirac two-component spin-angular-momentum eigenfunctions. Orthonormal basis matrices are given by

$$\hat{\boldsymbol{\epsilon}}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{\epsilon}}^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{\epsilon}}^{(-1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{383}$$

and they satisfy the eigenvalue equations

$$\boldsymbol{\tau}^2 \, \hat{\boldsymbol{\epsilon}}^{(\nu)} = 2 \, \hat{\boldsymbol{\epsilon}}^{(\nu)}, \tag{384}$$

$$\tau^3 \, \hat{\boldsymbol{\epsilon}}^{(\nu)} = \nu \, \hat{\boldsymbol{\epsilon}}^{(\nu)}, \tag{385}$$

where the 2 may be regarded as $s(s+1)$, with $s = 1$ as the spin eigenvalue. The matrix spherical harmonics are

$$\boldsymbol{Y}_{jl}^{m}(\hat{\boldsymbol{x}}) = \sum_{\nu} (l \ m-\nu \ 1 \ \nu | l \ 1 \ j \ m) Y_l^{m-\nu}(\hat{\boldsymbol{x}}) \, \hat{\boldsymbol{\epsilon}}^{(\nu)}, \tag{386}$$

with vector addition coefficients and spherical harmonics in the notation of [28]. The spherical harmonics satisfy the eigenvalue equations

$$\boldsymbol{L}^2 Y_l^m(\hat{\boldsymbol{x}}) = \hbar^2 l(l+1)\, Y_l^m(\hat{\boldsymbol{x}}), \tag{387}$$

$$L^3 Y_l^m(\hat{\boldsymbol{x}}) = \hbar\, m\, Y_l^m(\hat{\boldsymbol{x}}), \tag{388}$$

and by their construction, the matrix spherical harmonics are eigenfunctions of the total angular momentum:

$$\boldsymbol{J}^2 \mathbf{Y}_{jl}^m(\hat{\boldsymbol{x}}) = \hbar^2 j(j+1)\, \mathbf{Y}_{jl}^m(\hat{\boldsymbol{x}}), \tag{389}$$

$$J^3 \mathbf{Y}_{jl}^m(\hat{\boldsymbol{x}}) = \hbar\, m\, \mathbf{Y}_{jl}^m(\hat{\boldsymbol{x}}). \tag{390}$$

Explicit expressions in terms of spherical harmonics are

$$\mathbf{Y}_{jj}^m(\hat{\boldsymbol{x}}) = \begin{pmatrix} -\sqrt{\frac{(j+m)(j+1-m)}{2j(j+1)}}\, Y_j^{m-1}(\hat{\boldsymbol{x}}) \\ \frac{m}{\sqrt{j(j+1)}}\, Y_j^m(\hat{\boldsymbol{x}}) \\ \sqrt{\frac{(j-m)(j+1+m)}{2j(j+1)}}\, Y_j^{m+1}(\hat{\boldsymbol{x}}) \end{pmatrix}, \tag{391}$$

$$\mathbf{Y}_{jj+1}^m(\hat{\boldsymbol{x}}) = \begin{pmatrix} \sqrt{\frac{(j+1-m)(j+2-m)}{(2j+2)(2j+3)}}\, Y_{j+1}^{m-1}(\hat{\boldsymbol{x}}) \\ -\sqrt{\frac{(j+1-m)(j+1+m)}{(j+1)(2j+3)}}\, Y_{j+1}^m(\hat{\boldsymbol{x}}) \\ \sqrt{\frac{(j+2+m)(j+1+m)}{(2j+2)(2j+3)}}\, Y_{j+1}^{m+1}(\hat{\boldsymbol{x}}) \end{pmatrix}, \tag{392}$$

$$\mathbf{Y}_{jj-1}^m(\hat{\boldsymbol{x}}) = \begin{pmatrix} \sqrt{\frac{(j-1+m)(j+m)}{(2j-1)2j}}\, Y_{j-1}^{m-1}(\hat{\boldsymbol{x}}) \\ \sqrt{\frac{(j-m)(j+m)}{(2j-1)j}}\, Y_{j-1}^m(\hat{\boldsymbol{x}}) \\ \sqrt{\frac{(j-1-m)(j-m)}{(2j-1)2j}}\, Y_{j-1}^{m+1}(\hat{\boldsymbol{x}}) \end{pmatrix}. \tag{393}$$

These functions are orthonormal

$$\int \mathrm{d}\boldsymbol{\Omega}\, \mathbf{Y}_{j_2 l_2}^{m_2 \dagger}(\hat{\boldsymbol{x}})\, \mathbf{Y}_{j_1 l_1}^{m_1}(\hat{\boldsymbol{x}}) = \delta_{j_2 j_1}\delta_{l_2 l_1}\delta_{m_2 m_1}, \tag{394}$$

which follows from the relations

$$\int \mathrm{d}\boldsymbol{\Omega}\, Y_{l_2}^{m_2 *}(\hat{\boldsymbol{x}})\, Y_{l_1}^{m_1}(\hat{\boldsymbol{x}}) = \delta_{l_2 l_1}\delta_{m_2 m_1}, \tag{395}$$

$$\hat{\boldsymbol{\epsilon}}^{(\nu_2)\dagger}\, \hat{\boldsymbol{\epsilon}}^{(\nu_1)} = \delta_{\nu_2 \nu_1}, \tag{396}$$

$$\sum_\nu (l\ 1\ j_2\ m|l\ m-\nu\ 1\ \nu)(l\ m-\nu\ 1\ \nu|l\ 1\ j_1\ m) = \delta_{j_2 j_1}, \tag{397}$$

and they are complete

$$\sum_{jlm} \boldsymbol{Y}_{jl}^{m}(\hat{\boldsymbol{x}}_2)\boldsymbol{Y}_{jl}^{m\dagger}(\hat{\boldsymbol{x}}_1) = \boldsymbol{I}\,\delta(\cos\theta_2 - \cos\theta_1)\,\delta(\phi_2 - \phi_1), \tag{398}$$

based on the relations

$$\sum_{j} (l\ m-\nu_2\ 1\ \nu_2|l\ 1\ j\ m)(l\ 1\ j\ m|l\ m-\nu_1\ 1\ \nu_1) = \delta_{\nu_2\nu_1}, \tag{399}$$

$$\sum_{\nu} \hat{\boldsymbol{\epsilon}}^{(\nu)}\,\hat{\boldsymbol{\epsilon}}^{(\nu)\dagger} = \boldsymbol{I}, \tag{400}$$

$$\sum_{lm} Y_l^m(\hat{\boldsymbol{x}}_2)Y_l^{m*}(\hat{\boldsymbol{x}}_1) = \delta(\cos\theta_2 - \cos\theta_1)\,\delta(\phi_2 - \phi_1), \tag{401}$$

where θ_i, ϕ_i are the spherical coordinates of $\hat{\boldsymbol{x}}_i$.

An alternative set of matrix angular-momentum eigenfunctions is

$$\begin{aligned}
\boldsymbol{X}_1^{jm}(\hat{\boldsymbol{x}}) &= \frac{1}{\hbar\sqrt{j(j+1)}}\,\boldsymbol{L}_{\rm s}Y_j^m(\hat{\boldsymbol{x}}), &(402)\\
\boldsymbol{X}_2^{jm}(\hat{\boldsymbol{x}}) &= \frac{1}{\hbar\sqrt{j(j+1)}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\boldsymbol{L}_{\rm s}Y_j^m(\hat{\boldsymbol{x}}), &(403)\\
\boldsymbol{X}_3^{jm}(\hat{\boldsymbol{x}}) &= \hat{\boldsymbol{x}}_{\rm s}Y_j^m(\hat{\boldsymbol{x}}). &(404)
\end{aligned}$$

For $j = 0$, $\boldsymbol{X}_1^{00}(\hat{\boldsymbol{x}}) = \boldsymbol{X}_2^{00}(\hat{\boldsymbol{x}}) = 0$. From a comparison of Eqs. (391)-(393) to Eqs. (402)-(404), one has

$$\begin{aligned}
\boldsymbol{X}_1^{jm}(\hat{\boldsymbol{x}}) &= \boldsymbol{Y}_{jj}^{m}(\hat{\boldsymbol{x}}), &(405)\\
\boldsymbol{X}_2^{jm}(\hat{\boldsymbol{x}}) &= -\sqrt{\frac{j}{2j+1}}\,\boldsymbol{Y}_{jj+1}^{m}(\hat{\boldsymbol{x}}) - \sqrt{\frac{j+1}{2j+1}}\,\boldsymbol{Y}_{jj-1}^{m}(\hat{\boldsymbol{x}}), &(406)\\
\boldsymbol{X}_3^{jm}(\hat{\boldsymbol{x}}) &= -\sqrt{\frac{j+1}{2j+1}}\,\boldsymbol{Y}_{jj+1}^{m}(\hat{\boldsymbol{x}}) + \sqrt{\frac{j}{2j+1}}\,\boldsymbol{Y}_{jj-1}^{m}(\hat{\boldsymbol{x}}). &(407)
\end{aligned}$$

In view of the relations in Eqs. (405)-(407), the functions $\boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}})$ are eigenfunctions of $\boldsymbol{J}^2$ and J^3 with

$$\begin{aligned}
\boldsymbol{J}^2\boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}}) &= \hbar^2 j(j+1)\,\boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}}), &(408)\\
J^3\boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}}) &= \hbar\,m\,\boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}}). &(409)
\end{aligned}$$

This can be confirmed directly from the definitions in Eqs. (402)-(404) with the aid of the commutation relations

$$\begin{aligned}
\left[L^i, L^j\right] &= {\rm i}\hbar\epsilon_{ijk}L^k, &(410)\\
\left[L^i, x^j\right] &= {\rm i}\hbar\epsilon_{ijk}x^k &(411)
\end{aligned}$$

and the tau matrix identities in Sec. 3, which provide the operator identities

$$J^i \boldsymbol{L}_{\rm s} = \boldsymbol{L}_{\rm s} L^i, \tag{412}$$

$$\left[J^i, \boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\right] = 0, \tag{413}$$

$$J^i \hat{\boldsymbol{x}}_{\rm s} = \hat{\boldsymbol{x}}_{\rm s} L^i. \tag{414}$$

These functions are orthonormal

$$\int {\rm d}\boldsymbol{\Omega}\, \boldsymbol{X}_{i_2}^{j_2 m_2\dagger}(\hat{\boldsymbol{x}})\, \boldsymbol{X}_{i_1}^{j_1 m_1}(\hat{\boldsymbol{x}}) = \delta_{i_2 i_1}\delta_{j_2 j_1}\delta_{m_2 m_1}, \tag{415}$$

and they are complete

$$\sum_{ijm} \boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}}_2)\, \boldsymbol{X}_i^{jm\dagger}(\hat{\boldsymbol{x}}_1) = \boldsymbol{I}\,\delta(\cos\theta_2-\cos\theta_1)\,\delta(\phi_2-\phi_1), \tag{416}$$

where the latter fact may be seen from the completeness of the matrix spherical harmonics and the relation

$$\sum_i \boldsymbol{X}_i^{jm}(\hat{\boldsymbol{x}}_2)\, \boldsymbol{X}_i^{jm\dagger}(\hat{\boldsymbol{x}}_1) = \sum_l \boldsymbol{Y}_{jl}^{m}(\hat{\boldsymbol{x}}_2)\boldsymbol{Y}_{jl}^{m\dagger}(\hat{\boldsymbol{x}}_1). \tag{417}$$

The parity of the eigenfunctions is given by

$$\boldsymbol{X}_1^{jm}(-\hat{\boldsymbol{x}}) = (-1)^j \boldsymbol{X}_1^{jm}(\hat{\boldsymbol{x}}), \tag{418}$$

$$\boldsymbol{X}_2^{jm}(-\hat{\boldsymbol{x}}) = (-1)^{j+1} \boldsymbol{X}_2^{jm}(\hat{\boldsymbol{x}}), \tag{419}$$

$$\boldsymbol{X}_3^{jm}(-\hat{\boldsymbol{x}}) = (-1)^{j+1} \boldsymbol{X}_3^{jm}(\hat{\boldsymbol{x}}), \tag{420}$$

which follows from $Y_j^m(-\hat{\boldsymbol{x}}) = (-1)^j Y_j^m(\hat{\boldsymbol{x}})$.

8.2. Helicity eigenstates

Special cases of the transverse plane-wave solutions in Eq. (217) are circularly polarized states with the polarization vectors in Eq. (216). They can be grouped with the longitudinal plane-wave states in Eqs. (237) and (238) with the polarization vector in Eq. (236). These polarization vectors are summarized here as

$$\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \hat{\boldsymbol{\epsilon}}_0(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \hat{\boldsymbol{\epsilon}}_{-1}(\hat{\boldsymbol{e}}^3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where we have changed the label for λ from 2 to -1 for this section.

The states have a well-defined helicity; they are eigenfunctions of the operator for the projection of angular momentum in the direction of the wave vector $\boldsymbol{\mathcal{J}}\cdot\hat{\boldsymbol{k}}$ [5, 29]. In view of the relations

$$\boldsymbol{L}\cdot\hat{\boldsymbol{k}}\,{\rm e}^{\pm{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \tag{421}$$

$$\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) = \lambda\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \tag{422}$$

for the polarizations considered here, we have

$$\boldsymbol{\mathcal{J}}\cdot\hat{\boldsymbol{k}}\,\psi_{\boldsymbol{k},\lambda}^{(\pm)}(\boldsymbol{x}) = \lambda\hbar\,\psi_{\boldsymbol{k},\lambda}^{(\pm)}(\boldsymbol{x}) \tag{423}$$

for these states.

8.3. Transverse spherical photons

Transverse spherical wave functions are given by

$$\psi^{\mathrm{T}(\kappa,+)}_{\omega,jm}(\boldsymbol{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} f_{\omega,j}(r)\boldsymbol{X}^{jm}_1(\hat{\boldsymbol{x}}) \\ -\kappa\mathrm{i}\frac{c}{\omega}\boldsymbol{\tau}\cdot\boldsymbol{\nabla}f_{\omega,j}(r)\boldsymbol{X}^{jm}_1(\hat{\boldsymbol{x}}) \end{pmatrix}, \tag{424}$$

$$\psi^{\mathrm{T}(\kappa,-)}_{\omega,jm}(\boldsymbol{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{c}{\omega}\boldsymbol{\tau}\cdot\boldsymbol{\nabla}f_{\omega,j}(r)\boldsymbol{X}^{jm}_1(\hat{\boldsymbol{x}}) \\ \kappa\mathrm{i}f_{\omega,j}(r)\boldsymbol{X}^{jm}_1(\hat{\boldsymbol{x}}) \end{pmatrix}, \tag{425}$$

where $r = |\boldsymbol{x}|$ and $j \geq 1$. They are transverse because $\boldsymbol{\nabla}^\dagger_{\mathrm{s}}\,\boldsymbol{L}_{\mathrm{s}} = \boldsymbol{\nabla}^\dagger_{\mathrm{s}}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla} = 0$, so that

$$\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla})\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}) = \psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}), \tag{426}$$

$$\mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla})\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}) = 0. \tag{427}$$

The wave functions are eigenfunctions of angular momentum with eigenvalues given by [see Eq. (380)]

$$\boldsymbol{\mathcal{J}}^2\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}) = \hbar^2 j(j+1)\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}), \tag{428}$$

$$\mathcal{J}^3\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}) = \hbar\, m\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}), \tag{429}$$

and they are eigenfunctions of $\mathcal{H}$, with eigenvalue $\kappa\hbar\omega$

$$\mathcal{H}\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}) = -\mathrm{i}\,\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}) = \kappa\hbar\omega\,\psi^{\mathrm{T}(\kappa,\pi)}_{\omega,jm}(\boldsymbol{x}), \tag{430}$$

provided

$$\left(\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2}\right) f_{\omega,j}(r)\boldsymbol{X}^{jm}_1(\hat{\boldsymbol{x}}) = 0, \tag{431}$$

which is true if

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}\,r - \frac{j(j+1)}{r^2} + \frac{\omega^2}{c^2}\right) f_{\omega,j}(r) = 0. \tag{432}$$

Solutions of Eq. (432) are spherical Bessel functions given by [30]

$$f_{\omega,j}(r) \propto \begin{cases} j_j(\omega r/c) \\ h^{(1)}_j(\omega r/c) \end{cases}. \tag{433}$$

We employ the normalized solution

$$f_{\omega,j}(r) = \frac{\omega}{c}\sqrt{\frac{2}{\pi c}}\; j_j(\omega r/c) \tag{434}$$

for the wave functions; any other linear combination of spherical Bessel functions (with $j \geq 1$) is not integrable as $r \to 0$. The parity of the wave functions is

$$\mathfrak{P}\,\psi^{\mathrm{T}(\kappa,+)}_{\omega,jm}(\boldsymbol{x}) = (-1)^{j+1}\,\psi^{\mathrm{T}(\kappa,+)}_{\omega,jm}(\boldsymbol{x}), \tag{435}$$

$$\mathfrak{P}\,\psi^{\mathrm{T}(\kappa,-)}_{\omega,jm}(\boldsymbol{x}) = (-1)^{j}\,\psi^{\mathrm{T}(\kappa,-)}_{\omega,jm}(\boldsymbol{x}). \tag{436}$$

This provides the conventional parity and angular-momentum attributes for electric and magnetic multipole radiation. Namely, $\psi^{\mathrm{T}(\kappa,+)}_{\omega,jm}(\boldsymbol{x})$ is magnetic $2j$-pole or Mj radiation and $\psi^{\mathrm{T}(\kappa,-)}_{\omega,jm}(\boldsymbol{x})$ is electric $2j$-pole or Ej radiation.

Alternative forms for the lower three components in Eq. (424) or the upper three components in Eq. (425) are obtained by writing (see Appendix D)

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial r}\,r\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}} + \frac{1}{\hbar r}\left(\boldsymbol{L}_{\mathrm{s}}\,\hat{\boldsymbol{x}}^{\dagger}_{\mathrm{s}} + \hat{\boldsymbol{x}}_{\mathrm{s}}\,\boldsymbol{L}^{\dagger}_{\mathrm{s}}\right), \tag{437}$$

which yields

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,f_{\omega,j}(r)\boldsymbol{X}^{jm}_{1}(\hat{\boldsymbol{x}}) = \frac{1}{r}\frac{\partial}{\partial r}\,r\,f_{\omega,j}(r)\boldsymbol{X}^{jm}_{2}(\hat{\boldsymbol{x}}) - \frac{\sqrt{j(j+1)}}{r}\,f_{\omega,j}(r)\boldsymbol{X}^{jm}_{3}(\hat{\boldsymbol{x}}). \tag{438}$$

Relations among spherical Bessel functions provide

$$\frac{1}{r}\frac{\partial}{\partial r}\,r\,f_{\omega,j}(r) = \frac{\omega}{c}\,\frac{1}{2j+1}[(j+1)f_{\omega,j-1}(r) - j\,f_{\omega,j+1}(r)], \tag{439}$$

$$\frac{1}{r}\,f_{\omega,j}(r) = \frac{\omega}{c}\,\frac{1}{2j+1}[f_{\omega,j-1}(r) + f_{\omega,j+1}(r)], \tag{440}$$

which together with Eqs. (406) and (407) yield the second alternative form

$$\frac{c}{\omega}\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,f_{\omega,j}(r)\boldsymbol{X}^{jm}_{1}(\hat{\boldsymbol{x}}) = -\sqrt{\frac{j+1}{2j+1}}f_{\omega,j-1}(r)\,\boldsymbol{Y}^{m}_{jj-1}(\hat{\boldsymbol{x}}) + \sqrt{\frac{j}{2j+1}}f_{\omega,j+1}(r)\,\boldsymbol{Y}^{m}_{jj+1}(\hat{\boldsymbol{x}}). \tag{441}$$

This latter form is useful in calculating the wave function orthonormality and completeness relations.

An analogous longitudinal function is obtained by writing [Eq. (D.6)]

$$\boldsymbol{\nabla}_{\mathrm{s}} = \hat{\boldsymbol{x}}_{\mathrm{s}}\frac{\partial}{\partial r} - \frac{1}{\hbar r}\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\boldsymbol{L}_{\mathrm{s}} \tag{442}$$

and

$$\begin{aligned}\frac{\omega}{c}\boldsymbol{F}^{jm}_{\omega}(\boldsymbol{x}) &= \boldsymbol{\nabla}_{\mathrm{s}}\,f_{\omega,j}(r)\,Y^{m}_{j}(\hat{\boldsymbol{x}}) \\ &= \frac{\partial}{\partial r}\,f_{\omega,j}(r)\boldsymbol{X}^{jm}_{3}(\hat{\boldsymbol{x}}) - \frac{\sqrt{j(j+1)}}{r}\,f_{\omega,j}(r)\boldsymbol{X}^{jm}_{2}(\hat{\boldsymbol{x}}),\end{aligned} \tag{443}$$

so that

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,\boldsymbol{F}_\omega^{jm}(\boldsymbol{x}) \;=\; 0. \tag{444}$$

From the additional relation

$$\frac{\partial}{\partial r}\,f_{\omega,j}(r) \;=\; \frac{\omega}{c}\,\frac{1}{2j+1}[j\,f_{\omega,j-1}(r)-(j+1)f_{\omega,j+1}(r)], \tag{445}$$

together with Eqs. (406) and (407), one has

$$\boldsymbol{F}_\omega^{jm}(\boldsymbol{x}) = \sqrt{\frac{j}{2j+1}}\;f_{\omega,j-1}(r)\,\boldsymbol{Y}_{jj-1}^m(\hat{\boldsymbol{x}}) + \sqrt{\frac{j+1}{2j+1}}\;f_{\omega,j+1}(r)\,\boldsymbol{Y}_{jj+1}^m(\hat{\boldsymbol{x}}). \tag{446}$$

The orthonormality of the transverse wave functions is given by

$$\int \mathrm{d}\boldsymbol{x}\,\psi_{\omega_2,j_2m_2}^{\mathrm{T}(\kappa_2,\pi_2)\dagger}(\boldsymbol{x})\psi_{\omega_1,j_1m_1}^{\mathrm{T}(\kappa_1,\pi_1)}(\boldsymbol{x}) \;=\; \delta_{\kappa_2\kappa_1}\delta_{\pi_2\pi_1}\delta_{j_2j_1}\delta_{m_2m_1}\delta(\omega_2-\omega_1), \tag{447}$$

which takes into account the integral

$$\int_0^\infty \mathrm{d}r\,r^2\,f_{\omega_2,j}(r)\,f_{\omega_1,j}(r) \;=\; \delta(\omega_2-\omega_1). \tag{448}$$

The completeness relation for the transverse wave functions is

$$\int_0^\infty \mathrm{d}\omega\sum_{\kappa\pi jm}\psi_{\omega,jm}^{\mathrm{T}(\kappa,\pi)}(\boldsymbol{x}_2)\psi_{\omega,jm}^{\mathrm{T}(\kappa,\pi)\dagger}(\boldsymbol{x}_1) \;=\; \mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1), \tag{449}$$

which is shown in some detail by writing

$$\sum_{\kappa\pi}\psi_{\omega,jm}^{\mathrm{T}(\kappa,\pi)}(\boldsymbol{x}_2)\psi_{\omega,jm}^{\mathrm{T}(\kappa,\pi)\dagger}(\boldsymbol{x}_1) \;=\; \begin{pmatrix} \boldsymbol{S}_\omega^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S}_\omega^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1)\end{pmatrix}, \tag{450}$$

where

$$\begin{aligned}\boldsymbol{S}_\omega^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1) \;&=\; f_{\omega,j}(r_2)\boldsymbol{X}_1^{jm}(\hat{\boldsymbol{x}}_2)f_{\omega,j}(r_1)\boldsymbol{X}_1^{jm\dagger}(\hat{\boldsymbol{x}}_1)\\ &\quad+\frac{c^2}{\omega^2}\left[\boldsymbol{\tau}\cdot\boldsymbol{\nabla}_2 f_{\omega,j}(r_2)\boldsymbol{X}_1^{jm}(\hat{\boldsymbol{x}}_2)\right]\left[\boldsymbol{\tau}\cdot\boldsymbol{\nabla}_1 f_{\omega,j}(r_1)\boldsymbol{X}_1^{jm}(\hat{\boldsymbol{x}}_1)\right]^\dagger,\end{aligned} \tag{451}$$

and

$$\begin{aligned}\boldsymbol{S}_\omega^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1) \;&=\; \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\Big[\boldsymbol{S}_\omega^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1)+\boldsymbol{F}_\omega^{jm}(\boldsymbol{x}_2)\boldsymbol{F}_\omega^{jm\dagger}(\boldsymbol{x}_1)\Big]\\ &=\; \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\Big[f_{\omega,j}(r_2)\boldsymbol{Y}_{jj}^m(\hat{\boldsymbol{x}}_2)f_{\omega,j}(r_1)\boldsymbol{Y}_{jj}^{m\dagger}(\hat{\boldsymbol{x}}_1)\\ &\quad+f_{\omega,j-1}(r_2)\,\boldsymbol{Y}_{jj-1}^m(\hat{\boldsymbol{x}}_2)f_{\omega,j-1}(r_1)\,\boldsymbol{Y}_{jj-1}^{m\dagger}(\hat{\boldsymbol{x}}_1)\\ &\quad+f_{\omega,j+1}(r_2)\,\boldsymbol{Y}_{jj+1}^m(\hat{\boldsymbol{x}}_2)f_{\omega,j+1}(r_1)\,\boldsymbol{Y}_{jj+1}^{m\dagger}(\hat{\boldsymbol{x}}_1)\Big],\end{aligned} \tag{452}$$

which gives

$$\int_0^\infty \mathrm{d}\omega \sum_{jm} \boldsymbol{S}_\omega^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1) = \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\frac{1}{r_2r_1}\,\delta(r_2-r_1)$$
$$\times\sum_{jm}\Big[\boldsymbol{Y}_{jj}^m(\hat{\boldsymbol{x}}_2)\boldsymbol{Y}_{jj}^{m\dagger}(\hat{\boldsymbol{x}}_1)+\boldsymbol{Y}_{jj-1}^m(\hat{\boldsymbol{x}}_2)\boldsymbol{Y}_{jj-1}^{m\dagger}(\hat{\boldsymbol{x}}_1)+\boldsymbol{Y}_{jj+1}^m(\hat{\boldsymbol{x}}_2)\boldsymbol{Y}_{jj+1}^{m\dagger}(\hat{\boldsymbol{x}}_1)\Big]$$
$$= \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1), \tag{453}$$

based on

$$\int_0^\infty \mathrm{d}\omega\, f_{\omega,j}(r_2)\, f_{\omega,j}(r_1) \;=\; \frac{1}{r_2r_1}\,\delta(r_2-r_1). \tag{454}$$

8.4. Longitudinal spherical photons

Longitudinal spherical wave functions are

$$\psi_{k,jm}^{\mathrm{L}(+)}(\boldsymbol{x}) \;=\; \frac{1}{k}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\nabla}_{\mathrm{s}}\, g_{k,j}(r)Y_j^m(\hat{\boldsymbol{x}})\end{pmatrix}, \tag{455}$$

$$\psi_{k,jm}^{\mathrm{L}(-)}(\boldsymbol{x}) \;=\; \frac{1}{k}\begin{pmatrix}\boldsymbol{\nabla}_{\mathrm{s}}\, g_{k,j}(r)Y_j^m(\hat{\boldsymbol{x}})\\ \boldsymbol{0}\end{pmatrix}. \tag{456}$$

It follows from the identity $\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,\boldsymbol{\nabla}_{\mathrm{s}}=0$ that they are longitudinal

$$\mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla})\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}) \;=\; \psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}), \tag{457}$$

$$\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla})\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}) \;=\; 0 \tag{458}$$

and that they are eigenfunctions of $\mathcal{H}$, with eigenvalue 0

$$\mathcal{H}\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}) \;=\; 0, \tag{459}$$

with no condition on g. However, in order to have a complete set of longitudinal wave functions, a set of functions, indexed by the parameter k is specified here. The form of the plane-wave longitudinal solutions in Eqs. (237) and (238) and of the expansion of a plane wave in spherical waves suggest the choice

$$g_{k,j}(r) \;=\; k\sqrt{\frac{2}{\pi}}\; j_j(kr) \tag{460}$$

for the radial wave function, where k is a free parameter. This set of functions provides an infinite orthonormal set of degenerate ($\omega=0$) basis functions for each j.

From the form of $\boldsymbol{\nabla}_{\mathrm{s}}$ in Eq. (442) and the fact that $\boldsymbol{X}_2^{jm}(\hat{\boldsymbol{x}})$ and $\boldsymbol{X}_3^{jm}(\hat{\boldsymbol{x}})$ are eigenfunctions of angular momentum, it follows that the longitudinal spherical wave functions are also eigenfunctions of angular momentum

$$\boldsymbol{\mathcal{J}}^2\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}) \;=\; \hbar^2 j(j+1)\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}), \tag{461}$$

$$\mathcal{J}^3\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}) \;=\; \hbar\, m\,\psi_{k,jm}^{\mathrm{L}(\pi)}(\boldsymbol{x}). \tag{462}$$

They have parity given by

$$\mathfrak{P}\,\psi^{\mathrm{L}(+)}_{k,jm}(\boldsymbol{x}) \;=\; (-1)^{j+1}\,\psi^{\mathrm{L}(+)}_{k,jm}(\boldsymbol{x}), \tag{463}$$

$$\mathfrak{P}\,\psi^{\mathrm{L}(-)}_{k,jm}(\boldsymbol{x}) \;=\; (-1)^{j}\,\psi^{\mathrm{L}(-)}_{k,jm}(\boldsymbol{x}). \tag{464}$$

With spherical Bessel functions for the radial wave functions, we have, following Eqs. (442) to (446),

$$\frac{\boldsymbol{\nabla}_{\mathrm{s}}}{k}\,g_{k,j}(r)Y_j^m(\hat{\boldsymbol{x}}) \;=\; \sqrt{\frac{j}{2j+1}}\;g_{k,j-1}(r)\,\boldsymbol{Y}^m_{jj-1}(\hat{\boldsymbol{x}}) + \sqrt{\frac{j+1}{2j+1}}\;g_{k,j+1}(r)\,\boldsymbol{Y}^m_{jj+1}(\hat{\boldsymbol{x}}). \tag{465}$$

This identity facilitates calculation of the orthonormality relation for the longitudinal wave functions, which is

$$\int \mathrm{d}\boldsymbol{x}\,\psi^{\mathrm{L}(\pi_2)\dagger}_{k_2,j_2m_2}(\boldsymbol{x})\psi^{\mathrm{L}(\pi_1)}_{k_1,j_1m_1}(\boldsymbol{x}) \;=\; \delta_{\pi_2\pi_1}\delta_{j_2j_1}\delta_{m_2m_1}\delta(k_2-k_1), \tag{466}$$

with

$$\int_0^\infty \mathrm{d}r\,r^2\,g_{k_2,j}(r)\,g_{k_1,j}(r) \;=\; \delta(k_2-k_1). \tag{467}$$

The completeness relation for the longitudinal wave functions is

$$\int_0^\infty \mathrm{d}k\sum_{\pi jm}\psi^{\mathrm{L}(\pi)}_{k,jm}(\boldsymbol{x}_2)\psi^{\mathrm{L}(\pi)\dagger}_{k,jm}(\boldsymbol{x}_1) \;=\; \mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1), \tag{468}$$

which follows from

$$\sum_{\pi}\psi^{\mathrm{L}(\pi)}_{k,jm}(\boldsymbol{x}_2)\psi^{\mathrm{L}(\pi)\dagger}_{k,jm}(\boldsymbol{x}_1) \;=\; \begin{pmatrix} \boldsymbol{T}_k^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_k^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1)\end{pmatrix}, \tag{469}$$

where

$$\boldsymbol{T}_k^{jm}(\boldsymbol{x}_2,\boldsymbol{x}_1) \;=\; -\frac{\boldsymbol{\nabla}_{2\,\mathrm{s}}\boldsymbol{\nabla}^\dagger_{1\,\mathrm{s}}}{\boldsymbol{\nabla}_2^2}\,g_{k,j}(r_2)g_{k,j}(r_1)Y_j^m(\hat{\boldsymbol{x}}_2)Y_j^{m*}(\hat{\boldsymbol{x}}_1), \tag{470}$$

and

$$\boldsymbol{\nabla}^2 g_{k,j}(r)Y_j^m(\hat{\boldsymbol{x}}) \;=\; -k^2 g_{k,j}(r)Y_j^m(\hat{\boldsymbol{x}}). \tag{471}$$

Thus from

$$\int_0^\infty \mathrm{d}k\,g_{k,j}(r_2)g_{k,j}(r_1) \;=\; \frac{1}{r_2r_1}\,\delta(r_2-r_1), \tag{472}$$

we have

$$\int_0^\infty \mathrm{d}k \sum_{jm} \boldsymbol{T}_k^{jm}(\boldsymbol{x}_2, \boldsymbol{x}_1) \quad = \quad \boldsymbol{\mathit{\Pi}}_\mathrm{s}^\mathrm{L}(\boldsymbol{\nabla}_2)\delta(\boldsymbol{x}_2 - \boldsymbol{x}_1), \tag{473}$$

which provides Eq. (468). That result, together with Eq. (449) gives the full completeness relation.

As an illustration of a role of the spherical functions, we revisit the example of a point charge at the origin, for which

$$\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}) \quad = \quad -\frac{q}{4\pi\epsilon_0}\begin{pmatrix} \boldsymbol{\nabla}_\mathrm{s}\frac{1}{r} \\ \boldsymbol{0} \end{pmatrix}. \tag{474}$$

In view of the integral

$$\int_0^\infty \mathrm{d}k\, \frac{1}{k}\, g_{k,0}(r) \quad = \quad \sqrt{\frac{\pi}{2}}\,\frac{1}{r}, \tag{475}$$

one has

$$\mathit{\Psi}_\mathrm{p}(\boldsymbol{x}) \quad = \quad -\frac{q}{\sqrt{2}\,\pi\epsilon_0}\int_0^\infty \mathrm{d}k\, \psi_{k,00}^{\mathrm{L}(-)}(\boldsymbol{x}), \tag{476}$$

which is the analog, for spherical solutions, of Eq. (248) for plane wave solutions.

9. Maxwell Green function

A solution of the Maxwell equation for the electric and magnetic fields $\mathit{\Psi}(x)$ given a specified current source $\mathit{\Xi}(x)$, as they are related in Eq. (52)

$$\gamma_\mu\partial^\mu\, \mathit{\Psi}(x) \quad = \quad \mathit{\Xi}(x),$$

can be found with the aid of the 6×6 matrix Maxwell Green function $\mathcal{D}_\mathrm{M}(x_2 - x_1)$, given by

$$\begin{aligned} \mathcal{D}_\mathrm{M}(x_2 - x_1) \quad = \quad & \sum_{\lambda=0}^{2}\int \mathrm{d}\boldsymbol{k}\; \psi_{\boldsymbol{k},\lambda}^{(+)}(x_2)\overline{\psi}_{\boldsymbol{k},\lambda}^{(+)}(x_1)\,\theta(t_2 - t_1) \\ & - \sum_{\lambda=0}^{2}\int \mathrm{d}\boldsymbol{k}\; \psi_{\boldsymbol{k},\lambda}^{(-)}(x_2)\overline{\psi}_{\boldsymbol{k},\lambda}^{(-)}(x_1)\,\theta(t_1 - t_2), \end{aligned} \tag{477}$$

where $\psi_{\boldsymbol{k},\lambda}^{(\pm)}(x)$ is given by Eq. (217), (237) or (238), and Eq. (251). In view of the relations

$$\gamma_\mu\partial_2^\mu\, \psi_{\boldsymbol{k},\lambda}^{(\pm)}(x_2) \quad = \quad 0, \tag{478}$$

$$\gamma_\mu\partial_2^\mu\, \theta(t_2 - t_1) \quad = \quad \gamma_0\,\delta(ct_2 - ct_1), \tag{479}$$

$$\gamma_\mu\partial_2^\mu\, \theta(t_1 - t_2) \quad = \quad -\gamma_0\,\delta(ct_2 - ct_1) \tag{480}$$

and the completeness of the wave functions, we have

$$\gamma_\mu \partial_2^\mu \, \mathcal{D}_{\rm M}(x_2 - x_1) \;\; = \;\; \mathcal{I}\,\delta(x_2 - x_1) \tag{481}$$

and

$$\mathcal{D}_{\rm M}(x_2 - x_1)\,\gamma_\mu \overleftarrow{\partial}_1^{\,\mu} \;\; = \;\; -\mathcal{I}\,\delta(x_2 - x_1), \tag{482}$$

where

$$\delta(x_2 - x_1) \;\; = \;\; \delta(ct_2 - ct_1)\,\delta(\boldsymbol{x}_2 - \boldsymbol{x}_1). \tag{483}$$

In terms of the Green function, a solution for the electric and magnetic fields is

$$\Psi(x_2) \;\; = \;\; \int \mathrm{d}^4 x_1\, \mathcal{D}_{\rm M}(x_2 - x_1)\, \Xi(x_1), \tag{484}$$

as is confirmed by the application of $\gamma_\mu \partial_2^\mu$ to both sides. In Eq. (484) $\mathrm{d}^4 x_1 = c\,\mathrm{d}t_1 \mathrm{d}\boldsymbol{x}_1$. A separation into transverse or longitudinal solutions may be made by restricting the sum over polarizations to $\lambda = 1, 2$ for a transverse solution or $\lambda = 0$ for a longitudinal solution.

The Maxwell Green function also may be written as an integral over the four-vector k of the plane-wave solutions. For this it is useful to make the separation into transverse and longitudinal components. For the transverse part we have

$$\begin{aligned} \mathcal{D}_{\rm M}^{\rm T}(x_2 - x_1) \;\; = \;\; & \sum_{\lambda=1}^{2} \int \mathrm{d}\boldsymbol{k}\; \psi_{\boldsymbol{k},\lambda}^{(+)}(\boldsymbol{x}_2) \overline{\psi}_{\boldsymbol{k},\lambda}^{(+)}(\boldsymbol{x}_1)\, \mathrm{e}^{-\mathrm{i}\omega(t_2 - t_1)}\, \theta(t_2 - t_1) \\ & - \sum_{\lambda=1}^{2} \int \mathrm{d}\boldsymbol{k}\; \psi_{\boldsymbol{k},\lambda}^{(-)}(\boldsymbol{x}_2) \overline{\psi}_{\boldsymbol{k},\lambda}^{(-)}(\boldsymbol{x}_1)\, \mathrm{e}^{\mathrm{i}\omega(t_2 - t_1)}\, \theta(t_1 - t_2), \end{aligned} \tag{485}$$

and we employ the identities

$$\mathrm{e}^{-\mathrm{i}\omega(t_2 - t_1)}\, \theta(t_2 - t_1) \;\; = \;\; \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_0\, \frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2 - ct_1)}}{k_0(1 + \mathrm{i}\delta) - \omega/c}, \tag{486}$$

$$-\mathrm{e}^{\mathrm{i}\omega(t_2 - t_1)}\, \theta(t_1 - t_2) \;\; = \;\; \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_0\, \frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2 - ct_1)}}{k_0(1 + \mathrm{i}\delta) + \omega/c}, \tag{487}$$

where the limit $\delta \to 0^+$ for the integral is understood. We also have

$$\boldsymbol{\alpha} \cdot \boldsymbol{k}\, \psi_{\boldsymbol{k},\lambda}^{(\pm)}(\boldsymbol{x}_2) \;\; = \;\; |\boldsymbol{k}|\, \psi_{\boldsymbol{k},\lambda}^{(\pm)}(\boldsymbol{x}_2) = \frac{\omega}{c}\, \psi_{\boldsymbol{k},\lambda}^{(\pm)}(\boldsymbol{x}_2). \tag{488}$$

Together these relations yield [see Eq. (228)]

$$\mathcal{D}_{\rm M}^{\rm T}(x_2 - x_1) \;\; = \;\; \frac{\mathrm{i}}{2\pi} \sum_{\lambda=1}^{2} \int \mathrm{d}^4 k \left[\frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2 - ct_1)}}{k_0(1 + \mathrm{i}\delta) - \boldsymbol{\alpha} \cdot \boldsymbol{k}}\, \psi_{\boldsymbol{k},\lambda}^{(+)}(\boldsymbol{x}_2) \overline{\psi}_{\boldsymbol{k},\lambda}^{(+)}(\boldsymbol{x}_1) \right.$$

$$
\begin{aligned}
&\qquad + \frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}}{k_0(1+\mathrm{i}\delta)+\boldsymbol{\alpha}\cdot\boldsymbol{k}}\,\psi^{(-)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_2)\overline{\psi}^{(-)}_{\boldsymbol{k},\lambda}(\boldsymbol{x}_1)\Big]\\
&= \frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4k\;\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}}{\gamma^0k_0(1+\mathrm{i}\delta)-\boldsymbol{\gamma}\cdot\boldsymbol{k}}\begin{pmatrix}(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2 & \boldsymbol{0}\\ \boldsymbol{0} & (\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\end{pmatrix}\\
&= \frac{\mathrm{i}}{(2\pi)^4}\,\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}_2)\int_{\mathrm{C_F}}\mathrm{d}^4k\;\frac{\mathrm{e}^{-\mathrm{i}k\cdot(x_2-x_1)}}{\gamma^\mu k_\mu},
\end{aligned}
\tag{489}
$$

where $\mathrm{d}^4k=\mathrm{d}k_0\,\mathrm{d}\boldsymbol{k}$, and $\mathrm{C_F}$ indicates that the contour of integration over k_0 is the Feynman contour, which passes from $-\infty$ below the negative real axis, through 0, and above the positive real axis to $+\infty$; this is equivalent to including the factor $(1+\mathrm{i}\delta)$ multiplying k_0 in the denominator and integrating along the real axis.

For applications, it is useful to consider an alternative form for the transverse Green function. Taking into account the relation

$$
\begin{aligned}
\mathit{\Pi}^{\mathrm{T}}(\hat{\boldsymbol{k}})\frac{1}{\gamma^0k_0(1+\mathrm{i}\delta)-\boldsymbol{\gamma}\cdot\boldsymbol{k}} &= \mathit{\Pi}^{\mathrm{T}}(\hat{\boldsymbol{k}})\,\frac{\gamma^0k_0-\boldsymbol{\gamma}\cdot\boldsymbol{k}}{k_0^2-\boldsymbol{k}^2+\mathrm{i}\delta}\\
&= \begin{pmatrix} k_0\,(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2 & -\boldsymbol{\tau}\cdot\boldsymbol{k}\\ \boldsymbol{\tau}\cdot\boldsymbol{k} & -k_0\,(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\end{pmatrix}\frac{1}{k^2+\mathrm{i}\delta},
\end{aligned}
\tag{490}
$$

we have, from Eq. (489),

$$
\begin{aligned}
\mathcal{D}^{\mathrm{T}}_{\mathrm{M}}(x_2-x_1) &= \frac{\mathrm{i}}{(2\pi)^4}\int_{-\infty}^{\infty}\mathrm{d}k_0\;\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}\begin{pmatrix} k_0\,\boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\boldsymbol{\nabla}) & \mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\\ -\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla} & -k_0\,\boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\boldsymbol{\nabla})\end{pmatrix}\int\mathrm{d}\boldsymbol{k}\;\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2+\mathrm{i}\delta}\\
&= \frac{1}{8\pi^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}k_0\;\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}\begin{pmatrix} k_0\,\boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\boldsymbol{\nabla}) & \mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\\ -\mathrm{i}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla} & -k_0\,\boldsymbol{\mathit{\Pi}}^{\mathrm{T}}_{\mathrm{s}}(\boldsymbol{\nabla})\end{pmatrix}\frac{\mathrm{e}^{\mathrm{i}(k_0^2+\mathrm{i}\delta)^{1/2}|\boldsymbol{r}|}}{|\boldsymbol{r}|}\\
&\to \frac{1}{8\pi^2\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}k_0\;\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}\\
&\qquad\times\begin{pmatrix} k_0(\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})^2 & -(k_0^2+\mathrm{i}\delta)^{1/2}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\\ (k_0^2+\mathrm{i}\delta)^{1/2}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} & -k_0(\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})^2\end{pmatrix}\frac{\mathrm{e}^{\mathrm{i}(k_0^2+\mathrm{i}\delta)^{1/2}|\boldsymbol{r}|}}{|\boldsymbol{r}|},
\end{aligned}
\tag{491}
$$

where $\boldsymbol{r}=\boldsymbol{x}_2-\boldsymbol{x}_1$, the gradient $\boldsymbol{\nabla}$ is with respect to $\boldsymbol{r}$, and the branch of the square root in the exponent is determined by the condition $\mathrm{Im}(k_0^2+\mathrm{i}\delta)^{1/2}>0$, which specifies that $(k_0^2+\mathrm{i}\delta)^{1/2}\to|k_0|$ for real values of k_0. In the last line of Eq. (491), higher-order terms in $(k_0\,|\boldsymbol{r}|)^{-1}$ are not included, but the exact expression follows from the formulas in Appendix E.

For the longitudinal Green function, we write

$$
\begin{aligned}
\mathcal{D}^{\mathrm{L}}_{\mathrm{M}}(x_2-x_1) &= \int\mathrm{d}\boldsymbol{k}\;\psi^{(+)}_{\boldsymbol{k},0}(\boldsymbol{x}_2)\overline{\psi}^{(+)}_{\boldsymbol{k},0}(\boldsymbol{x}_1)\,\mathrm{e}^{-\epsilon(ct_2-ct_1)}\,\theta(t_2-t_1)\\
&\quad-\int\mathrm{d}\boldsymbol{k}\;\psi^{(-)}_{\boldsymbol{k},0}(\boldsymbol{x}_2)\overline{\psi}^{(-)}_{\boldsymbol{k},0}(\boldsymbol{x}_1)\,\mathrm{e}^{\epsilon(ct_2-ct_1)}\,\theta(t_1-t_2),
\end{aligned}
\tag{492}
$$

where damping factors with $\epsilon>0$ are added so that the Green function falls off for large time differences. In addition, we assume that the longitudinal wave functions are solutions of the Maxwell equation with an infinitesimal mass m_ϵ included, as given in Eq. (259), in order to be able to use the Feynman contour to specify the path of integration over k_0 in relation to the poles of the integrand. We employ the identities

$$\mathrm{e}^{-\mathrm{i}m_\epsilon c^2(t_2-t_1)/\hbar}\,\mathrm{e}^{-\epsilon(ct_2-ct_1)}\,\theta(t_2-t_1) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}}{k_0+\mathrm{i}\epsilon-m_\epsilon c/\hbar}, \tag{493}$$

$$-\mathrm{e}^{\mathrm{i}m_\epsilon c^2(t_2-t_1)/\hbar}\,\mathrm{e}^{\epsilon(ct_2-ct_1)}\,\theta(t_1-t_2) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}}{k_0-\mathrm{i}\epsilon+m_\epsilon c/\hbar} \tag{494}$$

and

$$\gamma^0\,\psi_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x}_2) = \kappa\,\psi_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x}_2), \tag{495}$$

$$\boldsymbol{\alpha}\cdot\boldsymbol{k}\,\psi_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x}_2) = 0 \tag{496}$$

to obtain

$$\begin{aligned}\mathcal{D}_{\mathrm{M}}^{\mathrm{L}}(x_2-x_1) &= \frac{\mathrm{i}}{2\pi}\int\mathrm{d}^4k\Bigg[\frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}}{k_0+\mathrm{i}\epsilon-m_\epsilon c/\hbar}\,\psi_{\boldsymbol{k},0}^{(+)}(\boldsymbol{x}_2)\overline{\psi}_{\boldsymbol{k},0}^{(+)}(\boldsymbol{x}_1)\\ &\qquad +\frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}}{k_0-\mathrm{i}\epsilon+m_\epsilon c/\hbar}\,\psi_{\boldsymbol{k},0}^{(-)}(\boldsymbol{x}_2)\overline{\psi}_{\boldsymbol{k},0}^{(-)}(\boldsymbol{x}_1)\Bigg]\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathrm{C_F}}\mathrm{d}^4k\frac{\mathrm{e}^{-\mathrm{i}k_0(ct_2-ct_1)}}{k_0-\gamma^0 m_\epsilon c/\hbar}\sum_{\kappa\to\pm}\psi_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x}_2)\overline{\psi}_{\boldsymbol{k},0}^{(\kappa)}(\boldsymbol{x}_1)\\ &= \frac{\mathrm{i}}{(2\pi)^4}\,\mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}_2)\int_{\mathrm{C_F}}\mathrm{d}^4k\,\frac{\mathrm{e}^{-\mathrm{i}k\cdot(x_2-x_1)}}{\gamma^\mu k_\mu-m_\epsilon c/\hbar}\,.\end{aligned} \tag{497}$$

Here the limit $\epsilon\to 0$ would be undefined without the mass term. A concise alternative expression for the longitudinal Green function is obtained by the substitution of the partial completeness relations that follow from Eqs. (243) and (244) into Eq. (492):

$$\mathcal{D}_{\mathrm{M}}^{\mathrm{L}}(x_2-x_1) = \mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}_2)\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_1)\begin{pmatrix}\boldsymbol{I}\,\theta(t_2-t_1) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\,\theta(t_1-t_2)\end{pmatrix}. \tag{498}$$

The transverse and longitudinal Green functions in Eqs. (489) and (497) differ only by the type of projection operator and the infinitesimal mass term in Eq. (497). However, such a mass term in the last line of Eqs. (489) would not change the relation of the path of integration over k_0 to the location of the poles of the integrand, so it could also be included in that expression. In particular, the poles in Eq. (490) at $k_0=\pm\left(\boldsymbol{k}^2-\mathrm{i}\delta\right)^{1/2}$ would move to $k_0=\pm\left[\boldsymbol{k}^2+(m_\epsilon c/\hbar)^2-\mathrm{i}\delta\right]^{1/2}$. These poles lie on curves in the second and fourth quadrants of the complex k_0 plane, whereas the Feynman contour passes through the first and third quadrants. Thus, we may write $\mathcal{D}_{\mathrm{M}}^{\mathrm{T}}(x_2-x_1)+\mathcal{D}_{\mathrm{M}}^{\mathrm{L}}(x_2-x_1)=\mathcal{D}_{\mathrm{M}}(x_2-x_1)$, with

$$\mathcal{D}_{\mathrm{M}}(x_2-x_1) = \frac{\mathrm{i}}{(2\pi)^4}\int_{C_F}\mathrm{d}^4k\,\frac{\mathrm{e}^{-\mathrm{i}k\cdot(x_2-x_1)}}{\gamma^\mu k_\mu-m_\epsilon c/\hbar}. \tag{499}$$

This result is a covariant Green function for the Maxwell equation which is of the same form as the well-known Green function for the Dirac equation. A formal coordinate-representation is

$$\mathcal{D}_{\rm M}(x_2-x_1) \;=\; \frac{1}{\gamma_\mu\partial_2^\mu}\,\delta(x_2-x_1). \tag{500}$$

The fields given by Eq. (484) represent a particular solution of the Maxwell equation. Any solution of Eq. (484) for $\Xi(x_1)=0$, such as the field of a static charge distribution, may be added to the particular solution, and the sum will be a solution with the same source function. In fact, even if the three-vector current density vanishes in the distant past and future, there could be a static charge distribution that persists, with or without a net total charge, for which the fields would be non-zero indefinitely. To deal with this case, we obtain an expression that takes into account the possible fields in the past and future by writing the time derivative

$$\begin{aligned}\frac{\partial}{\partial(ct_1)}\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_0\,\Psi(x_1) \;&=\; \int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_0\,\overleftarrow{\partial}{}_1^0\,\Psi(x_1)\\ &\quad+\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_0\partial_1^0\,\Psi(x_1),\end{aligned} \tag{501}$$

and for fields that vanish for large space-like distances, we write

$$\begin{aligned}\int {\rm d}\boldsymbol{x}_1\,\boldsymbol{\nabla}_1\cdot\,\mathcal{D}_{\rm M}(x_2-x_1)\boldsymbol{\gamma}\,\Psi(x_1) \;&=\; \int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}}_1\,\Psi(x_1)\\ &\quad+\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}_1\,\Psi(x_1)=0,\end{aligned} \tag{502}$$

where the result is zero, because it may be written as an integral over the bounding surface, by the Gauss-Ostrogradsky theorem. The sum of Eqs. (501) and (502) is

$$\begin{aligned}&\frac{\partial}{\partial(ct_1)}\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_0\,\Psi(x_1)\\ &\quad=\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_\mu\,\overleftarrow{\partial}{}_1^\mu\,\Psi(x_1)+\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_\mu\partial_1^\mu\,\Psi(x_1)\\ &\quad=-\int {\rm d}\boldsymbol{x}_1\,\delta(x_2-x_1)\,\Psi(x_1)+\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\,\Xi(x_1).\end{aligned} \tag{503}$$

Integration of Eq. (503) over t_1 from $t_{\rm i}$ to $t_{\rm f}$, where $t_{\rm i}<t_2<t_{\rm f}$, yields

$$\begin{aligned}\Psi(x_2) \;&=\; \int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_0\,\Psi(x_1)\Big|_{t_1=t_{\rm i}}-\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\gamma_0\,\Psi(x_1)\Big|_{t_1=t_{\rm f}}\\ &\quad+c\int_{t_{\rm i}}^{t_{\rm f}}{\rm d}t_1\int {\rm d}\boldsymbol{x}_1\,\mathcal{D}_{\rm M}(x_2-x_1)\,\Xi(x_1)\end{aligned} \tag{504}$$

or

$$\begin{aligned}\Psi(x_2) &= \int \mathrm{d}\boldsymbol{x}_1 \sum_{\lambda=0}^{2} \int \mathrm{d}\boldsymbol{k} \left[\psi^{(+)}_{\boldsymbol{k},\lambda}(x_2)\psi^{(+)\dagger}_{\boldsymbol{k},\lambda}(x_1)\, \Psi(x_1)\Big|_{t_1=t_\mathrm{i}} + \psi^{(-)}_{\boldsymbol{k},\lambda}(x_2)\psi^{(-)\dagger}_{\boldsymbol{k},\lambda}(x_1)\, \Psi(x_1)\Big|_{t_1=t_\mathrm{f}} \right] \\ &\quad + c\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t_1 \int \mathrm{d}\boldsymbol{x}_1\, \mathcal{D}_\mathrm{M}(x_2 - x_1)\, \Xi(x_1). \end{aligned} \tag{505}$$

As a consistency check of this expression, we note that it properly reduces to the expected result for the field of a constant charge distribution. In this case, the current source term vanishes, the initial and final fields are the same, and they are purely longitudinal. As a result, only longitudinal functions with no time dependence will contribute to the sum over states, which is just the longitudinal completeness relation, and Eq. (505) reduces to the proper identity.

10. Applications of the Maxwell Green function

The Maxwell Green function is used here to calculate the radiation fields of a point dipole source as an example of an application. Only the large distance transverse fields are considered, and they are given by

$$\Psi_\mathrm{d}(x_2) = \int \mathrm{d}^4 x_1\, \mathcal{D}^\mathrm{T}_\mathrm{M}(x_2 - x_1)\, \Xi_\mathrm{d}(x_1), \tag{506}$$

with the source term

$$\Xi_\mathrm{d}(x) = \begin{pmatrix} -\mu_0 c\, \boldsymbol{j}_\mathrm{s}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\omega_\mathrm{d} t} \\ \boldsymbol{0} \end{pmatrix}. \tag{507}$$

The classical source for dipole radiation is a charge q with position

$$\boldsymbol{x}_\mathrm{d}(t) = \boldsymbol{x}_0 \cos\omega_\mathrm{d} t \tag{508}$$

which produces a current density

$$\begin{aligned}\boldsymbol{j}_\mathrm{cl}(x) &= q\,\delta(\boldsymbol{x} - \boldsymbol{x}_\mathrm{d}(t))\, \dot{\boldsymbol{x}}_\mathrm{d}(t) \\ &\approx -\omega_\mathrm{d}\, \boldsymbol{d}\, \delta(\boldsymbol{x}) \sin\omega_\mathrm{d} t, \end{aligned} \tag{509}$$

where $\boldsymbol{d} = q\,\boldsymbol{x}_0$. This is the real part of

$$\boldsymbol{j}(x) = -\mathrm{i}\,\omega_\mathrm{d}\, \boldsymbol{d}\, \delta(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega_\mathrm{d} t}, \tag{510}$$

which is the source current for the radiation [17]. For the transverse Maxwell Green function, we use the expression on the last line of Eq. (491). Integration over t_1 yields a factor $2\pi\delta(k_0 c - \omega_\mathrm{d})$, and evaluation of the integration over k_0 follows. The result is

$$\begin{aligned} & c\int \mathrm{d}t_1\, \mathcal{D}^\mathrm{T}_\mathrm{M}(x_2 - x_1)\, \Xi_\mathrm{d}(x_1) \\ &= -\frac{\mu_0 c\, k}{4\pi\mathrm{i}} \begin{pmatrix} (\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})^2 & -\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} & -(\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{j}_\mathrm{s}(\boldsymbol{x}_1) \\ \boldsymbol{0} \end{pmatrix} \frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}|}}{|\boldsymbol{r}|}\, \mathrm{e}^{-\mathrm{i}\omega_\mathrm{d} t_2} + \dots\ , \end{aligned} \tag{511}$$

where $k = \omega_{\rm d}/c$. Since the source is point-like at the origin $\boldsymbol{x}_1 = 0$, $\boldsymbol{r} = \boldsymbol{x}_2$, and

$$\Psi_{\rm d}(x) = \frac{k^2}{4\pi\epsilon_0}\begin{pmatrix} (\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}})^2\,\boldsymbol{d}_{\rm s} \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\boldsymbol{d}_{\rm s}\end{pmatrix}\frac{{\rm e}^{{\rm i}k|\boldsymbol{x}|}}{|\boldsymbol{x}|}\,{\rm e}^{-{\rm i}\omega_{\rm d}t} + \dots\ . \tag{512}$$

The time-average differential radiated power, based on Eqs. (54) to (59) with a factor 1/2 from the time averaging [17] is

$$\begin{aligned}\frac{{\rm d}I_{\rm d}}{{\rm d}\Omega} &= \frac{1}{2}\,\boldsymbol{x}^2\,\hat{\boldsymbol{x}}\cdot\boldsymbol{S}(x) = \frac{c\epsilon_0}{4}\,\boldsymbol{x}^2\,\overline{\Psi}_{\rm d}(x)\boldsymbol{\gamma}\cdot\hat{\boldsymbol{x}}\,\Psi_{\rm d}(x) \\ &= \frac{ck^4}{32\,\pi^2\epsilon_0}\,\boldsymbol{d}_{\rm s}^\dagger(\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}})^2\,\boldsymbol{d}_{\rm s} = \frac{ck^4}{32\,\pi^2\epsilon_0}\left[\,\boldsymbol{d}^2 - (\hat{\boldsymbol{x}}\cdot\boldsymbol{d})^2\right],\end{aligned} \tag{513}$$

which is the well-known result.

A more realistic example is the radiation produced by a Dirac transition current, which is given by

$$\Xi_{\rm D}(x) = \begin{pmatrix} -\mu_0 c\,\boldsymbol{j}_{\rm s}^{if}(x) \\ \boldsymbol{0}\end{pmatrix} = \begin{pmatrix} \frac{2e}{\epsilon_0}\,\phi_f^\dagger(\boldsymbol{x})\,\boldsymbol{\alpha}_{\rm s}\,\phi_i(\boldsymbol{x})\,{\rm e}^{-{\rm i}\omega_{if}t} \\ \boldsymbol{0}\end{pmatrix}, \tag{514}$$

where ϕ_i and ϕ_f are the initial and final hydrogen atom Dirac wave functions, here $\boldsymbol{\alpha}$ is the 4×4 Dirac matrix, and

$$\omega_{if} = \frac{E_i - E_f}{\hbar} \tag{515}$$

is the frequency corresponding to the energy difference of the transition. The factor of 2 multiplying the matrix element accounts for the difference between a classical dipole moment and the quantum mechanical dipole moment operator in Eq. (520). (See the footnote on p. 407 of [17].) We have

$$\begin{aligned}&c\int {\rm d}t_1\,\mathcal{D}_{\rm M}^{\rm T}(x_2 - x_1)\,\Xi_{\rm D}(x_1) \\ &\qquad = \frac{{\rm i}k}{4\pi\epsilon_0 c}\begin{pmatrix} (\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})^2 & -\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{r}} & -(\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}})^2\end{pmatrix}\begin{pmatrix}\boldsymbol{j}_{\rm s}^{if}(\boldsymbol{x}_1) \\ \boldsymbol{0}\end{pmatrix}\frac{{\rm e}^{{\rm i}k|\boldsymbol{r}|}}{|\boldsymbol{r}|}\,{\rm e}^{-{\rm i}\omega_{if}t_2} + \dots\ ,\end{aligned} \tag{516}$$

where $k = \omega_{if}/c$. For distances far from the source atom, $|\boldsymbol{x}_2| \gg |\boldsymbol{x}_1|$, $\hat{\boldsymbol{k}} \approx \hat{\boldsymbol{r}} \approx \hat{\boldsymbol{x}}_2$, and in the exponent $k|\boldsymbol{r}| = k|\boldsymbol{x}_2| - \boldsymbol{k}\cdot\boldsymbol{x}_1 + \dots$, which yields

$$\begin{aligned}\Psi_{\rm D}(x_2) &= \int {\rm d}^4x_1\,\mathcal{D}_{\rm M}^{\rm T}(x_2 - x_1)\,\Xi_{\rm D}(x_1) \\ &= \frac{{\rm i}k}{4\pi\epsilon_0 c}\int {\rm d}\boldsymbol{x}_1\begin{pmatrix}(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\,\boldsymbol{j}_{\rm s}^{if}(\boldsymbol{x}_1) \\ \boldsymbol{\tau}\cdot\hat{\boldsymbol{k}}\,\boldsymbol{j}_{\rm s}^{if}(\boldsymbol{x}_1)\end{pmatrix}{\rm e}^{-{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}_1}\,\frac{{\rm e}^{{\rm i}k|\boldsymbol{x}_2|}}{|\boldsymbol{x}_2|}\,{\rm e}^{-{\rm i}\omega_{if}t_2} + \dots\ .\end{aligned} \tag{517}$$

The average radiated power is

$$
\begin{aligned}
\frac{\mathrm{d}I_\mathrm{D}}{\mathrm{d}\Omega} &= \frac{1}{2}\,\boldsymbol{x}_2^2\,\hat{\boldsymbol{x}}_{\boldsymbol{2}}\cdot\boldsymbol{S}(x_2) = \frac{c\epsilon_0}{4}\,\boldsymbol{x}_2^2\,\overline{\Psi}_\mathrm{D}(x_2)\boldsymbol{\gamma}\cdot\hat{\boldsymbol{k}}\,\Psi_\mathrm{D}(x_2) \\
&= \frac{k^2}{32\,\pi^2\epsilon_0 c}\int\mathrm{d}\boldsymbol{x}_1\,\boldsymbol{j}_\mathrm{s}^{if\dagger}(\boldsymbol{x}_1)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_1}\,(\boldsymbol{\tau}\cdot\hat{\boldsymbol{k}})^2\int\mathrm{d}\boldsymbol{x}_1'\,\boldsymbol{j}_\mathrm{s}^{if}(\boldsymbol{x}_1')\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_1'} \\
&= \hbar\omega_{if}\,\frac{\alpha kc}{2\pi}\sum_{\lambda=1}^{2}\int\mathrm{d}\boldsymbol{x}\,\phi_i^\dagger(\boldsymbol{x})\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\cdot\boldsymbol{\alpha}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\phi_f(\boldsymbol{x})\int\mathrm{d}\boldsymbol{x}'\,\phi_f^\dagger(\boldsymbol{x}')\,\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\cdot\boldsymbol{\alpha}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'}\phi_i(\boldsymbol{x}'),
\end{aligned}
\tag{518}
$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant. The radiated power integrated over directions of the vector $\hat{\boldsymbol{k}}$ may be interpreted as $\hbar\omega_{if}A_{if}$, where A_{if} is the radiative transition rate for $i\to f$, that is, the probability that the atom providing the source current makes a transition from state i to state f in one second. This gives

$$
A_{if} = \frac{\alpha kc}{2\pi}\int\mathrm{d}\Omega_{\boldsymbol{k}}\sum_{\lambda=1}^{2}\left\langle i\left|\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\cdot\boldsymbol{\alpha}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right|f\right\rangle\left\langle f\left|\hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}})\cdot\boldsymbol{\alpha}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right|i\right\rangle, \tag{519}
$$

which is the same as the relativistic radiative transition rate given by QED (see Appendix F). In the dipole approximation $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\to 1$, $\langle i\,|\boldsymbol{\alpha}|\,f\rangle = \mathrm{i}\,k\,\langle i\,|\boldsymbol{x}|\,f\rangle$, which follows from the identity $[H,\boldsymbol{x}] = [c\boldsymbol{\alpha}\cdot\boldsymbol{p},\boldsymbol{x}] = -\mathrm{i}\hbar c\boldsymbol{\alpha}$, where H is the Dirac Hamiltonian, and integration over $\hat{\boldsymbol{k}}$ yields the familiar result

$$
A_{if} \to \frac{4\alpha\omega_{if}^3}{3c^2}\,|\langle f\,|\boldsymbol{x}|\,i\rangle|^2\,. \tag{520}
$$

11. Summary

In Eq. (52), two of the Maxwell equations, Eqs. (2) and (3), are written in the form of the Dirac equation without a mass, but with the addition of a source term $\Xi(x)$:

$$
\gamma^\mu\partial_\mu\,\Psi(x) = \Xi(x),
$$

where the gamma matrices are 6×6 versions of the Dirac gamma matrices in Eq. (46), and

$$
\Psi(x) = \begin{pmatrix}\boldsymbol{E}_\mathrm{s}(x)\\ \mathrm{i}\,c\boldsymbol{B}_\mathrm{s}(x)\end{pmatrix}, \qquad \Xi(x) = \begin{pmatrix}-\mu_0 c\boldsymbol{J}_\mathrm{s}(x)\\ \boldsymbol{0}\end{pmatrix}
$$

from Eqs. (47) and (51). The source-free version of this equation, with $\Xi(x)=0$, can be written as a Schrödinger-like equation from Eq. (45) or (252)

$$
\mathrm{i}\hbar\,\frac{\partial}{\partial t}\,\Psi(x) = \mathcal{H}\,\Psi(x), \tag{521}
$$

where the Hamiltonian, Eq. (208),

$$\mathcal{H} = -\mathrm{i}\,\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}$$

is the analog of the Dirac Hamiltonian for the electron. The factors of $\hbar$ are not essential here, but they are introduced to provide the conventional units of frequency and energy. As with the Dirac wave functions, where all four components are necessary to describe an electron bound in an atom relativistically, all six of the components of the photon wave function apparently are necessary to properly account for the space-time properties of electromagnetic fields. There are three polarization degrees of freedom, two for radiation and one for electrostatic interactions, and relativistic covariance requires twice that many components. Alternatively stated, six complex functions are necessary to describe the six components of the electric and magnetic fields, and they are coupled by the Maxwell equation and Lorentz transformations.

According to Eq. (521), as in Eq. (258), the time dependence of the solution is given by

$$\Psi(x) = \mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar}\,\Psi(\boldsymbol{x}). \tag{522}$$

The time-independent solutions may be expanded in eigenfunctions of the Hamiltonian with eigenvalues E_n given by

$$\mathcal{H}\,\Psi_n(\boldsymbol{x}) \quad = \quad E_n\,\Psi_n(\boldsymbol{x}), \tag{523}$$

where n is a set of parameters that characterize the state represented by the wave function. For each eigenfunction, one has

$$\Psi_n(x) = \mathrm{e}^{-\mathrm{i}E_n t/\hbar}\,\Psi_n(\boldsymbol{x}). \tag{524}$$

The eigenfunctions are orthonormal

$$\int \mathrm{d}\boldsymbol{x}\;\Psi^\dagger_{n_2}(\boldsymbol{x})\,\Psi_{n_1}(\boldsymbol{x}) = \delta_{n_2 n_1}, \tag{525}$$

and they are complete

$$\sum_n \Psi_n(\boldsymbol{x}_2)\,\Psi^\dagger_n(\boldsymbol{x}_1) = \delta(\boldsymbol{x}_2 - \boldsymbol{x}_1). \tag{526}$$

The state index n includes continuous variables, so Eq. (525) has delta functions in those variables on the right-hand side, and the summation symbol in Eq. (526) includes integration over those variables.

States considered in detail in this paper are propagating plane waves in Secs. 7.1 and 7.2, standing plane waves in Sec. 7.7, and angular-momentum eigenstates in Secs. 8.3 and 8.4. The propagating plane-wave states are eigenfunctions of the momentum operator in Eq. (209)

$$\boldsymbol{\mathcal{P}} = -\mathrm{i}\,\hbar\,\mathcal{I}\,\boldsymbol{\nabla}.$$

This operator commutes with the Hamiltonian, $[\mathcal{H},\boldsymbol{\mathcal{P}}]=0$, and eigenstates of both energy and momentum are given in Eqs. (217), (237), and (238). These plane-wave states are further characterized by polarization vectors given in Eq. (210) or (235). Linear combinations of the traveling plane waves, combined to give standing-wave parity eigenfunctions, are in Eqs. (329), (330), and (339) to (342). The angular-momentum operator, Eq. (376), is

$$\boldsymbol{\mathcal{J}} \;=\; \boldsymbol{x}\times\boldsymbol{\mathcal{P}}+\hbar\,\boldsymbol{\mathcal{S}},$$

where the spin matrix, Eq. (154), is

$$\boldsymbol{\mathcal{S}} \;=\; \begin{pmatrix} \boldsymbol{\tau} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\tau} \end{pmatrix},$$

and $\boldsymbol{\tau}$ is given by Eqs. (15) to (17). One has $[\mathcal{H},\boldsymbol{\mathcal{J}}]=0$ in Eq. (377), and simultaneous eigenfunctions of energy, angular momentum squared $\boldsymbol{\mathcal{J}}^2$, third component of angular momentum $\mathcal{J}^3$, and parity are given in Eq (424), (425), (455), and (456). All three sets of eigenfunctions listed above are shown to be orthogonal and complete, as in Eqs. (525) and (526).

The eigenfunctions considered here are not normalizable wave functions. However, they provide basis functions for the expansion of a normalizable wave packet, as discussed in Sec. 7.8. For the sum

$$\Psi_f(\boldsymbol{x})=\sum_n f_n\,\Psi_n(\boldsymbol{x}), \tag{527}$$

from the orthonormality of the eigenfunctions one has

$$f_n=\int \mathrm{d}\boldsymbol{x}\,\Psi_n^\dagger(\boldsymbol{x})\,\Psi_f(\boldsymbol{x}) \tag{528}$$

and

$$\int \mathrm{d}\boldsymbol{x}\;\Psi_f^\dagger(\boldsymbol{x})\,\Psi_f(\boldsymbol{x})=\sum_n |f_n|^2=1 \tag{529}$$

for suitably chosen f_n. For the example of a Gaussian wave packet in Eq. (352), the expectation value of the Hamiltonian, in Eq. (357), is

$$\langle\,\Psi_f\,|\,\mathcal{H}\,|\,\Psi_f\rangle \;=\; \hbar\omega_0,$$

where $\omega_0=c|\boldsymbol{k}_0|$ is the frequency corresponding to the wave vector $\boldsymbol{k}_0$ of the wave packet. It is clear that Eq. (357) applies in more generality than just to the wave packet in Eq. (352). If the Gaussian shape function were replaced by any normalized real function, the expectation value of the Hamiltonian would still be exactly $\hbar\omega_0$. For the wave packet in Eq. (352), the expectation expectation value of the momentum operator, Eq. (358), is

$$\langle\,\Psi_f\,|\,\boldsymbol{\mathcal{P}}\,|\,\Psi_f\rangle \;=\; \hbar\boldsymbol{k}_0,$$

and the expectation value of the projection of the angular momentum in the direction of the wave vector is

$$\left\langle \Psi_f \left| \boldsymbol{\mathcal{J}} \cdot \hat{\boldsymbol{k}}_0 \right| \Psi_f \right\rangle \;=\; \hbar\, \hat{\boldsymbol{\epsilon}}_1^\dagger(\hat{\boldsymbol{k}}_0)\, \boldsymbol{\tau} \cdot \hat{\boldsymbol{k}}_0\, \hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0), \tag{530}$$

where the result depends on the polarization state represented by $\hat{\boldsymbol{\epsilon}}_1(\hat{\boldsymbol{k}}_0)$. For circular polarization, Eq. (216), the expectation value is $\pm\hbar$, while for linear polarization, Eq. (215), it is 0. The real part of the energy density is $\hbar\omega_0$ times the probability density

$$\mathrm{Re}\, \Psi_f^\dagger(x) \mathcal{H} \Psi_f(x) \;=\; \hbar\omega_0\, \Psi_f^\dagger(x)\, \Psi_f(x) \tag{531}$$

for the wave packet.

For any wave packet, represented by Ψ_f, the photon probability density four-vector defined in Eq. (368) is

$$q_f^\mu(x) \;=\; \overline{\Psi}_f(x)\gamma^\mu \Psi_f(x), \tag{532}$$

and the differential conservation of probability is given by Eq. (370)

$$\frac{\partial}{\partial t}\, q_f^0(x) + c\, \boldsymbol{\nabla} \cdot \boldsymbol{q}_f(x) \;=\; 0. \tag{533}$$

This is valid for any solution of the homogeneous Maxwell equation. By integrating over a closed volume and converting the divergence to an integral of the normal component of the vector over the surface, one obtains a statement of conservation of probability for the volume. If the surface of the volume is taken to infinity in all directions, where the wave function vanishes, this expression shows the time independence of the normalization of the wave function

$$\frac{\partial}{\partial t} \int \mathrm{d}\boldsymbol{x}\; \Psi_f^\dagger(x)\, \Psi_f(x) \;=\; 0. \tag{534}$$

Of course, this does not give meaningful results for a plane wave, because in this case, the probability density is constant over space and is not normalizable.

For electromagnetic fields and photons, Lorentz invariance is a necessary consideration. In Secs. 6.4 and 6.5 it is shown that

$$\gamma^\mu \partial_\mu \Psi'(x) \;=\; \Xi'(x),$$

where the primes indicate that the field and source have been transformed by either a rotation or a velocity boost. For rotations represented by the vector $\boldsymbol{u} = \theta\hat{\boldsymbol{u}}$, the transformed field, in Eq. (144), is

$$\Psi'(x) \;=\; \mathcal{R}(\boldsymbol{u})\, \Psi(R^{-1}(\boldsymbol{u})\, x),$$

where $R(\boldsymbol{u})$ is the coordinate rotation operator in Eq. (103) and

$$\mathcal{R}(\boldsymbol{u}) \;=\; \mathrm{e}^{-\mathrm{i}\boldsymbol{\mathcal{S}}\cdot\boldsymbol{u}}$$

in Eq. (153). The source term transforms in the same way. For velocity transformations, corresponding to the velocity $\boldsymbol{v} = c\tanh\zeta\,\hat{\boldsymbol{v}}$, Eq. (165) is

$$\Psi'(x) \;=\; \mathcal{V}(\boldsymbol{v})\,\Psi(V^{-1}(\boldsymbol{v})\,x),$$

where $V(\boldsymbol{v})$ is the coordinate velocity transformation operator in Eq. (123) and

$$\mathcal{V}(\boldsymbol{v}) \;=\; \mathrm{e}^{\zeta\boldsymbol{\mathcal{K}}\cdot\hat{\boldsymbol{v}}}$$

in Eq. (169), where

$$\boldsymbol{\mathcal{K}} \;=\; \begin{pmatrix} \mathbf{0} & \boldsymbol{\tau} \\ \boldsymbol{\tau} & \mathbf{0} \end{pmatrix}$$

in Eq. (170). The transformation of the source term under a velocity boost is noteworthy. In the presence of a non-zero source, the Maxwell equation is invariant, but the left- and right-hand sides do not transform separately. As shown in Sec. 6.5, the derivatives acting on the fields produce terms that combine with the original source term in such a way as to produce the velocity transformed source term, even though it is the three-vector current. In the absence of sources, the transformation reduces to a more conventional form.

In the case of photon wave functions, sources are taken to be absent and the wave functions are solutions of the homogeneous Maxwell equation. The Lorentz transformations of the plane-wave functions are explicitly shown in Secs. 7.5 and 7.6. As with the Dirac equation for an electron, the eigenvalues in Eq. (523) may be either positive or negative, and here they also may be zero, for both plane-wave and spherical-wave eigenfunctions. The negative eigenvalues, which are associated with relativistic invariance, are necessary in order to have a complete set of solutions satisfying Eq. (526). It is relevant to note that for the six-component wave packet in Eq. (352), there is an interaction between the upper-three components and lower-three components, evident in Eq. (354), that suppresses the role of the negative energy states.

In Sec. 5, Eqs. (86) and (87), orthogonal transverse and longitudinal projection operators are defined:

$$\mathit{\Pi}^{\mathrm{T}}(\boldsymbol{\nabla}) = \begin{pmatrix} \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) \end{pmatrix}, \qquad \mathit{\Pi}^{\mathrm{L}}(\boldsymbol{\nabla}) = \begin{pmatrix} \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla}) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla}) \end{pmatrix},$$

where from Eqs. (70) and (71)

$$\boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) = \frac{(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})^2}{\boldsymbol{\nabla}^2}, \qquad \boldsymbol{\mathit{\Pi}}_{\mathrm{s}}^{\mathrm{L}}(\boldsymbol{\nabla}) = \frac{\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{\nabla}_{\mathrm{s}}^{\dagger}}{\boldsymbol{\nabla}^2}\,.$$

These operators commute with the Hamiltonian, the momentum operator, and the angular-momentum operator

$$[\mathcal{H},\mathit{\Pi}] = [\boldsymbol{\mathcal{P}},\mathit{\Pi}] = [\boldsymbol{\mathcal{J}},\mathit{\Pi}] = 0, \tag{535}$$

where Π represents either projection operator, $\Pi^{\mathrm{T}}(\boldsymbol{\nabla})$ or $\Pi^{\mathrm{L}}(\boldsymbol{\nabla})$, so all the eigenstates considered in this paper are classified as being either transverse or longitudinal, with

$$\Pi^{\mathrm{T}}(\boldsymbol{\nabla})\,\Psi_n^{\mathrm{T}}(x) \;=\; \Psi_n^{\mathrm{T}}(x), \tag{536}$$

$$\Pi^{\mathrm{L}}(\boldsymbol{\nabla})\,\Psi_n^{\mathrm{L}}(x) \;=\; \Psi_n^{\mathrm{L}}(x), \tag{537}$$

respectively. The transverse states describe radiation and have non-zero eigenvalues in Eq. (523), while the longitudinal states correspond to electrostatic interactions, with an eigenvalue of zero. An exception is that the longitudinal states may have a non-zero eigenvalue if a hypothetical mass term is considered, as discussed in Sec. 7.4. The projection operators commute with rotations, but not, in general, with velocity boosts. However, as shown in Sec. 7.6.1, the velocity transformed transverse plane-wave states are also transverse. This corresponds to the fact that radiation may be treated relativistically independent of electrostatic interactions. On the other hand, as shown in Sec. 7.6.2, the velocity transformed longitudinal states have both longitudinal and transverse components, corresponding to the fact that moving charges may excite radiative transitions. Both transverse and longitudinal states are necessary in order to have a complete set, as in Eq. (526).

A solution of the inhomogeneous Maxwell equation may be obtained with the Maxwell Green function, as discussed in Sec. 9. The Green function satisfies the equation

$$\gamma_\mu\partial_2^\mu\,\mathcal{D}_{\mathrm{M}}(x_2-x_1) \;=\; \mathcal{I}\,\delta(x_2-x_1)$$

in Eq. (481), and a solution of the Maxwell equation is given by

$$\Psi(x_2) \;=\; \int \mathrm{d}^4x_1\,\mathcal{D}_{\mathrm{M}}(x_2-x_1)\,\Xi(x_1)$$

in Eq. (484). It is shown, by summing over the complete set of plane-wave solutions, that the Green function is

$$\mathcal{D}_{\mathrm{M}}(x_2-x_1) \;=\; \frac{\mathrm{i}}{(2\pi)^4}\int_{\mathrm{C_F}}\mathrm{d}^4k\,\frac{\mathrm{e}^{-\mathrm{i}k\cdot(x_2-x_1)}}{\gamma^\mu k_\mu - m_\epsilon c/\hbar}$$

in Eq. (499), which is the same form as the Dirac Green function, except that here it is a 6×6 matrix instead of a 4×4 matrix. In this equation, $\mathrm{C_F}$ is the Feynman contour and the infinitesimal mass is included to resolve an ambiguity in the longitudinal contribution.

In Sec. 10, applications of the Maxwell Green function are made, including a calculation of radiation from a Dirac-electron current source. In this example the six-component Maxwell formalism couples radiation to the Dirac current relativistically with a result that is the same as the result of a calculation that starts from Feynman-gauge QED.

12. Conclusion

We conclude that the criteria for properties of a single-photon wave function proposed in the introduction are met by the formalism described in the subsequent sections. In particular, the example of a photon wave packet provides a normalizable solution of the wave equation whose properties can be verified by explicit calculations. It yields the unanticipated result that for virtually any probability distribution, under rather mild assumptions about the form of the wave packet, the expectation value of the Hamiltonian is exactly $\langle \mathcal{H} \rangle = \hbar\omega_0$, where ω_0 is the frequency associated with the wave vector of the packet.

Appendix A. Velocity transformation of electromagnetic fields

The velocity transformation of electromagnetic fields is derived here without invoking potentials for completeness. With the aid of the identity $\boldsymbol{\nabla}_{\rm c}^\top \tilde{\boldsymbol{\tau}} \cdot c\boldsymbol{B} = (\boldsymbol{\nabla} \times c\boldsymbol{B})_{\rm c}^\top$, Eqs. (1) and (2) may be written as

$$\begin{aligned} \boldsymbol{\nabla}_{\rm c}^\top \boldsymbol{E}_{\rm c} &= \mu_0 c^2 \rho, && \text{(A.1)} \\ \frac{\partial \boldsymbol{E}_{\rm c}^\top}{\partial ct} - \boldsymbol{\nabla}_{\rm c}^\top \tilde{\boldsymbol{\tau}} \cdot c\boldsymbol{B} &= -\mu_0 c \boldsymbol{J}_{\rm c}^\top && \text{(A.2)} \end{aligned}$$

or

$$\partial_{\rm c}^\top g F = \mu_0 J^\top, \qquad \text{(A.3)}$$

where

$$F = \frac{1}{c}\begin{pmatrix} 0 & -\boldsymbol{E}_{\rm c}^\top \\ \boldsymbol{E}_{\rm c} & \tilde{\boldsymbol{\tau}} \cdot c\boldsymbol{B} \end{pmatrix} \qquad \text{(A.4)}$$

is the field tensor [see Eq. (38)] and

$$J = \begin{pmatrix} c\rho \\ \boldsymbol{J}_{\rm c} \end{pmatrix}. \qquad \text{(A.5)}$$

Since $V(\boldsymbol{v})\, g\, V(\boldsymbol{v}) = g$, Eq. (A.3) is equivalent to

$$\partial_{\rm c}^\top V(\boldsymbol{v})\, g\, V(\boldsymbol{v})\, F(x) V(\boldsymbol{v}) = \mu_0 J^\top(x) V(\boldsymbol{v}) \qquad \text{(A.6)}$$

or

$$\partial_{\rm c}^\top g\, V(\boldsymbol{v})\, F\left(V^{-1}(\boldsymbol{v})x\right) V(\boldsymbol{v}) = \mu_0 J^\top\!\left(V^{-1}(\boldsymbol{v})x\right) V(\boldsymbol{v}). \qquad \text{(A.7)}$$

Assuming the current transforms as a four-vector,

$$J'(x) = V(\boldsymbol{v})\, J\!\left(V^{-1}(\boldsymbol{v})x\right), \qquad \text{(A.8)}$$

Eq. (A.3) will be invariant if the field tensor transforms according to

$$F'(x) = V(\boldsymbol{v})\, F\!\left(V^{-1}(\boldsymbol{v})x\right) V(\boldsymbol{v}). \qquad \text{(A.9)}$$

Direct calculation yields[5]

$$F'(x') = \frac{1}{c}\begin{pmatrix} 0 & -\boldsymbol{E}_{\rm c}'^{\top}(x) \\ \boldsymbol{E}_{\rm c}'(x) & \tilde{\boldsymbol{\tau}}\cdot c\boldsymbol{B}'(x) \end{pmatrix}, \tag{A.10}$$

where $x' = V(\boldsymbol{v})\,x$ and

$$\boldsymbol{E}_{\rm c}' = \boldsymbol{E}_{\rm c}\cosh\zeta - \hat{\boldsymbol{v}}_{\rm c}\,\hat{\boldsymbol{v}}\cdot\boldsymbol{E}\,(\cosh\zeta-1) - \tilde{\boldsymbol{\tau}}\cdot\hat{\boldsymbol{v}}\,c\boldsymbol{B}_{\rm c}\sinh\zeta, \tag{A.11}$$

$$c\boldsymbol{B}_{\rm c}' = c\boldsymbol{B}_{\rm c}\cosh\zeta - \hat{\boldsymbol{v}}_{\rm c}\,\hat{\boldsymbol{v}}\cdot c\boldsymbol{B}\,(\cosh\zeta-1) + \tilde{\boldsymbol{\tau}}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{E}_{\rm c}\sinh\zeta. \tag{A.12}$$

These relations are equivalent (up to the velocity sign convention) to the electric and magnetic field transformations in [17], and in the spherical basis they are

$$\boldsymbol{E}_{\rm s}' = \boldsymbol{E}_{\rm s} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2\boldsymbol{E}_{\rm s}(\cosh\zeta-1) + {\rm i}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,c\boldsymbol{B}_{\rm s}\sinh\zeta, \tag{A.13}$$

$$c\boldsymbol{B}_{\rm s}' = c\boldsymbol{B}_{\rm s} + (\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}})^2c\boldsymbol{B}_{\rm s}(\cosh\zeta-1) - {\rm i}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{v}}\,\boldsymbol{E}_{\rm s}\sinh\zeta. \tag{A.14}$$

The Cartesian transformation in Eqs. (A.11) and (A.12) can be written as

$$\Psi_{\rm c}'(x') = {\rm e}^{-\zeta\tilde{\boldsymbol{\mathcal{K}}}\cdot\hat{\boldsymbol{v}}}\,\Psi_{\rm c}(x), \tag{A.15}$$

where

$$\tilde{\boldsymbol{\mathcal{K}}} = \begin{pmatrix} \boldsymbol{0} & \tilde{\boldsymbol{\tau}} \\ -\tilde{\boldsymbol{\tau}} & \boldsymbol{0} \end{pmatrix} \tag{A.16}$$

and

$$\Psi_{\rm c}(x) = \begin{pmatrix} \boldsymbol{E}_{\rm c}(x) \\ c\boldsymbol{B}_{\rm c}(x) \end{pmatrix}. \tag{A.17}$$

Similarly, the Cartesian form of the rotation transformation, corresponding to the operator in Eq. (153), is

$$\Psi_{\rm c}'(x') = {\rm e}^{\tilde{\boldsymbol{\mathcal{S}}}\cdot\boldsymbol{u}}\,\Psi_{\rm c}(x), \tag{A.18}$$

where

$$\tilde{\boldsymbol{\mathcal{S}}} = \begin{pmatrix} \tilde{\boldsymbol{\tau}} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{\tau}} \end{pmatrix}. \tag{A.19}$$

The components of the matrices in Eqs. (A.16) and (A.19) have the commutation relations

$$\left[\tilde{\mathcal{S}}^i, \tilde{\mathcal{S}}^j\right] = \epsilon_{ijk}\,\tilde{\mathcal{S}}^k, \tag{A.20}$$

$$\left[\tilde{\mathcal{S}}^i, \tilde{\mathcal{K}}^j\right] = \epsilon_{ijk}\,\tilde{\mathcal{K}}^k, \tag{A.21}$$

$$\left[\tilde{\mathcal{K}}^i, \tilde{\mathcal{K}}^j\right] = -\epsilon_{ijk}\,\tilde{\mathcal{S}}^k, \tag{A.22}$$

characteristic of the Lie algebra of Lorentz transformations.

[5]The identity $\epsilon_{ijk} = \epsilon_{ljk}\hat{v}^i\hat{v}^l + \epsilon_{ilk}\hat{v}^j\hat{v}^l + \epsilon_{ijl}\hat{v}^k\hat{v}^l$ may be useful here.

Appendix B. Inverse Laplacian

For cases where the integral definition of the inverse Laplacian converges poorly, we use a generalized definition that includes a damping factor to resolve ambiguity in the intermediate steps of the calculation. From the equation

$$\left(\boldsymbol{\nabla}^2 - \epsilon^2\right) \frac{\mathrm{e}^{-\epsilon\,|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} \quad = \quad -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{B.1}$$

one has

$$\frac{1}{\boldsymbol{\nabla}^2 - \epsilon^2}\,\delta(\boldsymbol{x}-\boldsymbol{x}') \quad = \quad -\frac{1}{4\pi}\,\frac{\mathrm{e}^{-\epsilon\,|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{B.2}$$

Multiplication by $f(\boldsymbol{x}')$ and integration over $\boldsymbol{x}'$ yields

$$\frac{1}{\boldsymbol{\nabla}^2 - \epsilon^2}\,f(\boldsymbol{x}) \quad = \quad -\frac{1}{4\pi}\int \mathrm{d}\boldsymbol{x}'\,\frac{\mathrm{e}^{-\epsilon\,|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,f(\boldsymbol{x}'). \tag{B.3}$$

We thus have, for example,

$$\frac{1}{\boldsymbol{\nabla}^2 - \epsilon^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \quad = \quad -\frac{1}{\boldsymbol{k}^2+\epsilon^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \to -\frac{1}{\boldsymbol{k}^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{B.4}$$

by direct calculation of the integral.

Appendix C. Coulomb matrix element

The calculation of the Coulomb field matrix element in Eq. (248) requires evaluation of the integral

$$\begin{aligned}\int \mathrm{d}\boldsymbol{x}\,\frac{\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \quad &= \quad \lim_{\epsilon\to 0}\int \mathrm{d}\boldsymbol{x}\,\mathrm{e}^{-\epsilon|\boldsymbol{x}|}\,\frac{\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &\to \quad -\mathrm{i}\,\frac{4\pi}{|\boldsymbol{k}|},\end{aligned} \tag{C.1}$$

which is defined with the convergence factor $\mathrm{e}^{-\epsilon|\boldsymbol{x}|}$. The inverse transformation requires the integral

$$\begin{aligned}\int \mathrm{d}\boldsymbol{k}\,\frac{\hat{\boldsymbol{k}}}{|\boldsymbol{k}|}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \quad &= \quad \lim_{\epsilon\to 0}\int \mathrm{d}\boldsymbol{k}\,\mathrm{e}^{-\epsilon|\boldsymbol{k}|}\,\frac{\hat{\boldsymbol{k}}}{|\boldsymbol{k}|}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &\to \quad \mathrm{i}\,\frac{2\pi^2\boldsymbol{x}}{|\boldsymbol{x}|^3},\end{aligned} \tag{C.2}$$

which confirms the result that

$$-\frac{\mathrm{i}\,q}{\sqrt{(2\pi)^3}\,\epsilon_0}\int \mathrm{d}\boldsymbol{k}\,\frac{1}{|\boldsymbol{k}|}\,\psi^{(+)}_{\boldsymbol{k},0}(\boldsymbol{x}) \quad = \quad \frac{q}{4\pi\epsilon_0|\boldsymbol{x}|^3}\begin{pmatrix}\boldsymbol{x}_{\mathrm{s}}\\ \boldsymbol{0}\end{pmatrix}. \tag{C.3}$$

Appendix D. Separation of the transverse and longitudinal gradient operators

The transverse gradient operator $\boldsymbol{\tau}\cdot\boldsymbol{\nabla}$ is separated into radial and angular parts by writing

$$\begin{aligned}\boldsymbol{\tau}\cdot\boldsymbol{\nabla} &= \boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}+\hat{\boldsymbol{x}}_{\rm s}\,\hat{\boldsymbol{x}}_{\rm s}^\dagger\,\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\\ &= \frac{\partial}{\partial r}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}-\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\overline{\boldsymbol{\nabla}}_{\rm s}\,\hat{\boldsymbol{x}}_{\rm s}^\dagger+\frac{1}{\hbar r}\,\hat{\boldsymbol{x}}_{\rm s}\boldsymbol{L}_{\rm s}^\dagger \end{aligned} \tag{D.1}$$

where the line over the gradient operator indicates that it does not act on the unit vector directly to the right. That term is

$$\begin{aligned}\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\overline{\boldsymbol{\nabla}}_{\rm s}\,\hat{\boldsymbol{x}}_{\rm s}^\dagger &= \boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\left(\boldsymbol{\nabla}_{\rm s}\hat{\boldsymbol{x}}_{\rm s}^\dagger+\frac{1}{r}\,\hat{\boldsymbol{x}}_{\rm s}\hat{\boldsymbol{x}}_{\rm s}^\dagger-\frac{1}{r}\,\boldsymbol{I}\right)\\ &= -\frac{1}{\hbar r}\,\boldsymbol{L}_{\rm s}\hat{\boldsymbol{x}}_{\rm s}^\dagger-\frac{1}{r}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}. \end{aligned} \tag{D.2}$$

Thus

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla} = \frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\boldsymbol{\tau}\cdot\,\hat{\boldsymbol{x}}\,+\,\frac{1}{\hbar r}\left(\boldsymbol{L}_{\rm s}\,\hat{\boldsymbol{x}}_{\rm s}^\dagger+\hat{\boldsymbol{x}}_{\rm s}\,\boldsymbol{L}_{\rm s}^\dagger\right). \tag{D.3}$$

Acting on $\boldsymbol{L}_{\rm s}$, the transverse gradient operator yields

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,\boldsymbol{L}_{\rm s} = \frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\boldsymbol{\tau}\cdot\,\hat{\boldsymbol{x}}\,\boldsymbol{L}_{\rm s}-\frac{1}{\hbar r}\hat{\boldsymbol{x}}_{\rm s}\,\boldsymbol{L}^2, \tag{D.4}$$

where $\boldsymbol{L}_{\rm s}^\dagger\boldsymbol{L}_{\rm s}=-\boldsymbol{L}^2$.

For the longitudinal gradient operator $\boldsymbol{\nabla}_{\rm s}$, the identity

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\,\frac{\partial}{\partial r}-\frac{\rm i}{\hbar r}\,\hat{\boldsymbol{x}}\times\boldsymbol{L} \tag{D.5}$$

provides

$$\boldsymbol{\nabla}_{\rm s} = \hat{\boldsymbol{x}}_{\rm s}\,\frac{\partial}{\partial r}-\frac{1}{\hbar r}\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\,\boldsymbol{L}_{\rm s}. \tag{D.6}$$

Appendix E. Exact transverse Green function

The exact transverse Maxwell Green function follows from Eq. (491) together with the relations

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,\frac{{\rm e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|} = -w\,\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}\,\frac{{\rm e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|}\left(1+\frac{1}{w|\boldsymbol{r}|}\right) \tag{E.1}$$

and

$$\frac{1}{\nabla^2}\,\frac{{\rm e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|} = -\frac{1}{4\pi}\int{\rm d}\boldsymbol{x}\,\frac{1}{|\boldsymbol{r}-\boldsymbol{x}|}\,\frac{{\rm e}^{-w|\boldsymbol{x}|}}{|\boldsymbol{x}|}=-\frac{1}{w^2\,|\boldsymbol{r}|}\left(1-{\rm e}^{-w|\boldsymbol{r}|}\right), \tag{E.2}$$

which yield

$$\frac{\nabla^i \nabla^j}{\nabla^2} \frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|} = \frac{r^i r^j}{|\boldsymbol{r}|^2} \frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|} + \left(3 \frac{r^i r^j}{|\boldsymbol{r}|^2} - \delta_{ij}\right) \left(\frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{w|\boldsymbol{r}|^2} + \frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{w^2|\boldsymbol{r}|^3} - \frac{1}{w^2|\boldsymbol{r}|^3}\right) \tag{E.3}$$

and

$$\boldsymbol{\Pi}_{\mathrm{s}}^{\mathrm{T}}(\boldsymbol{\nabla}) \frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|} = (\boldsymbol{\tau} \cdot \hat{\boldsymbol{r}})^2 \frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{|\boldsymbol{r}|} + \left[(\boldsymbol{\tau} \cdot \hat{\boldsymbol{r}})^2 - 2\, \hat{\boldsymbol{r}}_{\mathrm{s}}\, \hat{\boldsymbol{r}}_{\mathrm{s}}^{\dagger}\right] \left(\frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{w|\boldsymbol{r}|^2} + \frac{\mathrm{e}^{-w|\boldsymbol{r}|}}{w^2|\boldsymbol{r}|^3} - \frac{1}{w^2|\boldsymbol{r}|^3}\right). \tag{E.4}$$

Appendix F. Radiative decay in quantum electrodynamics

In QED, the radiative decay rate of an excited state may be obtained from the imaginary part of the radiative correction to the energy level of that state

$$\hbar \sum_f A_{if} = -2\, \mathrm{Im}(\Delta E_i), \tag{F.1}$$

where the sum is over all states with a lower unperturbed energy. This gives a correction to the level that, roughly speaking, results in an exponentially damped time dependence for the population of the state:

$$|\mathrm{e}^{-\mathrm{i}\,\Delta E\, t/\hbar}|^2 = \mathrm{e}^{-\sum_f A_{if} t}. \tag{F.2}$$

For one-photon decays, the rate is included in the second-order self-energy correction to the level. An expression derived from Feynman-gauge QED that includes some of the real part and all of the imaginary part of this level shift in hydrogen-like atoms is given by [31]

$$\begin{aligned}
\Delta E_i &= -\frac{\alpha \hbar^2 c^2}{4\pi^2} \int_{\hbar c k < E_i} \mathrm{d}\boldsymbol{k}\, \frac{1}{k} \left(\delta_{jl} - \frac{k^j k^l}{\boldsymbol{k}^2}\right) \left\langle \alpha^j \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{H - E_i + \hbar c k - \mathrm{i}\delta}\, \alpha^l \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right\rangle \\
&= -\frac{\alpha \hbar^2 c^2}{4\pi^2} \int_{\hbar c k < E_i} \mathrm{d}\boldsymbol{k}\, \frac{1}{k} \sum_{\lambda=1}^{2} \sum_f \left\langle i \left| \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \cdot \boldsymbol{\alpha}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right| f \right\rangle \\
&\quad \times \frac{1}{E_f - E_i + \hbar c k - \mathrm{i}\delta} \left\langle f \left| \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \cdot \boldsymbol{\alpha}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right| i \right\rangle,
\end{aligned} \tag{F.3}$$

where H is the Dirac Hamiltonian. The integrand is real, except for the imaginary infinitesimal in the denominator, for which

$$\mathrm{Im}\, \frac{1}{E_f - E_i + \hbar c k - \mathrm{i}\delta} \rightarrow \pi \delta(E_f - E_i + \hbar c k) \tag{F.4}$$

and hence

$$\sum_f A_{if} = \sum_f \frac{\alpha k c}{2\pi} \int \mathrm{d}\Omega_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \left\langle i \left| \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \cdot \boldsymbol{\alpha}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right| f \right\rangle \left\langle f \left| \hat{\boldsymbol{\epsilon}}_\lambda(\hat{\boldsymbol{k}}) \cdot \boldsymbol{\alpha}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right| i \right\rangle, \tag{F.5}$$

with the restriction $0 < E_f < E_i$ on the sum over f. The contribution to the decay rate from each final state f coincides with Eq. (519) for the transition rate A_{if}.

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