Motivation

- Geometric dimensions & tolerances are of concern in all aspects of product development.
- Designers are concerned with assemblability and function.
- Process planners are concerned with selection of set-ups, fixturing, machines and operation tolerances to minimize manufacturing cost and time.
- QA must verify that manufactured parts comply with design specifications.
- Miscommunication and misinterpretation between these groups can result in low acceptance rates or expensive rework.
- While 3D Computer aided Tolerance Analysis tools are available to designers, the same is not true for manufacturing process planning.
- A fundamental understanding of geometric variations, their accumulation, and their implications in design, manufacturing and inspection is needed.
- These are the motivations for developing mathematical models for GD&T.

The Challenge

- In engineering practice, tolerances are specified using national and international GD&T standards, such as ASME Y14.5M and ISO 1101.
- This standards are not based on any math foundation; they are a set of symbols, conventions and practices.
- If a model is to gain acceptance, it must be consistent with the standards.
- Many methods have been proposed for tolerance representation & analysis, based on elegant math models but failed to gain acceptance because they were not compatible with the standards.
- PARAMETRIC MODELS [Hillyard & Braid 78, Light & Gossard 82]
- OFFSET ZONES [Requicha 83, Requicha & Chan 84]
- VARIATIONAL SURFACES [Martinsen 93, Turner 90]
- VECTOR SPACES [Turner & Wozny 90]
- KINEMATIC MODELS [Chase & Magelby 98, Rivest 94, Kramer 92]
- DEGREE OF FREEDOM (DOF) MODELS [Bernstein 89, Clement 91, Zhang 92, Solomons 95, Kandikjian 98]; TTRS MODELS [Clement 91, Desroschers 99]
- Develop math models for GD&T consistent with the standard, i.e. retroactively fit a math model to the conventions in ASME Y14.5M.
- Therefore, it is important to understand the key concepts in GD&T standards before discussing tolerance analysis.

ASME Y14.5 Conventions: A Quick Look

Geometric variations have been decomposed into specific types because they affect function & assembly in different ways.

| Material Conditions (MMC, LMC) can enlarge position tolerance zones by the difference between MMC (or LMC) and actual size |

Datum order influences directions of measurements.

Bonus Tolerance & Shift:

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ASU Bi-Level Math Model

**TOPOLOGICAL MODEL** (DoF algebra; CTF graph):
- Similar to topology or control schema
- Models relationships between all feature control frames, datum reference frames (DRF) and their precedence (datum flow chain)
- Provides basis for geometric validation of D&T scheme, loop detection for analysis and DoFs
- Supported by DoF algebra

**METRIC MODEL** (T-Maps):
- Models the composite quantitative effect of all tolerances on a given feature
- Interaction of size, form, orientation, position is clearly identified
- Rule #1 is embedded in the formulation
- Relative volumes of regions can be used to study trade-offs in tolerance allocation (size vs form vs orientation...)

* US Patent No. 6,963,824

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**Topological Model: Basic Concepts**

Surface mapping: The degrees of freedom of all types of surfaces can be represented by combinations of points, lines, and planes establishing a mapping between surfaces and control frames.

Control frames, (D,T,R): Directed geometric relations R between datum D and target T rigid sets.

Degrees of freedom (DoFs) of an entity or rigid set: Translations (x,y,z) or rotations (α,β,γ) not constrained by geometric relations minus the invariant directions.

Invariant DoFs: An entity or rigid set is invariant in those transformations that have no effect on its location or orientation.

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**Topological Model: DoF Algebra**

- DRFs and TRFs are clusters of points, lines and planes with different geometric relations to each other (coincident, //, ⊥, …)
- DoF Algebra includes symbolic ops to determine free and invariant DoFs of entity clusters.
- This algebra was validated by applying it to all cases in the Y14.5.1.

**Algebraic Operators**

<table>
<thead>
<tr>
<th>Union Operation ∪</th>
<th>Intersection Op ∩</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁ ∪ b₁</td>
<td>a₁ ∩ b₁</td>
</tr>
<tr>
<td>1 1</td>
<td>1 1</td>
</tr>
<tr>
<td>1 0</td>
<td>0 1</td>
</tr>
<tr>
<td>0 1</td>
<td>0 1</td>
</tr>
</tbody>
</table>

**Combining DoFs for clusters**

[X_{fdof}] = [A_{fdof}] ∪ [B_{fdof}];
[X_{inv}] = [A_{inv}] ∩ [B_{inv}]

**Example:** Line-Plane (coincident):

the plane CS will be used as the cluster CS; the line CS needs to be transformed.

**Plane C:** C_{dof} = [001,110] and C_{inv} = [110,001]

**Line B:** B_{dof} = [110,110] and B_{inv} = [001,001]

[(BC)_{dof}] = [OP_{BC} B_{dof}] ∪ [C_{dof}]

[(BC)_{inv}] = [B_{inv}] ∩ [C_{inv}]

**Algebraic Relations**

- [A] ∪ [B] = [B] ∪ [A] ................. Commutative relation
- [A_{dof}] ∩ [A_{do}] = [∅]=[000,000] ........... Null set
- [A_{dof}] ∪ [A_{do}] = [I]=[111,111] ....... “Identity” vector
- [A_{inv}] = RCP {A_{do}} ............. Reciprocal relation (or Ā)

*Standard Associative, Distributive and Idempotence relations*
**DOF representation of Tolerances**

- DoF algebra models datum flow chains, proper DRF combinations and tolerance classes.
- The constrained DOFs are the intersection of the DOFs of the three tolerance elements.
- The target, DRF and tolerance classes are completely represented in terms of DOF vector.
- No matter what the target cluster is, the DOF vector of target entity is one of six combinations.

<table>
<thead>
<tr>
<th>No.</th>
<th>Target</th>
<th>DRFs</th>
<th>Tol. Class</th>
<th>Constrained DOFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(111,000)</td>
<td>(111,000)</td>
<td>(111,110)</td>
<td>(111,000)</td>
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<tr>
<td>2</td>
<td>(110,110)</td>
<td>(110,110)</td>
<td>(000,111)</td>
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<tr>
<td>6</td>
<td>(111,111)</td>
<td>(111,111)</td>
<td>(000,111)</td>
<td>(000,110)</td>
</tr>
</tbody>
</table>

**Metric Model For Planar Faces**

- Areal (barycentric) coordinates → A point in 2-D space is represented by 3 homogeneous coordinates.
- \( \sigma = \lambda_1 \sigma_1 + \lambda_2 \sigma_2 + \lambda_3 \sigma_3 \quad \lambda_1 + \lambda_2 + \lambda_3 = 1 \)

\( \sigma_1 = \{ \lambda_1, \lambda_2, \lambda_3 \} = \{1,0,0\} \)

\( \sigma_2 = \{ \lambda_1, \lambda_2, \lambda_3 \} = \{0,1,0\} \)

\( \sigma_3 = \{ \lambda_1, \lambda_2, \lambda_3 \} = \{0,0,1\} \)

By appropriate choice for \( \sigma_1, \sigma_2, \sigma_3 \), \( p, q, s \) are proportional to the scale for Cartesian frame placed on the E-space.

- Duality of space of points and planes:

\[ px + qy + rz + sw = 0 \]

Points \((x, y, z, w)\) lie on plane \((p, q, r, s)\)

All planes \((p, q, r, s)\) passing through the point \((x, y, z, w)\)
Form & Orientation Tolerances: Planar Features

FLOATING ZONES
- ORIENTATION zone \((t')\) translates \(\uparrow\downarrow\) and can rotate about \(x-\) or \(y-\) axes
- FORM zone \((t')\) translates \(\uparrow\downarrow\) and rotates about \(x-\) or \(y-\) axes

Addition of orientation tol \(t'\) to size reduces the allowable tilt
Orientation T-map can be obtained from size by truncating the \(\sigma_3\) axis

SIZE + ORIENTATION T-map

SIZE + FORM T-map
- As per Y14.5 Rule#1
  - Worst form occupies the entire zone
  - Perfect form occupies none
- Therefore, size + form is modeled by splitting into two planar T-maps that together must conform to size map

Tolerance Maps For Lines: 4-D Solid of Points
- 2D cross-sections of the T-Map
  - \(\lambda_4 = \lambda_5 = 0\)
  - \(\lambda_3 = \lambda_5 = 0\)
  - \(\lambda_2 = \lambda_4 = 0\)

3D cross-sections: Trade-off between position & form
- Perfect form
- Worst form

Material Modifiers in T-map models
- 4-D T-Maps: size is the 4th dimension
- The dipyramid now is the T-map for position of the medial plane.

Tolerance Analysis
- Purpose
  - Determine accumulation of geometric variations caused by all contributing elements (dimension, location, orientation, etc)
  - In general, the analyzed dimension \(A\) is a non-linear function of independent dimensions & geometric variations
- In parts variations are controlled by datum flow chains
- In assemblies tolerances accumulate (stack-up)
- Non-linear problem; hard to do with both dimensional & geometric tolerances

Types of analysis
- Worst case analysis – 100% interchange-ability
- Statistical analysis – selective assembly

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Tolerance Analysis with T-maps: Minkowski Sums

Variational possibilities – infinite combinations

Minkowski sum: $C = U_c$, where $c = a + b$ and $a \in A; b \in B$

Accumulation map

Individual Tolerance Maps

Worst case analysis with T-maps: Functional & Accumulation Maps

Functional Map

Fit accumulation map inside functional map

Accumulation Map

$t_f = t_2 + t_1$

$t_f^* = t_2 + d_2 t_1^* / d_1$

Clearance Distribution due to Position & Size of Mating Features

Circular Runout Model

- Circular runout is a composite tolerance that controls both circularity and concentricity (position), independent of size
- Applied to any axisymmetric X-sec

- Circular X-sec, involves two variables: circularity (annular zone) + eccentricity (a) An annular tolerance-zone of amount $t'$ which lies between the inner and outer boundaries $\gamma_1$ and $\gamma_2$ of radii $r_1$ and $r_o$, respectively. (b) Its 2D T-Map

- Planar (end) involves two variables: linear offset + angle (a) A cylindrical tolerance-zone of height $t$ which lies between the upper and lower boundaries of $y_1$ and $y_2$. (b) Its 2D T-Map.
Line Profile: parametric model

- Profile tolerances control the shape, size, and position of complex features, e.g. turbine blades and pump vanes.
- For line profiles, four variables are required to identify a variation of the theoretical shape within its tolerance-zone.

Example: A square line-profile
- For line profiles, each point in the T-Map represents one square with a given size and x-, y-, and θ-position in the tolerance-zone. Consequently, the T-Map is a 4-D geometric shape.

Line Profile: Decomposition model

1. Decompose the entire line-profile into its line, circular-arc, and/or free-form segments.
2. Define a local reference system for each of the segments, and create the primitive T-Map for each one.
3. Arbitrarily set a temporary reference frame for the entire profile and represent each primitive T-Map in this reference system.
4. Intersect the transformed primitive T-Maps in the temporary reference frame to get a tentative T-Map for the entire profile.
5. Find the maximum rotation center (pole) of the profile. Reset the origin of the reference frame to the pole and transform the tentative T-Map to its representation in this new frame.

T-map Catalog: Sample page

More than 50 T-map models have been developed so far based on combinations of target feature, tolerance type and datum type

<table>
<thead>
<tr>
<th>T-map</th>
<th>Geometry, tolerance, datum</th>
<th>T-map</th>
<th>Geometry, tolerance, datum</th>
</tr>
</thead>
</table>

The Tolerance Analysis Maze

Many variations of tolerance analysis approaches exist in practice

Min/Max charts  I-DEAS  VSA, eTolmate

<table>
<thead>
<tr>
<th>Dimensionality</th>
<th>Analysis</th>
<th>Tolerance classes</th>
<th>GDT standards</th>
<th>Level</th>
<th>Linearization</th>
<th>Automation</th>
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</thead>
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<tr>
<td>1-D</td>
<td>Worst case</td>
<td>dimensional</td>
<td>compatible</td>
<td>part</td>
<td>Linear</td>
<td>manual</td>
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<td>geometric</td>
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<td>assembly</td>
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<tr>
<td>3-D</td>
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<td>all</td>
<td>Not compatible</td>
<td>Parts + assembly</td>
<td>Non-linear</td>
<td>automated</td>
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<td>Any dist</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Worst case     | Dimensional      |                   |               |       |               |            | & statistical
Integrated GD&T: System Architecture

**Legend**
- GD&T Design Support Modules
- Tolerance Analysis Support Modules II (Charts)
- Tolerance Analysis Support Modules II (T-Maps)
- T-Maps (modeling, Minkowski Sum)
- GD&T Inspection Module

**GD&T in Design vs. Process plans**

**Designs (formal GD&T)**
- DRFs explicitly shown
- Formal GD&T frames
- Datum flow chain directly extracted
- Consolidated info, in single Drg.
- Drawings represent final parts
- Many tolerance analysis methods used (1D/2D/3D)

**Process plan (implied GD&T)**
- DRFs are implicit in setups, fixtures
- At most, +/- for dimensions, No GD&T
- Datum, and flow chain implicit, distributed
- Distributed info (in multiple steps/pages)
- Plans represent many transitions
- Mostly 1-D tolerance charts are used by process planners
• Process planners use their knowledge of machine accuracy, operation variability and fixturing elements to develop mfg plans
• Process planners must convert the GD&T schema to their setups, operation sequence and fixture plans (different datums)
• Stack analyses is typically done with 1D charts and plan documentation only contains conventional ± tolerances
• What if want to independently verify/audit process plan GD&T?
• That would require tolerance explication from process plans
• The T-map model can be used for both objectives

Tolerance Conversion
• Process plans typically call for multiple setups
• the datum flow chain used by manufacturing is different from design.
• This requires tolerance conversion and datum transfer.
• Example: design runout tolerances with bearing surfaces D, E; process plan for turning may call for the part to use surfaces E,G instead

The Tool & Mfg Engineers Handbook documents the manufacturing charts procedure for verifying design tolerances in process sequences
• This is just a 1D stack involving dimensional tolerances only. Trig functions are used to convert angular feature

Datum transformation
• establish relation between design tolerances and machining tolerance in transferring of the datum
• enables 3D tolerance analysis consistent with Y14.5 standard

• Minkowski Sum of the Manufacturing T-map and the Datum transformation T-map should fit into the Design T-map.
Tolerance Conversion: m-maps

**Procedure**
- Determine relationships between original datum flow and machining ops
- Generate T-Maps corresponding to variations that were controlled directly in design but have become indirect in manufacturing (m-maps)
- Chains can include transient features, as well
- M-map will depend on all the contributors in the stack and will need to be determined by a Minkowski sum,

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**References**