The photon-fluence scaling theorem for Compton-scattered radiation

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(Received 16 March 1981; accepted for publication 23 November 1981)

This paper concerns a method of scaling photon fluence from one scattering material to another when the photon energies are such that the dominant mode of interaction is Compton scattering. The theorem establishes a one-to-one correspondence between points in the two scattering media where the spectra of primary and scattered photons have the same distribution in energy and angle, and where the fluence ratio equals the square of the electron density ratio. Experimental tests were made with cobalt-60 gamma radiation using ionization-chamber measurements in graphite, acrylic plastic, polystyrene, and water phantoms. The experimental results are consistent with the equality of photon spectral shapes and angular distributions at corresponding points. The fluence ratios may differ by a few percent from the predicted values, depending on distance from the source.

I. INTRODUCTION

In the medical use of radiation it is common to encounter a situation where measurements of some dosimetric quantity have been made in a radiation beam in one material, and it is desired to convert those measurements to the values that would have been obtained in another material. The difficulty in doing that arises from the combination of purely geometrical inverse square attenuation, and attenuation due to physical interactions, the latter depending on the energy of the radiation and on the density and atomic number of the attenuating medium.

A number of individuals have discussed the problem in part. To our knowledge the first was O'Connor,[1] who was concerned with transit dose measurements in an inhomogeneous medium using x rays with a half-value layer of 1.5 mm Cu. He showed that, for a homogeneous body in an x-ray beam, the ratio of scattered to primary radiation is independent of density for water and materials that differ from water only in density. Casson[2] reported that absorbed dose in water can be obtained from absorbed dose in plastics, for photon energies that undergo Compton scatter, by scaling depths and field sizes according to electron densities. In a discussion of absorbed-dose corrections for inhomogeneity, Sontag and Cunningham[3] showed that primary and first-generation scattered radiation scaled inversely with electron density. Kutcher and Suntharalingam[4] tested the scaling of absorbed dose with electron density and reported that it was "adequate" for cobalt-60 γ radiation, and also appeared to serve at higher photon energies, within the accuracy of the measurements and calculations. Cunningham[5] called attention in a brief note to the earlier work of O'Connor, and also to the possible usefulness of the scaling principle for transferring to water absorbed dose to graphite determined from a graphite calorimeter.

Scaling photon attenuation calculations in terms of photon mean free paths is of course a well-established practice. In connection with studies of structural shielding against fallout radiation, Spencer noted that the effective thickness of shielding materials is best expressed in terms of electrons per unit area.[6] In a later study of the same subject, Spencer stated a scaling theorem very nearly the same as that discussed here, and gave a proof by comparison of transport equations.[7] That theorem involved scaling between materials of different densities but the same atomic number. If his "density" is interpreted to mean electron density (electrons per unit volume), Spencer's theorem becomes equivalent to that under discussion in this paper, which allows scaling between materials of different atomic numbers.

A theorem that allows accurate scaling from one medium to another, even under limited irradiation conditions, could well have many useful applications. It is the purpose of this note to present the scaling theorem, unencumbered by extraneous material, and to describe some experimental tests, using phantoms of several different materials.

II. THE SCALING THEOREM

The scaling theorem provides a relationship between the photon fluences in two media that are irradiated by the same source, provided the radiation interacts with the media by Compton scatter only.

Consider a point source of radiation S that irradiates part or all of a volume V, as in Fig. 1. The source can be inside or outside the volume V. Let ϵ be the electron density of V, in electrons per unit volume. Assume that (a) the radiation interactions in V are due only to Compton scatter, and (b) that the volume V is surrounded by empty space, so there is neither absorption nor scatter of radiation outside V (any void in V is considered to be outside V).

Let the volume V and all associated distances and dimensions be scaled according to the rule

$$x' = (\epsilon/\epsilon')x, \tag{1}$$

where x is a length, and ϵ' is the electron density of the second medium. Let V' be the volume corresponding to V that results from that scaling. Equation (1) is a rule for mapping the unprimed space that includes V into the primed space

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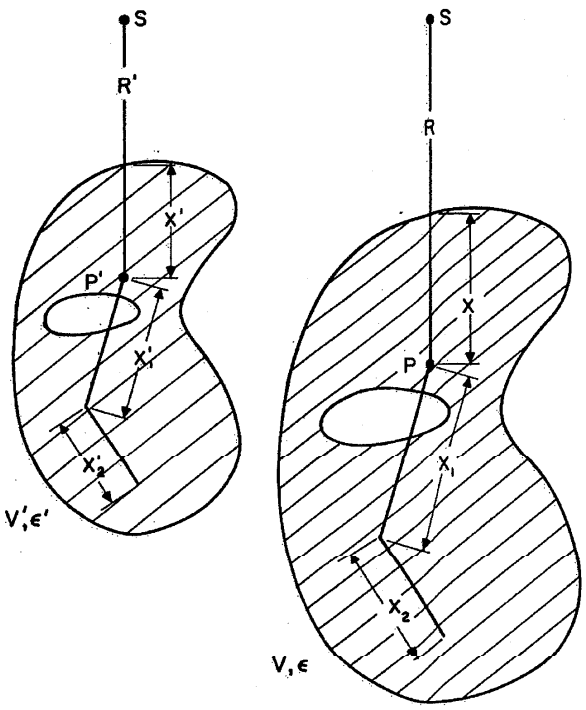

FIG. 1. Scaling of two volumes in the ratio of their electron densities. The figure illustrates the case that $\epsilon' = 1.33\epsilon$, and all lengths are scaled in the ratio 1 to 1.33. One possible configuration of photon trajectories from the source S is illustrated.

that includes V'. The scaling theorem states that *for every point P in volume V, there is a corresponding point P' in volume V' such that the photon spectral distributions are the same in energy and direction, and the photon fluences are related by*[8]

$$\Phi_{P'} = (\epsilon'/\epsilon)^2 \Phi_P. \tag{2}$$

It is convenient to think of the electron densities as constant, but it is only necessary to assume (c) that the electron densities at corresponding points P and P' are in a constant ratio. Likewise it is convenient to think of the source S as a point, but under some circumstances the theorem is valid for a source of arbitrary shape, provided the source dimensions also are suitably scaled. Combining Eqs. (1) and (2), it is seen that, for the arbitrary corresponding points P and P' in Fig. 1:

$$(R')^2 \Phi_{P'} = R^2 \Phi_P. \tag{3}$$

Thus a simple inverse square correction[9] yields equal photon fluences at corresponding points, regardless of the complexity of the paths by which photons reach the points. Equations (2) and (3) can be interpreted as referring to photon number fluence, or photon energy fluence.

Proof of the scaling theorem is found in the reference cited.[7] It is also possible to construct a proof without reference to the transport equation, by consideration of corresponding photon histories, as illustrated in Fig. 1.

Since for the radiations and materials of interest here, the ranges of the secondary electrons are appreciably less than the photon mean free path, the energy transport by electrons can for some purposes be neglected. Assuming this to be a satisfactory approximation, the absorbed dose at each point is a function of the photon fluence at that point only, and the scaling theorem predicts that absorbed dose at corresponding points will scale as the relative electron density squared. That is, the theorem predicts that, for fields subtending a given solid angle at the source, depth-dose curves in different materials will be the same in shape and in magnitude if the source-surface distance and the depth in each material are scaled with electron density according to Eq. (1), and if an inverse square factor is applied to account for the change in the source distance, in close analogy with Eq. (3). This implies that depth and field dimensions within the materials are measured in terms of electrons per unit area, and electron density means electrons per unit volume.

III. EXPERIMENTAL TEST OF THE SCALING THEOREM

The gamma radiation of cobalt-60 is a particularly favorable case for testing the scaling theorem in that pair production and the photoelectric effect are negligible at that photon energy in low-atomic-number materials, and the photoelectric coefficient is less than 2% of the scatter coefficient for scattered photons with energies as low as 100 keV. It should, however, be recognized that there is reason to expect that the scaling theorem will fail to some extent when applied to phantoms irradiated with a conventional cobalt-60 gamma-ray source. The source is not a point, there is scattering of radiation within the source head and collimator, and there is some absorption and scatter of the radiation by the intervening air.

The ionization measurements reported here were made in a cobalt-60 gamma-ray beam[10] in phantoms of water, polystyrene, acrylic plastic (polymethylmethacrylate), and graphite, using a graphite ionization chamber that has been described previously.[11] Table I shows the mean mass densities of the four materials and their electron densities relative to water. The source-detector distances used in three sets of measurements, designated A, B, and C, are also listed in Table I. The distances were chosen to satisfy Eq. (1). Four square collimator openings were used, giving field sizes at the detector ranging from a 52-mm square for Set A with a graphite phantom to a 222-mm square for Set C with the water phantom. (At a source distance of 1 m, these collimators gave square fields with sides of 80, 95, 115, and 145 mm.) Measurements were made as a function of phantom depth for scaled depths between 8 and 130 mm, where the scaled depth in each material corresponding to a linear depth x is

$$\bar{x} = (\epsilon/\epsilon_{\text{water}})x, \tag{4}$$

where $(\epsilon/\epsilon_{\text{water}})$ is the electron density relative to that of water.

Measurements were made for the four materials, the three sets of distances (A, B, and C), and the four collimator sizes, giving in all 48 depth–current curves. There were then 12 curves for each material, providing 12 independent tests of the scaling theorem. In order to assign scaled depths in the plastic and graphite phantoms, the mean mass density of

TABLE I. Measurement conditions and results.

Material		Water	Polystyrene	Acrylic	Graphite
Mean mass density (g/cm^3)		1	1.049	1.182	1.70
Relative electron density ($\epsilon/\epsilon_{water}$)		1	1.016	1.149	1.53
Source-detector distance (m)	A	1.000	0.984	0.870	0.654
	B	1.265	1.245	1.101	0.827
	C	1.530	1.506	1.332	1.000
Normalizing factor	A	1	1.004[a]	1.000	0.983
	B	1	1.002[a]	0.998	0.985
	C	1	1.006[a]	1.002	0.991
Root mean square deviation (%)		0.11	0.18	0.18	0.23

[a]These normalizing factors include no correction for excess absorption in the graphite chamber.

each plate was measured. It was found that the mean mass density of the individual plates of both acrylic and polystyrene varied by up to 0.4%, while the individual graphite plates showed mean density variations up to 3%.

The individual current measurements were adjusted to a reference temperature, pressure, and date; adjusted to a source-detector distance of 1 m by applying an inverse square correction according to Eq. (3); corrected for air absorption; and corrected for excess attenuation by the graphite chamber in the water and acrylic phantoms.[12] For each of the 12 tests, the depth–current curves for different materials showed close agreement in shape but some displacement in magnitude. In order to bring the plastic and graphite curves into coincidence with the water curves, arbitrary normalizing factors, independent of collimator size, were applied. These normalizing factors are shown in Table I.

In order to quantify the agreement between the curves, an arbitrary function (a third order polynomial modified with an exponential) was fitted to the normalized points. One such function was needed for each of the 12 tests, fitting from 20 to 23 measurements per test. The root mean square deviation of the measured points from the functions to which they were fitted is shown in Table I for each of the four materials. The individual measurements fitted the function within about $\pm$ 0.3%, except for graphite, for which some were larger, with one as large as 0.6%. In some cases the deviations showed a systematic trend with depth. At least for graphite, the pattern of the deviations indicated that they were caused by local fluctuations of mass density, rather than a difference in shape of the depth-current curves. It is believed that the agreement was sufficiently good so that, for purposes of radiation dosimetry, the corresponding depth–current curves can be considered to have the same shape.

IV. DISCUSSION

The scaling theorem is so well founded theoretically that it can be assumed to be valid under ideal conditions. The present work was undertaken to determine whether under practical conditions the scaling theorem gives results sufficiently accurate to meet the needs of the National Bureau of Standards in scaling absorbed dose from graphite to water. Since scaling from plastic to water is of general interest in medical physics practice, the measurements were extended to polystyrene and acrylic.

The theorem in question scales photon fluence, while the ionization chamber measures current; it is appropriate to inquire whether current measurements allow a valid conclusion concerning photon fluence in the phantom. Assuming that the ionization chamber wall is thick enough to provide full electron buildup, the chamber current is proportional to the photon fluence in the phantom, and the constant of proportionality is a function of the spectral quality and direction of the photon fluence. The question then takes the form of inquiring whether this constant of proportionality is the same at corresponding points in different materials. As already noted, corresponding depth–current curves were found to have essentially the same shape in the four materials. While it is in principle possible that the magnitude, spectral quality, and direction of the photon fluence could vary but in such a way as to keep the relative ionization chamber current the same at corresponding points in different materials, it is considered highly unlikely. Instead it appears reasonable to conclude that the agreement between normalized depth–current curves can be interpreted as showing that the constant of proportionality between chamber current and photon fluence is closely the same at corresponding points in different materials.

As noted above, the scaling theorem can be expected to fail to some extent in a practical situation, due to scatter and absorption of radiation. Figure 2 is a plot of the normalization factors as a function of source distance, after the water factors have been arbitrarily renormalized to 0.998, 1.007, and 1.009 for sets A, B, and C, respectively. All of the adjusted normalization factors fall within a few tenths percent of an arbitrarily-drawn smooth curve. Presumably the relationship in Fig. 2 indicates excess radiation due to scatter in the source, source head, and collimator; as expected, this is greatest near the source and tends to disappear with increasing distance. Because of this scattered radiation, the scaling of fluence by the inverse square rule of Eq. (3) can fail by several percent between 0.6 and 1.6 m, depending on the materials used. Because the renormalization factors are arbitrary, the relationship in Fig. 2 is not determined unambigu-

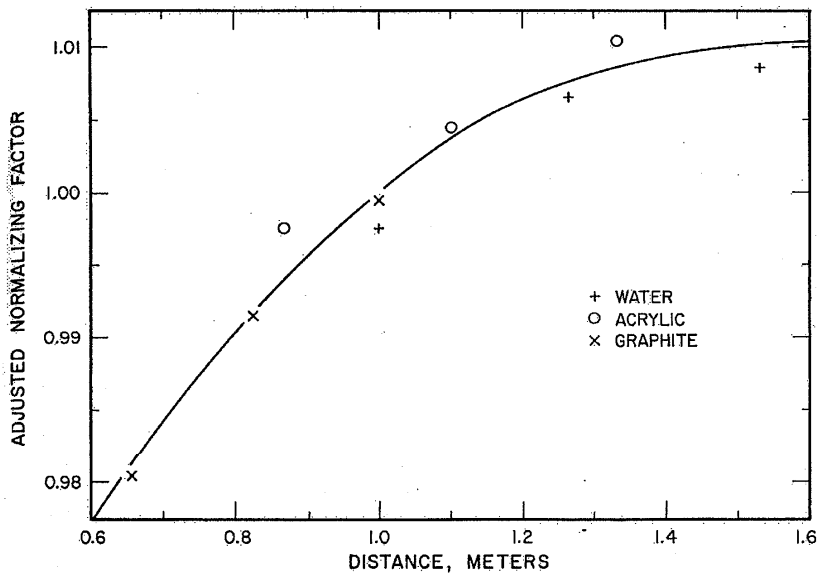

FIG. 2. Adjusted normalizing factors as a function of distance from the source. The water factors have been arbitrarily renormalized to the values shown, so that all the normalizing factors will fall near a smooth curve through the value unity at 1 m.

ously, and in any event it characterizes only the source used for these measurements.[10] Another cobalt-60 source would not necessarily show the same relationship.

Measurements performed in air in a cobalt-60 beam with a graphite ionization chamber and a set of nesting graphite sleeves, showed that about 2% of the ionization current comes from electrons arising outside the chamber wall. In the water phantom the chamber is inside an acrylic plastic tube, so that these extracameral electrons arise in acrylic in both the water and the acrylic phantoms. It is estimated that the extracameral current from electrons arising in acrylic plastic and polystyrene is at most 8% larger than the extracameral current from graphite would be, leading to an estimate of about 0.15% as the upper limit to the error caused by the lack of full electron buildup. Consideration of this effect would not affect the predicted normalization factors for the plastics, but would bring those for graphite slightly closer to unity.

It is concluded that, aside from normalizing factors, the measurements are consistent with the scaling theorem for the conditions studied, to an accuracy of a few tenths percent. Otherwise stated, it is believed that the measurements support the conclusion that photon spectral shapes and angular distributions are the same at corresponding points in water, polystyrene, acrylic plastic, and graphite, in a cobalt-60 gamma-ray beam. In scaling absorbed dose from graphite to water at NBS, the scaling theorem is used to identify corresponding points that will have the same spectral fluence. Because of uncertainty in the normalization, ionization measurements are made at the two corresponding points to obtain the desired ratio of the absorbed dose in water to that in graphite.[13] For scaling between plastic and water when using a cobalt-60 gamma-ray beam, it appears from Fig. 2 that the scaling theorem can be used directly, without resorting to ionization measurements and without incurring an uncertainty greater than a few tenths percent for polystyrene, and about one-half percent for acrylic plastic.

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[7]L. V. Spencer, *Gamma Ray Shielding Theory*, Vol. I of *Radiation Shielding*, edited by W. R. Kimel (Government Printing Office, Washington, D.C. 1966), p. I-67; and L. V. Spencer, A. B. Chilton, and C. M. Eisenhauer, *Structure Shielding Against Fallout Gamma Rays from Nuclear Detonations*, Nat. Bur. Stand. Spec. Publ. 570 (Washington, D.C. 1980), p. 261.
[8]An alternative formulation, used in Refs. 2 and 7, scales the source strength with the square of the ratio of the electron densities. Then the photon fluences at corresponding points are the same in energy, direction, and magnitude.
[9]Equation (3), which should not be confused with the usual inverse square law, does not predict that fluence in an absorbing medium has an inverse square dependence on distance.
[10]The cobalt-60 source was 20 mm in diameter and 20 mm thick, and was housed in an AECL Theratron F therapy head, which has a fixed source and a movable shutter. The beam was directed vertically downward.
[11]J. S. Pruitt and R. Loevinger, in *Measurements for the Safe Use of Radiation*, Nat. Bur. Stand. Spec. Publ. 456 (Washington, D.C. 1976), p. 37.
[12]Measurements made in the polystyrene phantom were not corrected for excess attenuation in the graphite chamber, since the necessary experiment on which to base the correction has not been performed.
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