Precision Calibration of Phase Meters

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Abstract—Using the calibration of a phase meter with a nominally linear response as an example, a statistical approach is discussed for predicting worst-case offsets of the meter response characteristic from the value of the reference standard. A linear calibration curve is used to model the meter response, and statistical tests are described which test the appropriateness of the model and whether the calculated calibration curve differs significantly from the ideal. Various levels of corrections to be applied can then be determined on the basis of these tests, and limits to offsets are calculated for each of the levels. By extending this approach, it is possible to predict limits of uncertainty when using the calibrated meter to make measurements.

I. INTRODUCTION

THIS paper discusses a statistical treatment of calibration data which leads to the prediction of measurement uncertainties after appropriate corrections are applied to the readings of the calibrated instrument. The method is illustrated using a phase meter as an example.

In any measurement, the "true" value of the measurand is hidden by random effects and systematic offsets inherent in the measuring instrument and the measurement process. The purpose of a calibration is to try to eliminate the systematic offset by determining suitable corrections which, when applied to the instrument reading, bring the measurement result into closer agreement with the reference standard. Since a degree of uncertainty is inherently associated with the process of calibration itself, the corrections for the systematic offset cannot be established precisely. However, it is possible to estimate limits to the uncertainty of the measurement result after the corrections have been applied. The statistical approaches that lead to these estimates are discussed in this paper.

II. CALIBRATION CURVE

The calibration of a measuring instrument can be represented mathematically by a "calibration curve" (Fig. 1) which relates the readings of the instrument under test to the corresponding values of the calibration standard. Since random fluctuations tend to mask the limiting mean of the instrument response at any particular test point, the corrections calculated based on predicted values derived from the calibration curve will, in general, give more reliable results than those obtained from the test data directly. A necessary condition is that the calibration curve
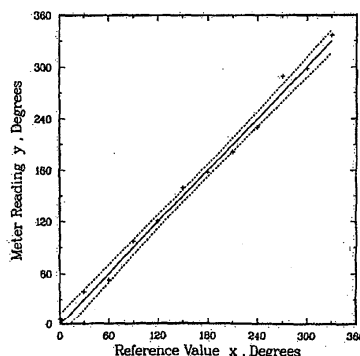
Fig. 1. Calibration curve. Dotted lines indicate 95 percent confidence bands.

models the instrument response correctly. Therefore, it is important to test whether the *a priori* assumption that the model fits the instrument response is justified.

Once the correctness of the model is established, the computed calibration curve can be compared with an ideal curve that represents an instrument which agrees perfectly with the standard. To do this, the authors examine whether there are statistically significant differences between the parameters of the computed calibration curve and corresponding parameters of an ideal calibration curve [1]. The outcome of such tests helps to decide what level of corrections, if any, will be necessary.

A. Illustrative Example

To illustrate the application of the above concepts, a simple instrument having a linear response is used as an example. However, the validity of the method is by no means restricted to linear systems and can be extended to more complex relationships. The formulas are derived for the statistical analysis of a nominally linear relationship between the phase angles indicated by a phase meter and the phase angle supplied by a calibrating standard signal source—a phase angle standard.

In this linear case, the calibration curve is a straight line which can be characterized by a slope and an intercept. The corrections to be applied to the phase meter readings can be derived from the linear equation. The extent to which corrections need to be applied must be regarded as a function not only of the calibration data but also of the accuracy specifications of the instrument. In general, there is no point in applying corrections if the uncorrected meter readings are already within the specified accuracy lim-

even though applying corrections will always reduce predicted offset of the measurement result that is computed from the statistical parameters. If the intercept of a linear calibration curve differs appreciably from zero, but the slope is not significantly different from its ideal value, a simple additive constant will bring the measurement result to within the specification limits. There is then no need to calculate individual corrections for every data point, although doing so may result in smaller numerical values for the predicted offset. In the following sections formulas are developed to evaluate the limits of the predicted offset for three levels of applied corrections. A comparison of the numerical values of these limits with the instrument specifications will guide the decision on selecting the appropriate level of corrections.

III. EXPERIMENTAL PROCEDURE USED FOR CALIBRATING A PHASE METER

A. Choice of Calibration Procedure

The example of a phase meter calibration is particularly suitable because the straightforward experimental procedure provides a good illustration of a generalized calibration method that could apply equally well to other types of instruments. The output reading of the meter is in the same units and of the same magnitude as the phase angle provided by the calibration source, and no intermediate steps or conversion factors are involved.

B. Circuit Configuration

A phase angle standard [2]-[4] which generates two sinusoidal signals adjustable in phase and independently adjustable in amplitude is used as a calibration source. The standard is designed so that the selected phase angle is known precisely and, therefore, can be used as the reference to which the readings on the phase meter are compared. For convenience, the phase angle standard can be operated via the IEEE-488 bus, allowing the test points to be selected under software control. Signals from the output of the calibration standard are applied directly to the input terminals of the instrument under test.

C. Test Point Selection

For the purpose of the calibration, a "range" is defined by the frequency and the amplitudes of the two test signals. In each range, measurements are made at several phase angles chosen to cover the desired span, usually from 0° to 360°. The exact number of test points is not important, as long as it is large enough to provide the appropriate accuracy for the calculation of the calibration curve. Experience has shown that for a phase meter with a 0.01° resolution, twelve points spread over the 360° span are a satisfactory compromise between the effort involved in making measurements and the accuracy obtained. The results of the measurements in each range are treated as a statistically independent population, and separate accuracy parameter values are computed for each range.

D. Measurement Procedure

For the statistical treatment, it is important to make replicate measurements, usually three or four, at each phase angle tested. This replication provides a measure of the variability of the readings due to the phase meter. Any variation in the output of the phase standard is generally at least an order of magnitude smaller, and is disregarded for the present discussion. The sequence of measurements at the selected phase-angle test points is randomized to minimize time dependent trends and thereby reduce a possible bias in the measurements.

The computer program determines the randomized sequence of phase angles to be tested, and the output of the phase standard is set accordingly. Readings from the phase meter are then recorded and stored in the computer.

IV. MODEL OF THE RESPONSE CHARACTERISTIC

A. Estimated Calibration Curve

Using a least squares fit to the data collected, a calibration curve is derived for the response characteristic of the meter under test for each range. For our example, assuming a linear response (phase reading versus phase standard), the model of the calibration data is a straight line of the form

$$y = a + bx + e$$

where

a and b are the intercept and slope of the straight line,
x phase angle given by the standard,
y reading on the phase meter
e term for the random effect.

If the subscript i ($i = 1, \cdots, k$) denotes the index of the test point, and the subscript j ($j = 1, \cdots, n$) the number of the replicate reading, then the estimated values (denoted by a caret) for the coefficient a and b of the calibration curve $\hat{y} = \hat{a} + \hat{b}x$ can be expressed as [1]

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$\hat{b} = \frac{\sum\limits_{i=1}^{k} \sum\limits_{j=1}^{n} (y_{ij} - \bar{y})(x_i - \bar{x})}{n \sum\limits_{i=1}^{k} (x_i - \bar{x})^2}$$

where the average $\bar{y}$ and the average $\bar{x}$ are

$$\bar{y} = \frac{\sum\limits_{i=1}^{k} \sum\limits_{j=1}^{n} y_{ij}}{nk} \quad \text{and} \quad \bar{x} = \frac{\sum\limits_{i=1}^{k} x_i}{k}.$$

B. Adequacy of the Model

To test whether the calibration data fit the linear model, the fitted value for each phase angle is compared to the average of the repeat measurements at corresponding phase angles by an "F-Test." This test provides a criterion to decide if the calibration data fit the linear model [5]. The calibration data are not consistent with the linear

model if

$$F' = \frac{S_L^2}{S_e^2} > F_{0.01}\{k - 2, k(n - 1)\}$$

where

$$S_L^2 = \frac{n}{k - 2} \sum_{i=1}^{k} (\hat{y}_i - \bar{y}_i)^2$$

$$S_e^2 = \frac{1}{k(n - 1)} \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij} - \bar{y}_i)^2$$

(see footnote 1)

and

$$\hat{y}_i = \hat{a} + \hat{b}x_i, \quad i = 1, \cdots, k$$

$$\bar{y}_i = \frac{\sum_{j=1}^{n} y_{ij}}{n}, \quad i = 1, \cdots, k$$

and $F_{0.01}\{k - 2, k(n - 1)\}$ is the upper one-percent point of the F distribution with $k - 2$ degrees of freedom in the numerator and $k(n - 1)$ degrees of freedom in the denominator. Use of a small significance level ($\alpha = 0.01$) in the test implies that we are only interested in detecting very substantial departures from linearity in the phase meter characteristic.[2]

Similarly, if the linear model is appropriate, levels of significance can be calculated for the coefficients of the calibration curve. These are based on the statistics t_1 and t_2 which test whether $a = 0$ and $b = 1$, respectively. The test statistics are

$$t_1 = \frac{\hat{a}}{S(\hat{a})} \qquad t_2 = \frac{\hat{b} - 1}{S(\hat{b})}$$

where $S(\hat{a})$ and $S(\hat{b})$ are the estimated standard deviations of the coefficients.

Using the tables of the Student's t distribution for $(nk - 2)$ degrees of freedom, the attained levels of statistical significance associated with t_1 and t_2 can be computed. A significance level near zero (<0.05) for t_1 indicates that the intercept is probably different from its "ideal" value of zero, and a significance value near zero for t_2 indicates that the slope is probably different from its "ideal" value of one.

V. CALCULATION OF PHASE METER UNCERTAINTY

When making a phase measurement, the reading obtained from the phase meter differs from the corresponding value of the standard by a systematic offset plus a random effect. As mentioned, the systematic offset can be

¹This variance accounts for the random effect, e, in the equation for the straight-line response characteristic.
²A special condition arises when the variability about the average is of the same order as the resolution of the meter, and consequently the readings at each test point are truncated to the same numerical value, or a value differing by only one significant digit. In this case the distribution of the deviations will be far from normal, and values of the F-test using tables based on a normal distribution cannot be applied.

reduced by an appropriate correction, while the random effect can only be reduced by averaging several readings. For the correction of the systematic component, we consider three cases:

Level 1. No correction applied:

$$\hat{x} = y$$

Level 2. A constant correction applied:

$$\hat{x} = y + C; \quad C = \bar{x} - \bar{y}$$

Level 3. Full correction applied using the calibration curve:

$$\hat{x} = (y - \hat{a})/\hat{b}.$$

For each case, we can estimate the limits to the uncertainty in the phase meter reading. We denote the systematic offset of an uncorrected reading Δ_x at a phase angle x by

$$\Delta_x = E(y \mid x) - x = a + (b - 1)x$$

where $E(y \mid x)$ is the expected phase meter reading, and the other symbols are defined as before.

It is evident that the offset is a function of the phase angle as well as the parameters of the calibration curve. In most cases, however, we would like to know the limits of the offset over the entire span of phase angles. For the straight line calibration curve, the equation for the upper and lower limits for the systematic offset at the point x_t, which can be derived [6] from the confidence bands of the calibration curve, shown as dotted lines in Fig. 1, are given by

$$u(\Delta_{x_t}) = A + (B - 1)x_t + R(x_t) \qquad \text{(1a)}$$

and

$$l(\Delta_{x_t}) = A + (B - 1)x_t - R(x_t) \qquad \text{(1b)}$$

where

$$R(x_t) = s \sqrt{2F_{0.05}(2, nk - 2)}$$

$$\left[\frac{1}{nk} + \frac{(x_t - \bar{x})^2}{n \sum_{i=1}^{k} (x_i - \bar{x})^2} \right]^{1/2}$$

In the equations x_t denotes the phase angle given by the standard, s is the standard deviation of fit of the straight line, the coefficients A and B are assigned appropriate values for level 1 and level 2 as shown below, and F is the value of the F distribution for the upper 5-percent point with two degrees of freedom in the numerator and $nk - 2$ degrees of freedom in the denominator. Fig. 1 is a plot of the characteristic curve for a phase meter and shows the upper and lower limits of likely systematic offsets. Note that the largest values occur at $0°$ and at $360°$.

A. Estimated Limits for the Systematic Offset

For the three levels of applied corrections, the upper and lower limits of the systematic offset can be calculated

the magnitude of the largest offset for each level $\hat{\Delta}_i$ be found. These values can then be compared to performance specification limits for the instruments in order to determine what level of corrections need to be applied. Should be noted, however, that the $\hat{\Delta}_i$'s account only for the uncertainty due to the calibration process. If it is desired to include the additional uncertainty that arises from the user's measurement with the meter, then the standard deviation for the user's measurement process must be taken into account as shown below.

The limits of systematic offset can be estimated as follows:

Level 1, no corrections applied:
Using (1a) and (1b), setting $A = \hat{a}$, $B = \hat{b}$

$$\hat{\Delta}_1 = \text{maximum of} \left\{ \left| l(\Delta_0) \right|, \left| u(\Delta_0) \right|, \right.$$
$$\left. \left| l(\Delta_{360}) \right|, \left| u(\Delta_{360}) \right| \right\}.$$

Level 2, constant correction C applied:
Using (1a) and (1b), setting $A = \hat{a} + C$, $B = \hat{b}$

$$\hat{\Delta}_2 = \text{maximum of} \left\{ \left| l(\Delta_0) \right|, \left| u(\Delta_0) \right|, \right.$$
$$\left. \left| l(\Delta_{360}) \right|, \left| u(\Delta_{360}) \right| \right\}.$$

Level 3, full correction applied:
Using $u(\Delta_{x_i}) = +R(x_i)/\hat{b}$ and $l(\Delta_{x_i}) = -R(x_i)/\hat{b}$

$$\hat{\Delta}_3 = \text{maximum of} \left\{ \left| l(\Delta_0) \right|, \left| u(\Delta_0) \right|, \right.$$
$$\left. \left| l(\Delta_{360}) \right|, \left| u(\Delta_{360}) \right| \right\}.$$

Estimated Limits for Phase Meter Reading Uncertainty

To obtain an overall estimate of the uncertainty of a phase meter reading, the variability of replicated readings must be included as well as the systematic offset (relative to the standard) given above. The estimate of the standard deviation for the user's measurement process, s_p, must be calculated from the data obtained under the test conditions in the user's laboratory. This standard deviation may well be different than that calculated from the calibration data. The value for the standard deviation s_p may now be added to the $\hat{\Delta}$ limits of the systematic offsets for the three levels of corrections applied to provide a bound $\hat{E}$ to the uncertainty of the meter reading relative to the value supplied by the standard.

Level 1, no corrections applied:

$$\hat{E}_1 = \hat{\Delta}_1 + s_p \cdot t_{\alpha/2}(\nu_p).$$

Level 2, a constant correction A applied:

$$\hat{E}_2 = \hat{\Delta}_2 + s_p \cdot t_{\alpha/2}(\nu_p).$$

Level 3, full correction applied:

$$\hat{E}_3 = \hat{\Delta}_3 + s_p \cdot t_{\alpha/2}(\nu_p).$$

where,

s_p standard deviation of repeat measurements
ν_p degrees of freedom associated with s_p

$t_{\alpha/2(\nu_p)}$ $1 - \alpha/2$ percentile of the Student's t distribution with ν_p degrees of freedom.

The standard deviation s_p should have at least 15 degrees of freedom. Additional repeat measurement can be combined for a pooled value of the standard deviation by computing the square root of the weighted average (weight $= \nu_i$) of the variances of each set of repeat measurements. A sample calculation is shown in the Appendix.

VI. CONCLUSION

A statistical procedure has been described for the calibration of a phase meter with a nominally linear response. The systematic offset of the meter reading relative to the values provided by the calibration standard can be predicted from a calibration curve. Three levels of correction are considered which will bring the meter readings to within the specified accuracy. The level is selected depending on how closely the actual calibration agrees with an ideal calibration curve. The overall uncertainty of the phase meter reading can be estimated by applying the appropriate level of corrections as well as a term for the random effects of the measurement process.

APPENDIX

A. Sample Calculation

The predicted values of the phase meter reading are obtained by fitting the calibration data for the set of current, voltage, and frequency conditions to a linear equation which models the average response of the phase meter. In the equations shown below $\hat{y}$ is the predicted value of the phase meter response for a phase angle value x given by the standard:

Test Conditions:	Input A: 100 V	Input B: 100 V	Frequency: 60 Hz
Predicted value	$\hat{y} = -0.019402 + 0.999749\,x$		
Standard errors	(0.006952)	(0.000036)	
Significance levels[3]	(0.009)	(0.000)	
Residual Standard Deviation	$s = 0.02218$		
Lack of fit	$F' = 0.466$		
Significance level[4]	(0.895)		$2s = 0.04436$

Level of Correction	Correction Equation	Limit to Offset[5]
No correction	$\hat{x} = y$	0.091
Constant correction	$\hat{x} = y + 0.021944$	0.070
Complete calibration curve	$\hat{x} = 1.000251 \cdot y + 0.019407$	0.020

Assuming arbitrarily for this example that with 20 degrees of freedom the user's standard deviation is 20 percent

[3]Significance levels are derived using the statistical t-tests to decide if the intercept and slope of the linear model are different from zero and one, respectively. A level near zero (less than or equal to 0.05) indicates that the associated parameter is probably different from the ideal value.
[4]The significance level of F' is associated with an objective statistical test for the adequacy of a linear model relating the phase meter under test and the NBS assigned values. Levels near zero indicate that the assumption of a straight line relationship may be incorrect.
[5]Phase meter offset relative to the reference standard.

larger than that calculated from the calibration data, then $s_p = 0.027$, and for a the confidence factor $\alpha = 0.05$, the estimated uncertainty of the phase meter readings becomes:

Level of Correction	Correct Equation	Limit to Uncertainty[6]
No correction	$\hat{x} = y$	0.147
Constant correction	$\hat{x} = y + 0.021944$	0.126
Complete calibration curve	$\hat{x} = 1.000251 \cdot y + 0.019407$	0.076

[6]Uncertainty of phase meter reading relative to the reference standard for a given (user's) standard deviation.

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