NBSIR 75-750

The Calibration of Indexing Tables by Subdivision

Charles P. Reeve

Institute for Basic Standards
National Bureau of Standards
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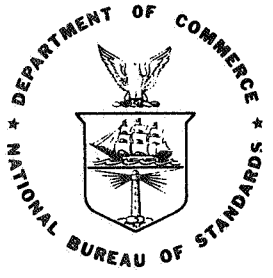
Final



U S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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U.S. DEPARTMENT OF COMMERCE, Elliot L. Richardson, *Secretary*
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NATIONAL BUREAU OF STANDARDS, Ernest Ambler, *Acting Director*

Contents

1. Introduction

The indexing table plays a vital role in the calibration of angle standards[1,2,3][1]. An object which is wrung or clamped to an indexing table can be rotated through certain angles very precisely. The smallest angular increment varies with different indexing tables but is most commonly one degree. The deviation from nominal of any angular interval in a high quality indexing table is usually no more than 0.25 second, and the short term repeatability of any setting is usually less than 0.05 second.

In most angular calibrations the indexing table plays one of three roles:

(1) Is used simply to rotate an object through some nominal angle whose precise value need not be known.

(2) Is calibrated simultaneously with an angular standard in a routine calibration process.

(3) Is used in the calibration of an angular standard where some of its angles need to be known precisely beforehand.

The second role is seen in the calibration of polygons. For example, a 30^{o} polygon may be calibrated by comparing each of its twelve angles to the twelve 30^{o} angles of the indexing table and then computing a least squares solution for the 24 unknowns. The third role occurs when it is impossible or impractical to employ a self-calibrating algorithm such as in the calibration of a small angle block (less than 15^{o}) by direct comparison to a known interval of an indexing table. In that case it is necessary to do a preliminary calibration of the indexing table angle. One way of accomplishing this is by applying a measurement algorithm usually called the "method of subdivision". The mathematics of this method are presented in great detail.

2. The Method of Subdivision

Angles on the same indexing table cannot easily be compared with each other, so it is more convenient to calibrate two indexing tables (denoted by A and B) simultaneously. That way each angle on one table can be compared with several on the other to give a redundant set of observations. This idea is incorporated in the method of subdivision which consists of two types of measurement designs, "complete closure" and "partial closure". The complete closure design is used to subdivide

[1]Figures in brackets indicate literature references at the end of the paper.

the entire 360 degrees of each table into n_0 equal segments $A_1, \ldots, A_{n_0}$ and $B_1, \ldots, B_{n_0}$ respectively where

$$\sum_{i=1}^{n_0} A_i = \sum_{i=1}^{n_0} B_i = 360^\circ .$$

Then by partial closure one segment on each table, say A_1 and B_1, is subdivided into n_1 equal segments $\dot{A}_1, \ldots, \dot{A}_{n_1}$ and $\dot{B}_1, \ldots, \dot{B}_{n_1}$ respectively where

$$\sum_{i=1}^{n_1} \dot{A}_i = A_1 \text{ and } \sum_{i=1}^{n_1} \dot{B}_i = B_1.$$

Similarly $\dot{A}_1$ and $\dot{B}_1$ can be divided into n_2 equal segments $\ddot{A}_1, \ldots, \ddot{A}_{n_2}$ and $\ddot{B}_1, , \ldots, \ddot{B}_{n_2}$ respectively where

$$\sum_{i=1}^{n_2} \ddot{A}_i = \dot{A}_1 \text{ and } \sum_{i=1}^{n_2} \ddot{B}_i = \dot{B}_1.$$

This process can be continued until the desired level is reached.

As an example, if it were desired to know the value of the $0^\circ - 1^\circ$ interval on each table, they could be calibrated using the complete closure design with $n_0 = 12$ and two partial closure designs with $n_1 = 6$ and $n_2 = 5$. Thus each 30° interval would be calibrated, then $0^\circ - 30^\circ$ on each table would be subdivided into 5° intervals, and the $0^\circ - 5^\circ$ interval on each table would be subdivided into 1° intervals. This subdivision is illustrated in figure 1.

It should be noted that only those increments which are common to both tables can be calibrated. For instance, if table A has a smallest increment of one degree and table B has a smallest increment of ten minutes, then the smallest increment which can be calibrated on each table is one degree.

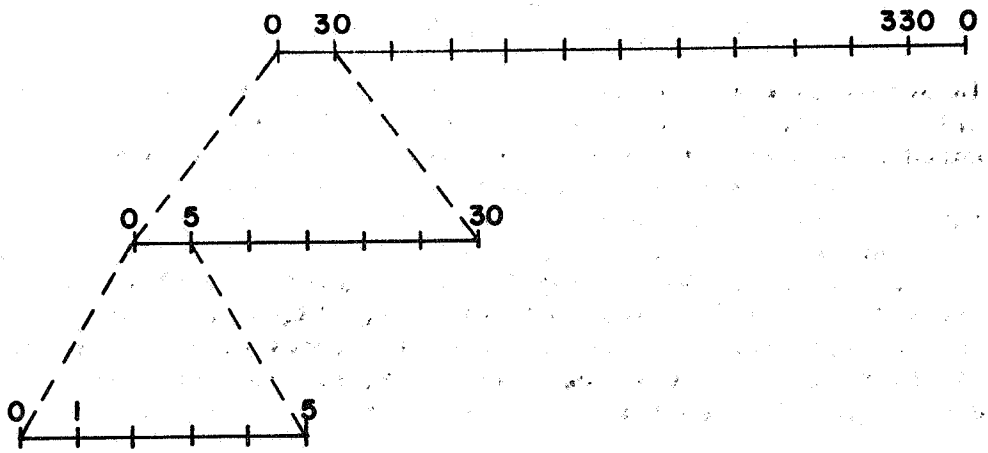


Figure 1.

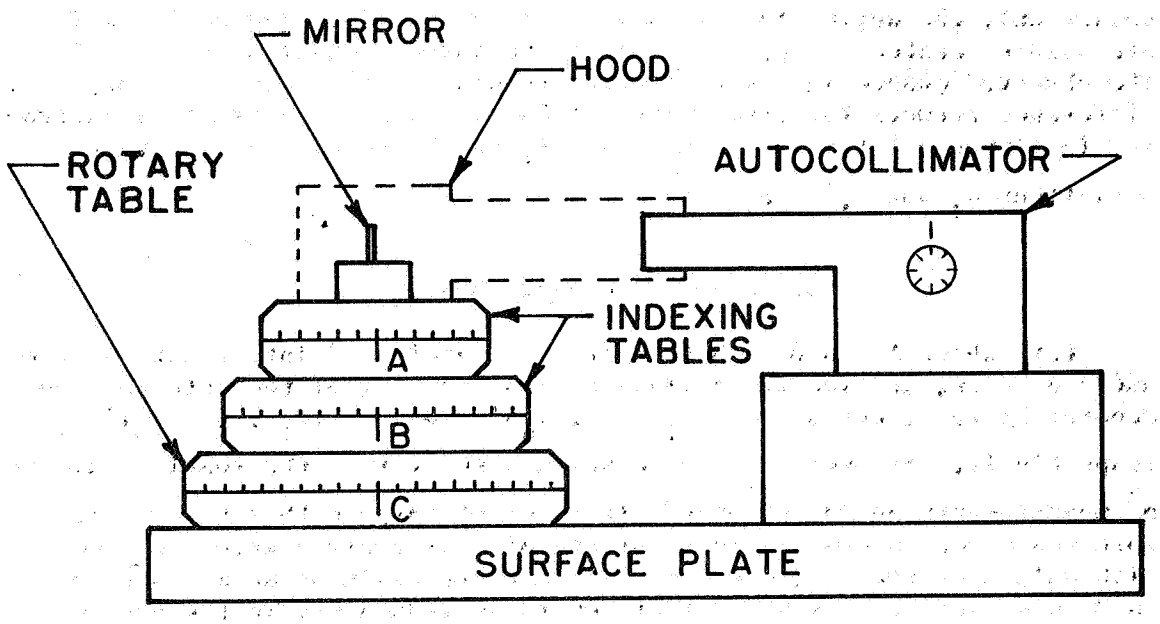


Figure 2.

3. Preparation for Measurement

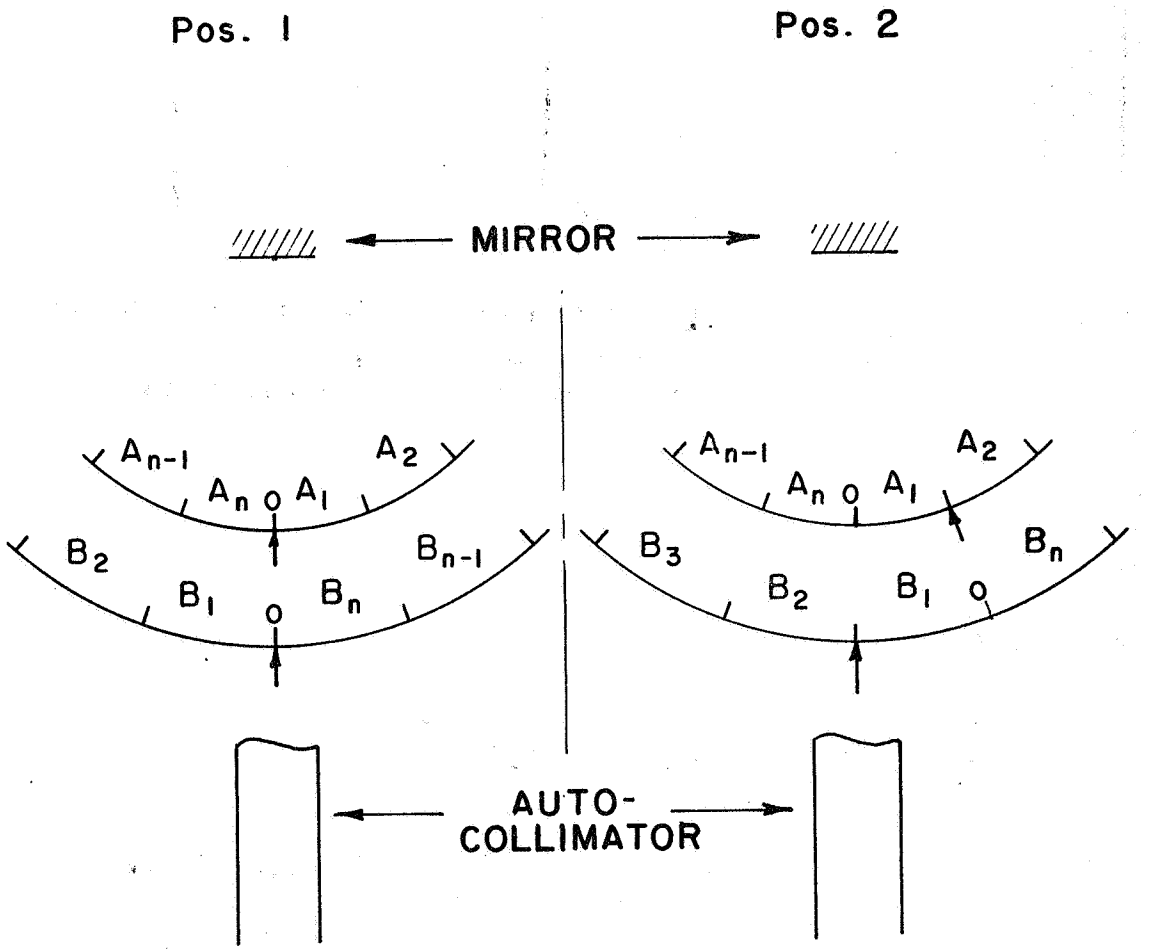
In preparation for measurement table A should be mounted on table B as nearly concentric as possible, It is then helpful if this assembly is mounted concentrically on table C which is either an indexing table or a rotary table. This table is not to be calibrated but serves only to rotate the other two tables between sets of measurements as required by the measurement design. A mirror is then mounted on table A approximately at the center. An autocollimator is mounted so that the face of the mirror is centered in its field of view. Adjustments are made so that the autocollimator reading is near the center of its scale when each of the tables is set in its zero position. The autocollimator should be adjusted so that it reads horizontal angle only.

If table C is not available then either an adjustable mirror must be used or the autocollimator must be shifted between sets of measurements. The whole assembly should be clamped to a surface plate as shown in figure 2. The process of clamping the tables should be done with a minimum of distortion, A hood should be constructed over the autocollimator and mirror so that no outside light can interfere with the autocollimator reading.

The assembly is then ready for the measurement process which involves only the appropriate rotations of the tables. Tables A and B are always rotated in opposite directions through equal nominal angles. The observed change in autocollimator reading is then equal to the difference between the true values of the two angles. This is illustrated in figure 3 for the angles A_1 and B_1 which are nominally N with deviations α_1 and β_1 respectively.

4. Complete Closure Design

Let tables A and B each be initially subdivided into n intervals of 360/n degrees, and let the deviations from nominal of the intervals be denoted by the vectors $\alpha = (\alpha_1 \ldots, \alpha_n)'$ and $\beta = (\beta_1, \ldots \beta_n)'$ respectively. If every α were compared with every β the result would be n^2 measurements on 2n unknowns. In cases of large n this may require more measurements than would be practical. If every α were compared with only m of the β's, where $m \leq n$, then there would be mn measurements on 2n unknowns, so for $m > 2$ there would be redundancy in the system. A convenient measurement algorithm is to take n blocks of m differences of the form



READING

$y_1 = \Delta + \epsilon_1$

READING

$y_2 = \Delta + (N+\alpha_1) - (N+\beta_1) + \epsilon_2$

$= \Delta + \alpha_1 - \beta_1 + \epsilon_2$

$$z = y_2 - y_1 = \alpha_1 - \beta_1 - \epsilon_1 + \epsilon_2$$

Figure 3.

$$\begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \vdots \\ \alpha_m - \beta_m \end{bmatrix}, \quad \begin{bmatrix} \alpha_2 - \beta_3 \\ \alpha_3 - \beta_4 \\ \vdots \\ \alpha_{m+1} - \beta_{m+2} \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_n - \beta_{n-1} \\ \alpha_1 - \beta_n \\ \vdots \\ \alpha_{m-1} - \beta_{m-2} \end{bmatrix}$$

(note that $\beta_{2n-1} = \beta_{n-1}$) where with each succeeding block the α subscript is increased by one and the β subscript by two. The i^{th} block of m differences is generated by taking m+1 observations according to the following scheme:

$$y_{i1} = \Delta_i + \varepsilon_1$$

$$y_{i2} = \Delta_i + \alpha_i - \beta_{2i-1} + \varepsilon_2$$

$$y_{i3} = \Delta_i + \alpha_i + \alpha_{i+1} - \beta_{2i-1} - \beta_{2i} + \varepsilon_3$$

$$\vdots$$

$$y_{i,m+1} = \Delta_i + \alpha_i + \ldots + \alpha_{i+m-1} - \beta_{2i-1} - \ldots - \beta_{2i+m-2} + \varepsilon_{m+1}$$

where Δ_i is the initial reading of the autocollimator and the ε's are independent error values from a distribution whose mean is zero and whose variance is σ^2. The subscripts of β are reduced modulo n. The complete closure design will be denoted by C(n,m).

Before proceeding, a word should be said about sign convention. This model assumes that the two tables are numbered in opposite directions, and that increasing the angle of table A gives a positive deflection of the autocollimator. If the tables do not conform to this convention, they can be made to conform by reversing the assignment of the angles or by reversing the sign of the observations or both.

The new random variables for the i^{th} block, $z_i = (z_{i1}, \ldots z_{im})'$, are formed by

$$z_{i1} = y_{i2} - y_{i1} = \alpha_i - \beta_{2i-1} + \varepsilon_2 - \varepsilon_1$$

$$z_{i2} = y_{i3} - y_{i2} = \alpha_{i+1} - \beta_{2i} + \varepsilon_3 - \varepsilon_2$$

$$\vdots$$

$$z_{im} = y_{i,m+1} - y_{im} = \alpha_{i+m-1} - \beta_{2i+m-2} + \varepsilon_{m+1} - \varepsilon_m,$$

or in matrix notation $z_i = My_i$ where

$$M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

Since $\mathrm{Var}(y_i) = \sigma^2 I_{m+1}$, then $\mathrm{Var}(z_i) = \sigma^2 V_m$ where

$$V_m = MM' = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & & 0 & 0 \\ 0 & -1 & 2 & -1 & & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Let $z = (z_1\ z_2\ \ldots\ z_n)'$. Then the least squares estimation takes the form

$$E(z) = X \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where

$$X = \begin{bmatrix} D_m P_n^0 & -D_m P_n^0 \\ D_m P_n^1 & -D_m P_n^2 \\ \vdots & \vdots \\ D_m P_n^{n-1} & -D_m P_n^{n-2} \end{bmatrix}$$

and $D_m = [I_m \ \theta_{m,n-m}]$ where I_m is the m x m identity matrix and $\theta_{m,n-m}$ is the m x (n-m) zero matrix, and P_n^1 is the n x n permutation matrix which is given by

$$P_n^1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} .$$

The full variance-covariance matrix of the observations is given by the nm x nm block diagonal matrix

$$W = \begin{bmatrix} V_m & & & \\ & V_m & & \\ & & \ddots & \\ & & & V_m \end{bmatrix} .$$

Then

$$W^{-1} = \begin{bmatrix} V_m^{-1} & & & \\ & V_m^{-1} & & \\ & & \ddots & \\ & & & V_m^{-1} \end{bmatrix}$$ where

$$V_m^{-1} = \frac{1}{m+1}\begin{bmatrix} m & m-1 & m-2 & & 2 & 1 \\ m-1 & 2(m-1) & 2(m-2) & \cdots & 4 & 2 \\ m-2 & 2(m-2) & 3(m-2) & & 6 & 3 \\ & \vdots & & & \vdots & \\ 2 & 4 & 6 & & 2(m-1) & m-1 \\ 1 & 2 & 3 & \cdots & m-1 & m \end{bmatrix}.$$

The normal equations (incorporating the restraints

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 0)$$ take the form

$$\begin{bmatrix} X'W^{-1}X & \pm & 0 \\ & 0 & \pm \\ \pm' \quad 0 & 0 & 0 \\ 0 \quad \pm' & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X'W^{-1}z \\ 0 \\ 0 \end{bmatrix}$$

where λ_1 and λ_2 are Lagrangian multipliers entering in the minimization process and $\pm = (1\ 1\ \ldots\ 1)'$. (The normal equations are developed further in Appendix A in a form suitable for computer programming.)

The estimates are given by

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X'W^{-1}X & \pm & 0 \\ & 0 & \pm \\ \pm' \quad 0 & 0 & 0 \\ 0 \quad \pm' & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} X'W^{-1}z \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C & a & b \\ a' & c & d \\ b' & d & c \end{bmatrix} \begin{bmatrix} X'W^{-1}z \\ 0 \\ 0 \end{bmatrix}$$

where C is the variance-covariance matrix of the estimates.

The predicted values are given by

$$\hat{z} = X \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} ,$$

and the deviations by

$$d = z - \hat{z}.$$

The estimate of σ is given by

$$\hat{\sigma} = \sqrt{d'W^{-1}d/(mn-2n+2)}$$

and the standard deviation of the estimates by

$$\hat{\sigma}_{\alpha_i} = \hat{\sigma}\sqrt{C_{ii}} \quad \text{and} \quad \hat{\sigma}_{\beta_j} = \hat{\sigma}\sqrt{C_{j+n,\, j+n}}, \quad \text{and}$$

the covariance of α_i and β_j is given by

$$\hat{\sigma}_{\alpha_i\beta_j} = \hat{\sigma}\, C_{i,\, n+j} .$$

For the cumulative values let

$$\hat{\phi}_k = \sum_{i=1}^{k} \hat{\alpha}_i \quad \text{and} \quad \hat{\psi}_k = \sum_{i=1}^{k} \beta_i .$$

The standard deviations are given by

$$\hat{\sigma}_{\phi_k} = \hat{\sigma}\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{k} C_{ij}} \quad \text{and} \quad \hat{\sigma}_{\psi_k} = \hat{\sigma}\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{k} C_{n+i,\, n+j}}$$

There is no error entering from the restraints since they represent an exact relationship.

5. Partial Closure Design

Let A_i and B_i be two calibrated intervals of the same nominal angle on tables A and B respectively and let their observed deviations from nominal be given by m_A and m_B respectively. Let the corresponding variance-covariance matrix of the values be given by

$$S = \begin{bmatrix} \sigma^2_{m_A} & \sigma_{m_A m_B} \\ \sigma_{m_A m_B} & \sigma^2_{m_B} \end{bmatrix} .$$

The intervals may be subdivided into n segments denoted by $\dot{\alpha} = (\dot{\alpha}_1 \ldots \dot{\alpha}_n)'$ and $\dot{\beta} = (\dot{\beta}_1 \ldots \dot{\beta}_n)'$ by using the partial closure design under the restraint that

$$\sum_{i=1}^{n} \dot{\alpha}_i = m_A \quad \text{and} \quad \sum_{i=1}^{n} \dot{\beta}_i = m_B .$$

Each $\dot{\alpha}$ can be compared to each $\dot{\beta}$ so that there are n^2 measurements of the 2n unknowns. One measurement algorithm which is convenient forms the 2n-1 groups of differences

$$[\dot{\alpha}_1 - \dot{\beta}_n], \begin{bmatrix} \dot{\alpha}_1 - \dot{\beta}_{n-1} \\ \dot{\alpha}_2 - \dot{\beta}_n \end{bmatrix}, \ldots, \begin{bmatrix} \dot{\alpha}_1 - \dot{\beta}_1 \\ \dot{\alpha}_2 - \dot{\beta}_2 \\ \vdots \\ \dot{\alpha}_{n-1} - \dot{\beta}_{n-1} \\ \dot{\alpha}_n - \dot{\beta}_n \end{bmatrix}, \begin{bmatrix} \dot{\alpha}_2 - \dot{\beta}_1 \\ \dot{\alpha}_3 - \dot{\beta}_2 \\ \vdots \\ \dot{\alpha}_n - \dot{\beta}_{n-1} \end{bmatrix}, \ldots, [\dot{\alpha}_n - \dot{\beta}_1]$$

where the central block has n differences and the adjacent blocks decrease in size by 1 until the end blocks have only 1 difference. Each block of k differences requires k+1 measurements. For example the central block is generated by n+1 measurements according to the following scheme:

$$y_{n1} = \Delta_n + \varepsilon_1$$

$$y_{n2} = \Delta_n + \dot{\alpha}_1 - \dot{\beta}_1 + \varepsilon_2$$

$$\vdots$$

$$y_{n,n+1} = \Delta_n + \dot{\alpha}_1 + \ldots + \dot{\alpha}_n - \dot{\beta}_1 - \ldots - \dot{\beta}_n + \varepsilon_{n+1}$$

where Δ_n is the initial reading of the autocollimator and the ε's are independent error values from a distribution whose mean is zero and whose variance is σ^2. This partial closure design will be denoted by P(n). The new random variables $z_n = (z_{n1} \ldots z_{nn})'$ are formed by

$$z_{n1} = y_{n2} - y_{n1} = \dot{\alpha}_1 - \dot{\beta}_1 + \varepsilon_2 - \varepsilon_1$$

$$z_{n2} = y_{n3} - y_{n2} = \dot{\alpha}_2 - \dot{\beta}_2 + \varepsilon_3 - \varepsilon_2$$

$$\vdots$$

$$z_{nn} = y_{n,n+1} - y_{nn} = \dot{\alpha}_n - \dot{\beta}_n + \varepsilon_{n+1} - \varepsilon_n$$

or in matrix notation $z_n = My_n$ where

$$M = \begin{bmatrix} -1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & & -1 & 1 \end{bmatrix}$$

Since $\mathrm{Var}(y_n) = \sigma^2 I_{n+1}$, then $\mathrm{Var}(z_n) = \sigma^2 V_n$ where V_n is defined as in the previous section.

Let $z' = (z_1' \; z_2' \; \ldots \; z_{2n-1}')$ be the complete set of difference measurements. The size of the vector z_k is $k \times 1$ if $k \leq n$ and $(2n-k) \times 1$ if $k > n$. Corresponding to each z_k is a variance-covariance matrix V_k which is $k \times k$ if $k \leq n$ and $(2n-k) \times (2n-k)$ if $k > n$. For $k = 1,2$, and 3 the matrices take the form

$$V_1 = [2], \; V_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \; V_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} .$$

The least squares estimation takes the form

$$E(z) = X \begin{pmatrix} \cdot \\ \alpha \\ \cdot \\ \beta \end{pmatrix}$$

where

$$X = \begin{bmatrix} D_1 & & -D_1 P_n^{n-1} \\ D_2 & & -D_2 P_n^{n-2} \\ \vdots & & \vdots \\ D_n & & -D_n P_n^{0} \\ D_{n-1} P_n^{1} & & -D_{n-1} \\ \vdots & & \vdots \\ D_1 P_n^{n-1} & & -D_1 \end{bmatrix}$$

and $D_k = \begin{bmatrix} I_k & \theta_{k,n-k} \end{bmatrix}$ where I_k, $\theta_{k,n-k}$, and P_n^1 are defined as in the previous section. The variance-covariance matrix of observations is given by

$$W = \begin{bmatrix} V_1 & & & & & & \\ & V_2 & & & & & \\ & & \ddots & & & & \\ & & & V_n & & & \\ & & & & V_{n-1} & & \\ & & & & & \ddots & \\ & & & & & & V_1 \end{bmatrix}$$

where W is a block diagonal matrix whose blocks vary in size from 1 x 1 to n x n. Then

$$W^{-1} = \begin{bmatrix} V_1^{-1} & & & & & & \\ & V_2^{-1} & & & & & \\ & & \ddots & & & & \\ & & & V_n^{-1} & & & \\ & & & & V_{n-1}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & V_1^{-1} \end{bmatrix}$$

where V_k^{-1} is the same as in the previous section. For example,

$$V_1^{-1} = \frac{1}{2}[1],\ V_2^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ V_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \ldots .$$

The normal equations (incorporating the restraints $\sum_{i=1}^{n} \dot{\alpha}_i = m_A$ and $\sum_{i=1}^{n} \dot{\beta}_i = m_B$) take the form

$$\begin{bmatrix} X'W^{-1}X & \pm & 0 \\ \pm' & 0 & 0 \\ 0 & \pm' & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X'W^{-1}z \\ m_A \\ m_B \end{bmatrix}$$

where λ_1, λ_2, and $\pm$ are defined as before. (The normal equations are developed further in Appendix B in a form suitable for computer programming.)

The estimates are given by

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X'W^{-1}X & \pm & 0 \\ & 0 & \pm \\ \pm' & 0 & 0 \\ 0 & \pm' & 0 \end{bmatrix}^{-1} \begin{bmatrix} X'W^{-1}z \\ m_A \\ m_B \end{bmatrix} = \begin{bmatrix} C & a & b \\ a' & c & d \\ b' & d & c \end{bmatrix} \begin{bmatrix} X'W^{-1}z \\ m_A \\ m_B \end{bmatrix}$$

where C is the variance-covariance matrix of the estimates.

The predicted values are given by

$$\hat{z} = X \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix}$$

and the deviations by

$$d = z - \hat{z}.$$

The estimate of σ is given by

$$\hat{\sigma} = \sqrt{d'W^{-1}d/(n^2-2n+2)}.$$

Since the random error entering through the restraints is non-zero it must be taken into account when computing the total standard deviation of the estimates. The complete expression for the variances and covariances of the estimates is given by

$\text{Var} \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \hat{\sigma}^2 C + (a\ b)\ S \begin{pmatrix} a' \\ b' \end{pmatrix}$. Thus

$$\hat{\sigma}^2_{\dot{\alpha}_i} = \hat{\sigma}^2 C_{ii} + a_i^2 \hat{\sigma}^2_{m_A} + b_i^2 \hat{\sigma}^2_{m_B} + 2a_i b_i \hat{\sigma}_{m_A m_B} ,$$

$$\hat{\sigma}^2_{\dot{\beta}_j} = \hat{\sigma}^2 C_{j+n,\, j+n} + a^2_{j+n} \hat{\sigma}^2_{m_A} + b^2_{j+n} \hat{\sigma}^2_{m_B} + 2a_{j+n} b_{j+n} \hat{\sigma}_{m_A m_B} ,$$

and the covariance of $\dot{\alpha}_i$ and $\dot{\beta}_j$ is

$$\hat{\sigma}_{\dot{\alpha}_i \dot{\beta}_j} = \hat{\sigma}^2 C_{i,n+j} + a_i a_{n+j} \hat{\sigma}^2_{m_A} + (a_i b_{n+j} + a_{n+j} b_i) \hat{\sigma}_{m_A m_B}$$

$$+ b_i b_{n+j} \hat{\sigma}^2_{m_B} .$$

For the cumulative values, let

$$\hat{\phi}_k = \sum_{i=1}^{k} \hat{\alpha}_i \text{ and } \hat{\psi}_k = \sum_{i=1}^{k} \hat{\beta}_i .$$

The variances are given by

$$\hat{\sigma}^2_{\dot{\phi}_k} = \hat{\sigma}^2 \sum_{i=1}^{k} \sum_{j=1}^{k} C_{ij} + \left(\sum_{i=1}^{k} a_i \right)^2 \hat{\sigma}_{m_A} + \left(\sum_{i=1}^{k} b_i \right)^2 \hat{\sigma}^2_{m_B} +$$

$$2 \left(\sum_{i=1}^{k} a_i \right) \left(\sum_{i=1}^{k} b_i \right) \hat{\sigma}_{m_A m_B} , \text{ and } \hat{\sigma}^2_{\dot{\psi}_k} = \hat{\sigma}^2 \sum_{i=1}^{k} \sum_{j=1}^{k} C_{i+n,\, j+n} +$$

$$\left(\sum_{i=1}^{k} a_{i+n} \right)^2 \hat{\sigma}^2_{m_A} + \left(\sum_{i=1}^{k} b_{i+n} \right)^2 \hat{\sigma}^2_{m_B} + 2 \left(\sum_{i=1}^{k} a_{i+n} \right) \left(\sum_{i=1}^{k} b_{i+n} \right) \hat{\sigma}_{m_A m_B} .$$

6. Sources of Error

In the mathematical models just described the only errors accounted for were the random errors of measurement and the propagation of these errors into the following series. The error of a single measurement which was given as σ is the combination of random errors from three sources:

1) random setting error in indexing table A

2) random setting error in indexing table B

3) random error of the autocollimator reading

Since some of the physical parameters of the system are unknown and cannot be modeled, there are systematic errors in the measured values. These probably are small, but it has not been determined if they are negligible. Factors such as the autocollimator reading being biased by vertical angle, imperfections in the mirror, effects of varying temperature, and foreign particles in the setting mechanisms of the indexing tables can influence the accuracy of calibrated values. There is evidence that over a short period of time, such as a week, that these effects are negligible in comparison to the random errors.

There are several reasons why long term changes in the indexing table angles may not be negligible. If the table is moved from one location to another it may be clamped down in a different manner thus resulting in slight distortions. The grease which lubricates the tables may collect small amounts of dust over long periods of time and as this dust works its way into the teeth of the tables it causes a variability in the setting of different angles.

Over a period of years the values of two indexing tables at NBS have been observed to drift slowly, but whenever a table was dismantled, cleaned, and reassembled there was usually a significant change in the angular values. The conclusion to be drawn is that over short periods of time systematic errors can probably be ignored while over longer periods of time, such as several months, they become significant and it is better to recalibrate the table before using it.

7. Applications of Designs

As described in section 2, a combination of the two types of designs allows the calibration of intervals of various sizes by subdivision. However, each design can be useful by itself. When the complete closure design is used alone, it subdivides the two tables into n equal angles each. The standard deviation of all the angles on indexing table A are equal, and likewise for indexing table B.

Table 1 lists the standard deviation coefficients, k_A and k_B, of the angles for all values of n up to 36 and for all appropriate values of m. Note that the standard deviation coefficients for the angles on indexing tables A and B are not always equal for a given design. The standard deviations of the individual angles are given by $s_{\alpha_i} = k_A s$ and $s_{\beta_i} = k_B s$ where s is the estimated standard deviation of a single measurement.

The standard deviation coefficients for the cumulative angles are also given. They are the maximum values of the k's which correspond to all possible sums of consecutive angles.

Table 1 is given primarily as an aid in determining which value of m to use when subdividing two indexing tables into n equal parts each. Assuming that there is a desired upper limit for the standard deviation of each angle (s_{max}) and that an approximate value of the observed standard deviation (s) is known, then any value of m is acceptable whose corresponding k value is less than the ratio s_{max}/s. For example, given the parameters $s \approx .03$, $s_{max} = .02$ and $n = 24$, then $k \leq .667$; therefore, $m \geq 6$ for individual values and $m \geq 10$ for cumulative values. If $s_{max} = .05$ then $k \leq 1.667$, and $m \geq 3$ for individual values and $m \geq 7$ for cumulative values.

The least squares solution to each complete closure design is restrained by the fact that the sum of the angles on each table is exactly zero, therefore there is no error entering the measurement process from that source.

Table 2 lists the standard deviation coefficients, $k = k_A = k_B$, of the angles which are calibrated by the partial closure designs for values of n up to ten. In this case the standard deviation coefficients are the same for the coresponding angles on indexing tables A and B, that is, $k_A = k_B$ for each position. As in the first case $s_\alpha = s_\beta = ks$, but now there is an error term entering from the restraint since the values of the angles being subdivided are not known exactly. This error is spread out over the smaller angles and generally will be small compared with the random measurement error within the series.

The combination of a complete closure design and one or more partial closure designs gives a flexible system which should be adequate to economically measure any possible combination of indexing table angles which one could imagine. The only requirement for using these tools is a modest amount of imagination. An example is given in the next section.

TABLE 1.

STANDARD DEVIATION COEFFICIENTS OF ANGLES ON INDEXING TABLES A AND B CALIBRATED WITH COMPLETE CLOSURE DESIGNS.

		...INDIVIDUAL..		....CUMULATIVE....	
N	M	K(A)	K(B)	MAX K(A)	MAX K(B)
4	3	.707	.707	.707	.707
4	4	.654	.654	.654	.654
5	3	.775	.775	.775	.775
5	4	.633	.633	.633	.633
5	5	.592	.592	.592	.592
6	3	.845	.782	.867	.913
6	4	.677	.687	.677	.687
6	5	.577	.577	.577	.577
6	6	.544	.545	.544	.545
8	3	.956	.866	1.163	1.414
8	4	.741	.725	.830	.881
8	5	.610	.596	.625	.627
8	6	.550	.554	.550	.554
8	7	.500	.500	.500	.500
8	8	.477	.478	.477	.478
9	3	1.003	.923	1.296	1.689
9	4	.741	.693	.856	.956
9	5	.620	.605	.663	.680
9	6	.555	.547	.564	.565
9	7	.509	.509	.509	.509
9	8	.471	.471	.471	.471
9	9	.451	.451	.451	.451
10	3	1.040	.949	1.429	2.025
10	4	.788	.764	.990	1.258
10	5	.631	.606	.706	.753
10	6	.562	.560	.596	.609
10	7	.514	.509	.520	.520
10	8	.479	.481	.479	.481
10	9	.447	.447	.447	.447
10	10	.430	.430	.430	.430
12	3	1.121	1.027	1.789	2.779
12	4	.827	.802	1.201	1.626
12	5	.648	.619	.811	.946
12	6	.574	.567	.662	.710
12	7	.519	.510	.554	.565
12	8	.487	.483	.499	.503
12	9	.455	.452	.458	.458
12	10	.431	.432	.431	.432
12	11	.408	.408	.408	.408
12	12	.394	.395	.394	.395

TABLE 1. (CONT.)

STANDARD DEVIATION COEFFICIENTS OF ANGLES ON INDEXING TABLES A AND B CALIBRATED WITH COMPLETE CLOSURE DESIGNS.

N	M	...INDIVIDUAL... K(A)	K(B)	....CUMULATIVE.... MAX K(A)	MAX K(B)
15	3	1.231	1.145	2.363	3.913
15	4	.834	.777	1.389	2.071
15	5	.669	.642	.974	1.284
15	6	.581	.562	.753	.896
15	7	.528	.517	.627	.688
15	8	.487	.480	.542	.565
15	9	.457	.454	.487	.494
15	10	.433	.430	.448	.449
15	11	.413	.412	.419	.419
15	12	.396	.395	.398	.398
15	13	.380	.380	.380	.380
15	14	.365	.365	.365	.365
15	15	.355	.355	.355	.355
18	3	1.329	1.240	2.966	5.255
18	4	.937	.908	1.898	3.187
18	5	.690	.659	1.154	1.700
18	6	.598	.589	.893	1.217
18	7	.533	.520	.701	.850
18	8	.493	.489	.603	.691
18	9	.461	.455	.526	.562
18	10	.435	.434	.477	.495
18	11	.414	.412	.438	.443
18	12	.397	.397	.411	.414
18	13	.381	.380	.388	.389
18	14	.368	.368	.371	.372
18	15	.356	.355	.357	.357
18	16	.344	.345	.344	.345
18	17	.333	.333	.333	.333
18	18	.325	.326	.325	.326
20	3	1.391	1.304	3.450	6.205
20	4	.970	.941	2.185	3.726
20	5	.703	.671	1.295	2.002
20	6	.606	.596	.988	1.410
20	7	.537	.524	.758	.976
20	8	.495	.491	.644	.772
20	9	.462	.456	.556	.622
20	10	.437	.435	.500	.535
20	11	.415	.412	.454	.469
20	12	.397	.397	.422	.430
20	13	.382	.380	.397	.399
20	14	.368	.368	.377	.379
20	15	.356	.355	.361	.361
20	16	.345	.345	.347	.348
20	17	.335	.334	.336	.336

TABLE 1. (CONT.)

STANDARD DEVIATION COEFFICIENTS OF ANGLES ON INDEXING TABLES A AND B CALIBRATED WITH COMPLETE CLOSURE DESIGNS.

N	M	...INDIVIDUAL.. K(A)	K(B)	....CUMULATIVE.... MAX K(A)	MAX K(B)
20	18	.325	.326	.325	.326
20	19	.316	.316	.316	.316
20	20	.309	.310	.309	.310
24	3	1.507	1.424	4.444	8.246
24	4	1.033	1.004	2.785	4.976
24	5	.729	.696	1.602	2.658
24	6	.622	.611	1.195	1.866
24	7	.545	.531	.886	1.266
24	8	.500	.496	.736	.980
24	9	.465	.458	.617	.759
24	10	.439	.437	.548	.637
24	11	.417	.413	.492	.543
24	12	.399	.398	.452	.482
24	13	.383	.381	.418	.434
24	14	.369	.368	.393	.402
24	15	.356	.355	.373	.376
24	16	.345	.345	.357	.358
24	17	.335	.334	.342	.343
24	18	.326	.326	.330	.331
24	19	.318	.317	.320	.320
24	20	.310	.310	.311	.311
24	21	.303	.302	.303	.303
24	22	.296	.296	.296	.296
24	23	.289	.289	.289	.289
24	24	.283	.283	.283	.283
30	3	1.665	1.588	6.086	11.635
30	4	1.121	1.093	3.782	7.085
30	5	.766	.733	2.120	3.759
30	6	.643	.632	1.552	2.650
30	7	.556	.542	1.113	1.768
30	8	.507	.503	.901	1.359
30	9	.469	.462	.731	1.021
30	10	.442	.440	.633	.837
30	11	.419	.415	.553	.683
30	12	.400	.399	.501	.592
30	13	.384	.381	.458	.515
30	14	.370	.369	.426	.466
30	15	.357	.356	.399	.422
30	16	.346	.346	.377	.393
30	17	.336	.335	.358	.367
30	18	.326	.326	.343	.349
30	19	.318	.317	.330	.333
30	20	.310	.310	.319	.321

TABLE 1. (CONT.)

STANDARD DEVIATION COEFFICIENTS OF ANGLES ON INDEXING TABLES A AND B CALIBRATED WITH COMPLETE CLOSURE DESIGNS.

N	M	...INDIVIDUAL.. K(A)	K(B)	....CUMULATIVE.... MAX K(A)	MAX K(B)
30	21	.303	.302	.309	.310
30	22	.296	.296	.300	.301
30	23	.290	.290	.293	.293
30	24	.284	.284	.286	.286
30	25	.278	.278	.279	.279
30	26	.273	.273	.273	.274
30	27	.268	.268	.268	.268
30	28	.263	.263	.263	.263
30	29	.258	.258	.258	.258
30	30	.254	.254	.254	.254
36	3	1.810	1.737	7.956	15.379
36	4	1.203	1.175	4.919	9.348
36	5	.801	.768	2.715	4.981
36	6	.664	.653	1.965	3.499
36	7	.567	.552	1.377	2.335
36	8	.515	.509	1.096	1.780
36	9	.473	.466	.867	1.328
36	10	.445	.442	.737	1.073
36	11	.421	.417	.629	.862
36	12	.401	.400	.560	.731
36	13	.385	.382	.502	.621
36	14	.370	.370	.462	.548
36	15	.358	.356	.428	.487
36	16	.346	.346	.402	.445
36	17	.336	.335	.379	.409
36	18	.327	.326	.361	.381
36	19	.318	.317	.344	.358
36	20	.310	.310	.331	.340
36	21	.303	.302	.319	.324
36	22	.296	.296	.309	.312
36	23	.290	.290	.299	.301
36	24	.284	.284	.291	.292
36	25	.278	.278	.284	.284
36	26	.273	.273	.277	.278
36	27	.268	.268	.271	.271
36	28	.263	.263	.265	.266
36	29	.259	.259	.260	.260
36	30	.254	.254	.256	.256
36	31	.250	.250	.251	.251
36	32	.247	.247	.247	.247
36	33	.243	.243	.243	.243
36	34	.239	.239	.239	.239
36	35	.236	.235	.236	.236
36	36	.233	.233	.233	.233

TABLE 2.

STANDARD DEVIATION COEFFICIENTS OF ANGLES ON INDEXING TABLES A AND B CALIBRATED WITH PARTIAL CLOSURE DESIGNS WHERE K(A)=K(B).

POS	N=2	N=3	N=4	N=5	N=6
1	.791	.677	.601	.546	.504
2	.791	.764	.667	.601	.552
3	-	.677	.667	.599	.549
4	-	-	.601	.601	.549
5	-	-	-	.546	.552
6	-	-	-	-	.504

POS	N=7	N=8	N=9	N=10
1	.471	.443	.420	.400
2	.513	.482	.456	.434
3	.511	.480	.454	.432
4	.510	.479	.453	.431
5	.511	.479	.453	.431
6	.513	.480	.453	.431
7	.471	.482	.454	.431
8	-	.443	.455	.432
9	-	-	.420	.434
10	-	-	-	.400

These designs are applicable to other areas where angle measurements are involved. The complete closure method is applicable to the calibration of a polygon. When a polygon is substituted for the mirror its exterior angles are calibrated. If the polygon has twelve sides or less it is not too burdensome to take the full number of measurements. For a twelve sided polygon the C(12,12) design could be used requiring (12 x 13) = 156 measurements, a reasonable number. However, if a 36 sided polygon were calibrated by the C(36,36) series it would require (36 x 37) = 1332 measurements which normally would not be economical. Instead, a design such as C(36,6) could be used requiring a more modest (36 x 7) = 252 measurements. In cases where n is large and m is small the error in cumulative values greatly exceeds the error in the individual values. This may be an important factor in polygon calibrations, so the value of m must be carefully chosen according to the required precision.

Rotary tables may be used in place of indexing tables, but the observed standard deviation will be much larger because rotary table setting errors are generally much larger than indexing table setting errors.

8. Example

The first application of the method of subdivision was in connection with the absolute measurement of a set of angle blocks. Angular intervals on each indexing table which needed to be calibrated were $0^\circ - 60^\circ$, $0^\circ - 30^\circ$, $0^\circ - 15^\circ$, $0^\circ - 5^\circ$, and $0^\circ - 1^\circ$. On table A these angles were denoted by $\alpha_1+\alpha_2$, α_1, $\dot{\alpha}_1+\dot{\alpha}_2+\dot{\alpha}_3$, $\dot{\alpha}_1$, and $\ddot{\alpha}_1$ respectively. The corresponding angles on table B were similarly denoted using β instead of α. The most natural subdivision scheme seemed to be complete closure on the 30° intervals and two partial closures on 5° and 1° intervals. The three designs chosen were C(12,6), P(6) and P(5).

The observed y values from the three series are given in table 3. The computed values for the individual angles and their corresponding standard deviations are given in table 4 along with the observed standard deviation of a single measurement, s, and the degrees of freedom, df.

9. Conclusion

Computer programs have been written for the calibration of indexing tables by the methods described in this paper. The programs have been tested and successfully implemented. As a result several angle calibrations have been made more efficient by reducing both the time required to make the measurements and the time required to reduce the data.

TABLE 3.

THREE ECHELONS OF MEASUREMENTS, Y, TAKEN IN THE CALIBRATION OF INDEXING TABLES A AND B. MEASUREMENTS TAKEN BY ROWS (VALUES IN SECONDS).

SERIES 1	8.24	8.17	7.88	7.96	8.16	8.24	8.18
	6.79	6.61	6.73	6.99	6.99	6.91	6.85
C(12,6)	6.63	6.80	6.95	7.00	6.91	6.78	6.82
	5.31	5.49	5.50	5.50	5.37	5.44	5.39
COMPLETE	5.66	5.77	5.66	5.61	5.58	5.54	5.49
CLOSURE	5.78	5.70	5.64	5.57	5.53	5.47	5.69
	5.27	5.12	5.13	5.01	5.11	5.35	5.29
	4.79	4.77	4.67	4.72	4.90	4.95	4.86
	5.59	5.60	5.53	5.73	5.80	5.69	5.38
	3.63	3.67	3.90	3.94	3.80	3.59	3.66
	3.31	3.61	3.67	3.54	3.29	3.30	3.44
	4.16	4.27	4.13	3.84	3.95	4.11	4.15
SERIES 2	4.40	4.37					
	4.50	4.49	4.44				
P(6)	3.09	3.08	3.06	3.09			
	4.06	4.09	4.02	4.01	3.99		
PARTIAL	3.52	3.50	3.46	3.40	3.43	3.42	
CLOSURE	6.66	6.63	6.54	6.58	6.51	6.48	6.43
	4.88	4.85	4.83	4.71	4.66	4.61	
	5.18	5.15	5.13	5.04	4.95		
	5.88	5.88	5.84	5.75			
	5.20	5.13	5.09				
	5.44	5.37					
SERIES 3	3.76	3.78					
	3.51	3.47	3.45				
P(5)	4.63	4.58	4.65	4.64			
	3.99	4.00	3.96	3.97	3.96		
PARTIAL	6.03	5.99	6.07	6.03	5.99	5.98	
CLOSURE	5.16	5.15	5.14	5.14	5.18		
	5.38	5.35	5.37	5.36			
	5.41	5.38	5.38				
	4.35	4.32					

TABLE 4.

COMPUTED VALUES FOR INDIVIDUAL ANGLES ON INDEXING TABLES A AND B (VALUES IN SECONDS).

S/DF	INTERVAL	TABLE A ESTIMATE	TABLE B ESTIMATE	STD DEV
SERIES 1	0-30	-.117	.037	.019
	30-60	-.247	.019	.019
	60-90	.101	.011	.019
.037/50	90-120	.179	-.021	.019
	120-150	.052	-.064	.019
	150-180	-.065	.049	.019
	180-210	-.091	-.008	.019
	210-240	.007	.004	.019
	240-270	-.061	-.022	.019
	270-300	-.013	.018	.019
	300-330	.213	-.039	.019
	330-0	.042	.015	.019
SERIES 2	0-5	.020	-.002	.016
	5-10	-.020	.004	.017
	10-15	-.012	.042	.016
.030/26	15-20	-.025	.030	.016
	20-25	-.036	-.014	.017
	25-30	-.044	-.023	.016
SERIES 3	0-1	.004	.005	.018
	1-2	.016	-.023	.019
	2-3	-.015	.018	.019
.031/17	3-4	-.003	-.006	.019
	4-5	.018	.004	.018

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Appendix A

Reduction of the Normal Equations of the Complete Closure Model

The normal equations given in section 4 can be expressed in a form which is useful in programming the model for the computer. Instead of performing the matrix multiplications the normal equations are formed directly so the amount of memory needed is substantially reduced. The product $X'W^{-1}$ is given by

$$X'W^{-1} = \begin{bmatrix} P_n^n D'_m V_m^{-1} & P_n^{n-1} D'_m V_m^{-1} & \cdots & P_n^1 D'_m V_m^{-1} \\ -P_n^n D'_m V_m^{-1} & -P_n^{n-2} D'_m V_m^{-1} & \cdots & -P_n^2 D'_m V_m^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} M_1 & M_2 & \cdots & M_n \\ -\dot{M}_1 & -\dot{M}_2 & \cdots & -\dot{M}_n \end{bmatrix}$$

where M_i and $\dot{M}_i$ and n x m matrices. Now

$$(P_n^a D'_m V_m^{-1})_{ij} = (P_n^a \begin{bmatrix} V_m^{-1} \\ \theta_{n-m,m} \end{bmatrix})_{ij} = (P_n^a T_{nm})_{ij} = (T_{nm})_{i+a,j}$$

where the subscripts are reduced modulo n. The coefficent matrix of the observations, $X'W^{-1}$, is then given by

$$(M_k)_{ij} = (T_{nm})_{i+n-k+1,j} \text{ and}$$

$$(\dot{M}_k)_{ij} = (T_{nm})_{i+n-2k+2,j} \text{ where } i = 1,n;\ j = 1,m;\ k = 1,n.$$

Let Q and $\dot{Q}$ be n x 1 vectors defined by

$$\begin{bmatrix} Q \\ -\dot{Q} \end{bmatrix} = \begin{bmatrix} M_1 & M_2 & \cdots & M_n \\ -\dot{M}_1 & -\dot{M}_2 & \cdots & -\dot{M}_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

where each z_i is an m x 1 vector of observations. Then Q and $\dot{Q}$ can be expressed by

$$(Q)_i = \sum_{k=1}^{n} \sum_{j=1}^{m} (M_k)_{ij} (z_k)_j = \sum_{k=1}^{n} \sum_{j=1}^{m} (T_{nm})_{i+n-k+1,j} \, (z_k)_j, \quad \text{and}$$

$$(\dot{Q})_i = \sum_{k=1}^{n} \sum_{j=1}^{m} (\dot{M}_k)_{ij} (z_k)_j = \sum_{k=1}^{n} \sum_{j=1}^{m} (T_{nm})_{i+n-2k+2,j} \, (z_k)_j$$

where i = 1,n.

The product $X'W^{-1}X$ is given by

$$X'W^{-1}X = \begin{bmatrix} \sum_{k=0}^{n-1} P_n^{n-k} D_m' V_m^{-1} D_m P_n^k & -\sum_{k=0}^{n-1} P_n^{n-k} D_m' V_m^{-1} D_m P_n^{2k} \\ -\sum_{k=0}^{n-1} P_n^{n-2k} D_m' V_m^{-1} D_m P_n^k & \sum_{k=0}^{n-1} P_n^{n-2k} D_m' V_m^{-1} D_m P_n^{2k} \end{bmatrix}$$

$$= \begin{bmatrix} N & -\dot{N} \\ -\dot{N}' & \ddot{N} \end{bmatrix} \text{ where } N, \dot{N} \text{ and } \ddot{N} \text{ are n x n matrices.}$$

Now $\left(P^a_n D'_m V_m^{-1} D_m P^b_n\right)_{ij} = \left(P^a_n \begin{bmatrix} V_m^{-1} & \theta_{m,n-m} \\ \theta_{n-m,m} & \theta_{n-m,n-m} \end{bmatrix} P^b_n\right)_{ij} =$

$\left(P^a_n U_{nm} P^b_n\right)_{ij} = (U_{nm})_{i+a,j-b}$ where the subscripts are reduced modulo n. The matrix of normal equations, $X'W^{-1}X$, is then given by

$$(N)_{ij} = \sum_{k=0}^{n-1} (U_{nm})_{i+n-k,j-k},$$

$$(\dot{N})_{ij} = \sum_{k=0}^{n-1} (U_{nm})_{i+n-k,j-2k}, \text{ and}$$

$$(\ddot{N})_{ij} = \sum_{k=0}^{n=1} (U_{nm})_{i+n-2k,j-2k} \text{ where } i = 1,n;\ j = 1,n.$$

After augmenting with the two restraints the normal equations take the form

$$\begin{bmatrix} N & -\dot{N} & \pm & 0 \\ -\dot{N}' & \ddot{N} & 0 & \pm \\ \pm' & 0 & 0 & 0 \\ 0 & \pm' & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} Q \\ -\dot{Q} \\ 0 \\ 0 \end{bmatrix}.$$

Appendix B

Reduction of the Normal Equations of the Partial Closure Method

The normal equations in section 5 can be treated in the same manner as those in section 4. The product $X'W^{-1}$ is given by

$$X'W^{-1} = \begin{bmatrix} D_1' V_1^{-1} & \cdots & D_n' V_n^{-1} & P_n^{n-1} D_{n-1}' V_{n-1}^{-1} & \cdots & P_n' D_1' V_1^{-1} \\ -P_n' D_1' V_1^{-1} & \cdots & -P_n^n D_n' V_n^{-1} & -D_{n-1}' V_{n-1}^{-1} & \cdots & -D_1' V_1^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} M_1 & \cdots & M_n & \dot{M}_{n-1} & \cdots & \dot{M}_1 \\ -\dot{M}_1 & \cdots & -\dot{M}_n & -M_{n-1} & \cdots & -M_1 \end{bmatrix}$$

where M_k and $\dot{M}_k$ are n x k matrices. Now

$$(D_a' V_a^{-1})_{ij} = \left(\begin{bmatrix} V_a^{-1} \\ \theta_{n-a,a} \end{bmatrix} \right)_{ij} = (T_{na})_{ij} \quad \text{and}$$

$$(P_n^a D_a' V_a^{-1})_{ij} = (P_n^a T_{na})_{ij} = (T_{na})_{i+a,j}$$

where the subscripts are reduced modulo n. The coefficient matrix of the observations, $X'W^{-1}$, is given by

$$(M_k)_{ij} = (T_{nk})_{ij} \quad \text{and}$$

$$(\dot{M}_k)_{ij} = (T_{nk})_{i+k,j} \quad \text{where } i = 1,n;\ j = 1,k.$$

Let Q and $\dot{Q}$ be n x 1 vectors defined by

$$\begin{bmatrix} Q \\ \dot{Q} \end{bmatrix} = \begin{bmatrix} M_1 & \cdots & M_n & \dot{M}_{n-1} & \cdots & \dot{M}_1 \\ -\dot{M}_1 & \cdots & -\dot{M}_n & -M_{n-1} & \cdots & -M_1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \\ \vdots \\ z_{2n-1} \end{bmatrix}$$

where the z_k are vectors of observations of size k x 1 if $k \leq n$ and (2n-k) x 1 if $k > n$. Then Q and $\dot{Q}$ can be expressed by

$$(Q)_i = \sum_{k=1}^{n} \sum_{j=1}^{k} (M_k)_{ij} (z_k)_j + \sum_{k=1}^{n-1} \sum_{j=1}^{k} (\dot{M}_k)_{ij} (z_{2n-k})_j$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{k} (T_{nk})_{ij} (z_k)_j + \sum_{k=1}^{n-1} \sum_{j=1}^{k} (T_{nk})_{i+k,j} (z_{2n-k})_j, \quad \text{and}$$

$$(\dot{Q})_i = \sum_{k=1}^{n} \sum_{j=1}^{k} (\dot{M}_k)_{ij} (z_k)_j + \sum_{k=1}^{n-1} \sum_{j=1}^{k} (M_k)_{ij} (z_{2n-k})_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{k} (T_{nk})_{i+k,j} (z_k)_j + \sum_{k=1}^{n-1} \sum_{j=1}^{k} (T_{nk})_{ij} (z_{2n-k})_j$$

where i = 1,n.

The product $X'W^{-1}X$ is given by

$$X'W^{-1}X = \begin{bmatrix} N & -\dot{N} \\ -\dot{N}' & \ddot{N} \end{bmatrix} \quad \text{where}$$

$$N = \sum_{k=1}^{n} D_k' V_k^{-1} D_k + \sum_{k=1}^{n-1} P_n^{n-k} D_{n-k}' V_{n-k}^{-1} D_{n-k} P_n^k ,$$

$$\dot{N} = \sum_{k=1}^{n} D_k' V_k^{-1} D_k P_n^{n-k} + \sum_{k=1}^{n-1} P_n^{n-k} D_{n-k}' V_{n-k}^{-1} D_{n-k}, \quad \text{and}$$

$$\ddot{N} = \sum_{k=1}^{n} P_n^k D_k' V_k^{-1} D_k P^{n-k} + \sum_{k=1}^{n-1} D_{n-k}' V_{n-k}^{-1} D_{n-k} .$$

$$\text{Now } (P_n^a D_b' V_b^{-1} D_b P_n^c)_{ij} = (P_n^a \begin{bmatrix} V_b^{-1} & \theta_{b,n-b} \\ \theta_{n-b,b} & \theta_{n-b,n-b} \end{bmatrix} P_n^c)_{ij} =$$

$$(P_n^a U_{nb} P_n^c)_{ij} = (U_{nb})_{i+a,j-c}$$

where the subscripts are reduced modulo n. The matrix of normal equations, $X'W^{-1}X$, is then given by

$$(N)_{ij} = \sum_{k=1}^{n} (U_{nk})_{ij} + \sum_{k=1}^{n-1} (U_{n,n-k})_{i+n-k,j-k} ,$$

$$(\dot{N})_{ij} = \sum_{k=1}^{n} (U_{nk})_{i,j-n+k} + \sum_{k=1}^{n-1} (U_{n,n-k})_{i+n-k,j}, \quad \text{and}$$

$$(\ddot{N})_{ij} = \sum_{k=1}^{n} (U_{nk})_{i+k,j-n+k} + \sum_{k=1}^{n-1} (U_{n,n-k})_{ij}$$

where $i = 1,n$; $j = 1,n$.

After augmenting with the two restraints the normal equations take the form

$$\begin{bmatrix} N & -\dot{N} & \pm & 0 \\ -\dot{N}' & \ddot{N} & 0 & \pm \\ \pm' & 0 & 0 & 0 \\ 0 & \pm' & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \begin{bmatrix} Q \\ -\dot{Q} \\ m_A \\ m_B \end{bmatrix}$$

where m_A and m_B are restraint values from the previous series.