NBSIR 73-151

ON UNCERTAINTY IN MASS MEASUREMENT

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Washington, D. C. 20234

March 1973

Interim

Prepared for
Optical Physics Division
Institute for Basic Standards
National Bureau of Standards
Washington, D. C. 20234

U. S. DEPARTMENT OF COMMERCE, Frederick B. Dent, Secretary

NATIONAL BUREAU OF STANDARDS, Richard W. Roberts, Director

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INTRODUCTION

For every measurement there must be included a statement of uncertainty. This makes the measured value usable, defining the expected consistency of the value with the total measurement system. It provides an interval about the measured value in which the limiting mean of a large number of measurements is expected to be. The uncertainty statement is based on a knowledge of three factors: the standard, the unknown, and the measurement process.

THE STANDARD

It is assumed that the standard is a well defined object and that its mass has been determined in a well characterized process which validly describes the expected consistency of the mass value with reference to the total measurement system. Barring any real change, the actual mass of the standard is constant and it is either greater than, equal to, or less than its assigned mass estimate. Its uncertainty, E, describes the limits that the mass estimate will differ from the actual mass of the standard. Any error in the assigned mass of this standard will be systematic, always the same amount, in the same direction, although the direction and magnitude is not known.

THE UNKNOWN

It is possible that properties of the unknown, such as static, magnetic or hygroscopic characteristics, will cause the total collection of data made over an extended period of time to vary more than measurements made over a short period of time reflecting essentially the same conditions for each measurement.

The mass estimate of the unknown may also show systematic shifts due to location, season, etc. These shifts are often due to incorrect assumptions in the treatment of the raw data and remain undetected until the unknown is transferred to another facility which has conditions significantly different than those of the location where the original determinations were made. Because these shifts cannot usually be estimated by the measuring facility itself, they will not be treated in this paper.

THE MEASUREMENT PROCESS

All mass measurements are measures of differences between an unknown and a standard. In its simplest form, it is a determination of the difference between two objects of the same nominal size. The measurement process determines this difference. The balance, the unknown, the standard, the weighing procedure, the operator, the location and the treatment of the raw data all define the measurement process. It is assumed that for each weighing, a set procedure will be used for measuring the difference between the unknown and the standard. For this discussion, it will also be assumed that the difference is in mass units and that corrections for buoyant effects have been made.

Parameters of the Measurement Process

A collection of difference measurements can be thought of as a set of random variables taken from some probability distribution centered on a limiting mean value. Each measure of the difference between the unknown and the standard will lie above or below this limiting mean which is estimated by the mean of the observed differences. For independent measurements the variability of these values about the mean is described by the standard deviation of the process, σ. The mean of n observed differences has a standard deviation of $\sigma\sqrt{n}$.

Estimation of the Standard Deviation of the Process

In some cases, a standard deviation can be assumed from prior knowledge. However, in most cases, especially where the unknown is not immune to environmental or other factors, an estimate of the standard deviation of the measurement process should be computed from repeated measurements of the difference using the equation:

$$s = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n - 1}}$$

where n is the number of determinations of the difference made

X_i is an individual determination of the difference

$\bar{X}$ is the mean of the collection of differences

The estimate of the standard deviation for this process is based on n-1 degrees of freedom. As more determinations of this difference are made, the number of degrees of freedom from which the standard deviation has been computed will increase, producing a more reliable estimate of the standard deviation.

<u>Pooled Estimates of the Standard Deviation of the Process</u>

It is not always feasible to measure any one object enough times to establish a reliable estimate of the standard deviation of the process. However, knowledge obtained from the measurements of a given unknown can be applied to like measurements of similar objects. For example, if the mass of several unknowns of the same material and nominal size are to be measured using the same procedure, an estimate of the standard deviation of the process can be made from each group of difference measurements. If one is careful to be sure that each group of measurements reflects varying conditions so that each estimate of the standard deviation is a true estimate of the long term variability of the process, then the individual estimates of standard deviation can be pooled as follows:

$$s = \sqrt{\frac{\sum_{i=1}^{n}(n_i - 1)s_i^2}{\Sigma(n_i - 1)}}$$

where s_i is the individual estimate of the standard deviation

n_i is the number of difference determinations determining the estimate.

This is a pooled standard deviation of the process based on

$$n = \sum_{i=1}^{n}(n_i - 1)$$

degrees of freedom. The standard deviation of the mean of the difference between one object and the standard would be $s/\sqrt{n_i}$ where s is the pooled estimate of the standard deviation of the process and n_i is the number of measurements made of the difference between the unknown in question and the standard.

Since the standard deviation of the process may reflect, in part, the variability of the unknown, one must verify that the processes can be assumed to be the same so that estimates of standard deviation obtained from all groups of similar measurements can be pooled legitimately.

A limit to the possible effect of the random variation can be obtained using normal distribution theory by taking a multiple, t, of the standard deviation of the distribution. This multiplier depends on the state of our knowledge of the underlying distribution: for most cases when the standard deviation, σ, is well known the multiplier, 3.0, is used. For other cases one may use the factor from the standard t distribution which corrects for the smallness of the number of observations in the estimation of σ.

COMPUTATION OF UNCERTAINTY

The total uncertainty of the mass estimate of the unknown is the sum of the uncertainty of the difference measurement (the random component) and the uncertainty of the mass estimate of the standard (E, the systematic component) just as the mass of the unknown is the sum of the mass estimate of the difference between the unknown and the standard, and the mass estimate of the standard. Repeated measurements of the difference can decrease the random component of the uncertainty, however only a redetermination of the mass of the standard can decrease the systematic component of uncertainty.

Propagation of Error

In many measurement processes, the mass of the unknown is not determined directly against a standard, but is measured indirectly through a chain of difference measurements. The following example is offered as an explanation of the logic in the propagation of error through a chain of measurements. It is not expected that the uncertainty of a complex chain of measurements can be computed from this example particularly if complicated least square solutions are involved in determining the uncertainty of the values obtained.*

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* Solutions for all common weighing designs are available free of charge from the Office of Measurement Services at the National Bureau of Standards, Washington, D. C. 20234.

In the simplest form, this chain is a series of sum and difference measurements. For example, if we begin with a starting standard, T (with assigned value K), say at 100 grams, then one could determine the value of two unknown 50 gram weights, first by measuring the difference, X, between the $\Sigma 100=(50)_1+(50)_2$ and the standard T, and then by measuring the difference, Y, between the two 50 gram weights. The table shows the values along with the standard deviations associated with the measurement.

Observation	Quantity Measured	Standard Deviation
X	$[(50)_1 + (50)_2] - T$	σ_X
Y	$(50)_1 - (50)_2$	σ_Y

The value for $(50)_1$ and $(50)_2$ are given by

Quantity	Mass Value
$(50)_1$	$A = (\frac{1}{2}X + \frac{1}{2}Y) + \frac{1}{2}K$
$(50)_2$	$B = (\frac{1}{2}X - \frac{1}{2}Y) + \frac{1}{2}K$

The usual formula for the standard deviation of a linear combination, (ax+by), of independent random variables is

$$\sigma_{(ax+by)} = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2} \quad **$$

Because the value K is provided by an outside laboratory and is not determined in the measurement process described above, it is not a random variable of that process. Therefore, the standard deviation of the value A is

$$(\text{s.d. of A}) = \sqrt{(\tfrac{1}{2})^2(\sigma_X)^2 + (\tfrac{1}{2})^2(\sigma_Y)^2}$$

** The formulas for the computation of the variance of other functional forms of random variables can be found in Table 2-3, Statistical Concepts in Metrology by Harry H. Ku, Precision Measurement and Calibration, NBS Special Publication 300, Vol. 1, 5.2, pp. 315-339, 1969.

The uncertainty in K will be the same for all uses of the standard and cannot be reduced by making more measurements in the home laboratory. A proportional part of the uncertainty in K will be the systematic component of the uncertainty in A. The proportionality factor is computed from the ratio of the nominal size of the unknown, in this case 50g, to the nominal size of the starting standard, in this case 100g. The uncertainty in A is the sum of the random and systematic components of uncertainty

$$\text{Uncertainty in A} = 3(\text{s.d. of A}) + (50/100)(\text{Uncertainty in K})$$

Similarly for B

$$(\text{s.d. of B}) = \sqrt{(\tfrac{1}{2})^2(\sigma_X)^2 + (-\tfrac{1}{2})^2(\sigma_Y)^2} = \sqrt{(\tfrac{1}{2})^2(\sigma_X)^2 + (\tfrac{1}{2})^2(\sigma_Y)^2}$$

$$\text{Uncertainty in B} = 3(\text{s.d. of B}) + (50/100)(\text{Uncertainty in K})$$

If the $(50)_2$ weight were really a summation $50=(25)_1+(25)_2$, then one would need an additional measurement Z of the difference between the two 25 gram weights. We will presume the standard deviation of this measurement to be σ_Z. The weighing design is quite simple:

Quantity Measured	Value
$(25)_1 + (25)_2$	B
$(25)_1 - (25)_2$	Z

Unknown	Estimate of Unknown
$(25)_1$	$C = \tfrac{1}{2}(B+Z) = \tfrac{1}{2}[(\tfrac{1}{2}X-\tfrac{1}{2}Y)+\tfrac{1}{2}K] + \tfrac{1}{2}Z$
$(25)_2$	$D = \tfrac{1}{2}(B-Z) = \tfrac{1}{2}[(\tfrac{1}{2}X-\tfrac{1}{2}Y)+\tfrac{1}{2}K] - \tfrac{1}{2}Z$

If one does not ever repeat the original measurements X and Y when redetermining values for the 25 gram weights, the situation is just like the case when the value B has been assigned by an outside laboratory. In this case, Z is the only random variable and the standard deviation of C is

$$\sqrt{(\tfrac{1}{2})^2(\sigma_Z)^2} = \tfrac{1}{2}\sigma_Z$$

and of D is

$$\sqrt{(-\tfrac{1}{2})^2(\sigma_Z)^2} = \tfrac{1}{2}\sigma_Z$$

The systematic component of the uncertainty in C and D is a proportional part of the uncertainty in B. The uncertainty in C and D would be computed as follows:

$$\text{Uncertainty in C} = 3(\tfrac{1}{2})\sigma_Z + (25/50)(\text{Uncertainty in B})$$

$$\text{Uncertainty in D} = 3(\tfrac{1}{2})\sigma_Z + (25/50)(\text{Uncertainty in B})$$

However, if the whole series of measurements is repeated to redetermine the values for the 25 gram weights, then X, Y and Z are all random variables and the standard deviation of C is

$$(\text{s.d. of C}) = \sqrt{(\tfrac{1}{2})^2(\text{s.d.of B})^2+(\tfrac{1}{2})^2(\sigma_Z)^2}$$

$$= \sqrt{(\tfrac{1}{2})^2[(\tfrac{1}{2})^2(\sigma_X)^2+(\tfrac{1}{2})^2(\sigma_Y)^2]+(\tfrac{1}{2})^2(\sigma_Z)^2}$$

and of D is

$$(\text{s.d. of D}) = \sqrt{(\tfrac{1}{2})^2(\text{s.d.of B})^2+(-\tfrac{1}{2})^2(\sigma_Z)^2}$$

$$= \sqrt{(\tfrac{1}{2})^2[(\tfrac{1}{2})^2(\sigma_X)^2+(\tfrac{1}{2})^2(\sigma_Y)^2]+(\tfrac{1}{2})^2(\sigma_Z)^2}$$

The assigned value of T is still constant and a proportional part of its uncertainty, now (25/100)(Uncertainty in K), is the systematic component of the uncertainty in C and D. In this case the uncertainty of C would be

Uncertainty in C =

$$3\sqrt{(\tfrac{1}{2})^2[(\tfrac{1}{2})^2(\sigma_X)^2+(\tfrac{1}{2})^2(\sigma_Y)^2]+(\tfrac{1}{2})^2(\sigma_Z)^2} + (25/100)(\text{Uncertainty in K})$$

and the uncertainty in D would be

Uncertainty i [illegible] -

$$3\sqrt{(\tfrac{1}{2})^2[(\tfrac{1}{2})^2(\sigma_X)^2+(\tfrac{1}{2})^2(\sigma_Y)^2]+(\tfrac{1}{2})^2(\sigma_Z)^2} + (25/100)(\text{Uncertainty in K})$$

The above example is analagous to the situation where one is measuring a set of weights in a 5 3 2 1 series where a value for the $\Sigma 10 = 5+3+2$ has been determined in a 50 30 20 10 $\Sigma 10$ series. In the general form, the uncertainty of a given object is computed as follows:

$$3\sqrt{d^2\sigma_w^{\,2}+(k)^2(\sigma_\Sigma)^2} + KE$$

where d is a design factor obtained from the least square solution of the last series of measurements

σ_w is the standard deviation of the measurement process used for the last series

k is the ratio of the nominal size of the object in question to the nominal size of the summation measured in the preceding series

σ_Σ is the standard deviation of the summation

K is the ratio of the nominal size of the object in question to the nominal size of the starting standard and

E is the uncertainty in the assigned value of the starting standard.

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FORM NBS-114A (1-71)

U.S. DEPT. OF COMM. BIBLIOGRAPHIC DATA SHEET	1. PUBLICATION OR REPORT NO. NBSIR 73-151	2. Gov't Accession No.	3. Recipient's Accession No. COM-73-10866
4. TITLE AND SUBTITLE ON UNCERTAINTY IN MASS MEASUREMENT			5. Publication Date 6. Performing Organization Code
7. AUTHOR(S) Janet R. Donaldson			8. Performing Organization NBSIR 73-151
9. PERFORMING ORGANIZATION NAME AND ADDRESS NATIONAL BUREAU OF STANDARDS DEPARTMENT OF COMMERCE WASHINGTON, D.C. 20234			10. Project/Task/Work Unit No. 2320190 11. Contract/Grant No.
12. Sponsoring Organization Name and Address National Bureau of Standards Department of Commerce Washington, D. C. 20234			13. Type of Report & Period Covered 1967 - 1973 Interim 14. Sponsoring Agency Code

15. SUPPLEMENTARY NOTES

16. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)

This paper is a review of the factors which determine the total uncertainty of a measured mass value. It explains how to determine the standard deviation of the measurement process, how to combine error terms to provide a realistic estimate of the uncertainty, and how these error terms are propagated through a chain of mass measurements.

17. KEY WORDS (Alphabetical order, separated by semicolons) Mass measurement; random variability; standard deviation; systematic error; uncertainty

18. AVAILABILITY STATEMENT	19. SECURITY CLASS (THIS REPORT)	21. NO. OF PAGES
[X] UNLIMITED.	UNCLASSIFIED	
[ ] FOR OFFICIAL DISTRIBUTION. DO NOT RELEASE TO NTIS.	20. SECURITY CLASS (THIS PAGE) UNCLASSIFIED	22. Price

USCOMM-DC 66244-P71