Generalizations of Orthogonal polynomials in Askey Schemes

Howard S. Cohl*

*Applied and Computational Mathematics Division,
National Institute of Standards and Technology, Gaithersburg, Maryland

Department of Mathematics and Statistics Colloqium
Department of Mathematics and Statistics

American University
Washington, D.C.

April 15, 2014

Acknowedgements (co-authors)

Generalized hypergeometric orthogonal polynomials

- **Hans Volkmer**, University of Wisconsin-Milwaukee
- **Michael A. Baeder**, student Harvey Mudd College

Basic hypergeometric orthogonal polynomials

- **Roberto Costas-Santos**, Universidad de Alcalá, Spain
- **Philbert Hwang**, student at Poolesville High School

Orthogonal polynomials

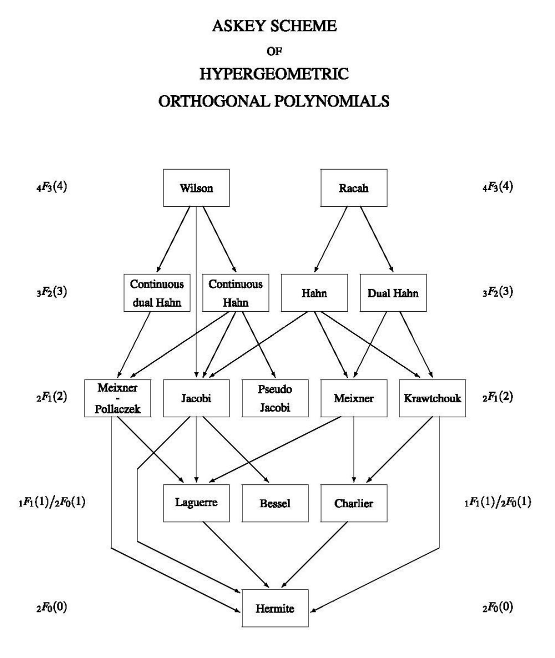
- Gabor Szegő, "Orthogonal Polynomials" (1959)
- A **polynomial** is an expression of **finite length** constructed from **variables** and **constants**, i.e.,

$$p_n(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^4 + \cdots + c_n z^n$$

- An **orthogonal (orthonormal) set of polynomials** $\{p_n(x)\}$ is defined by the relations

$$\int_a^b p_m(x) p_n(x) w(x) dx = \delta_{m,n}.$$

- **Askey scheme** – a way of organizing **orthogonal polynomials** of **hypergeometric type** into a **hierarchy**


ASKEY SCHEME
OF
HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS
$_4F_3(4)$
Wilson
Racah
$_4F_3(4)$
$_3F_2(3)$
Continuous dual Hahn
Continuous Hahn
Hahn
Dual Hahn
$_3F_2(3)$
$_2F_1(2)$
Meixner - Pollaczek
Jacobi
Pseudo Jacobi
Meixner
Krawtchouk
$_2F_1(2)$
$_1F_1(1)/_2F_0(1)$
Laguerre
Bessel
Charlier
$_1F_1(1)/_2F_0(1)$
$_2F_0(0)$
Hermite
$_2F_0(0)$

Classical continuous orthogonal polynomials

Orthog. polynomial	**Symbol**	$w(x)$	(a,b)
Wilson	W_n	$\left\lvert\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right\rvert^2$	$(0,\infty)$
Cts. dual Hahn	S_n	$\left\lvert\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right\rvert^2$	$(0,\infty)$
Continuous Hahn	p_n	$\lvert\Gamma(a+ix)\Gamma(b+ix)\rvert^2$	$(-\infty,\infty)$
Meixner-Pollaczek	$P_n^{(\lambda)}$	$e^{(2\phi-\pi)x}\lvert\Gamma(\lambda+ix)\rvert^2$	$(-\infty,\infty)$
Jacobi	$P_n^{(\alpha,\beta)}$	$(1-x)^\alpha(1+x)^\beta$	$(-1,1)$
Laguerre	L_n	$e^{-x}x^\alpha$	$(0,\infty)$
Hermite	H_n	e^{-x^2}	$(-\infty,\infty)$

Factorials and generalized hypergeometric series

Euler's gamma function and factorial for non-negative integers

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re}\, z > 0, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbf{N}_0 = 0, 1, 2, \ldots$$

Pochhammer symbol: the rising factorial in the complex plane

$$(a)_n := (a)(a+1)\ldots(a+n-1), \quad (a)_0 := 1, \quad a \in \mathbf{C}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \notin -\mathbf{N}_0 = 0, -1, -2, \ldots$$

Generalized hypergeometric series

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!}$$

Example: Definition of the (continuous) Wilson polynomials

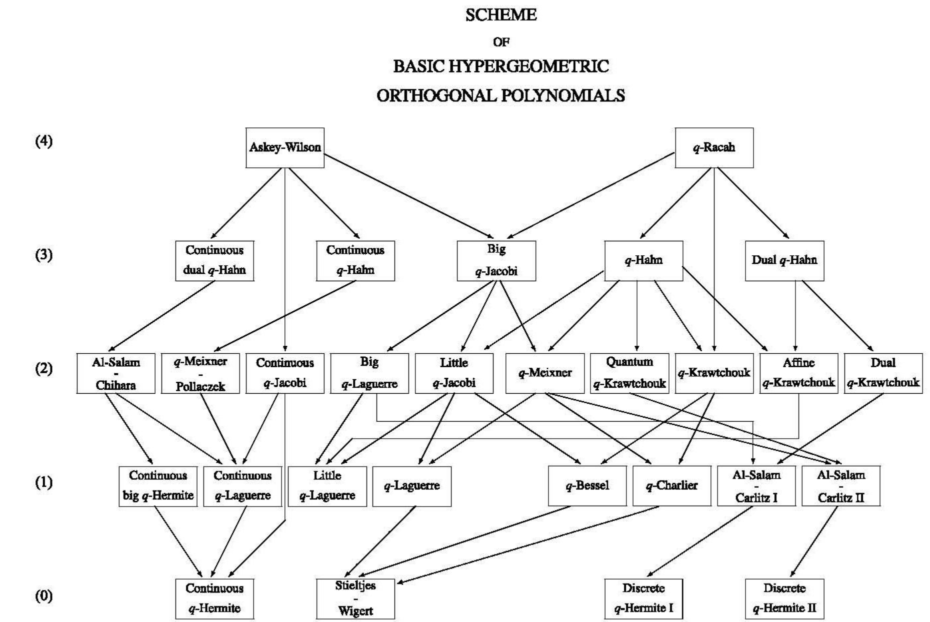
No symmetry in the **parameters**

$$W_n(x^2;a,b,c,d)$$
$$= (a+b)_n(a+c)_n(a+d)_n\, {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix\\ a+b, a+c, a+d\end{matrix};1\right)$$

Symmetry in the **parameters**

$$W_n(x^2;a,b,c,d) = \frac{(a-ix)_n(b-ix)_n(c-ix)_n(d-ix)_n}{(-2ix)_n}$$

$$\times\, {}_7F_6\left(\begin{matrix}-n, 2ix-n, ix-\frac{1}{2}n+1, a+ix, b+ix, c+ix, d+ix\\ ix-\frac{1}{2}n, 1-n-a+ix, 1-n-b+ix, 1-n-c+ix, 1-n-d+ix\end{matrix};1\right)$$


SCHEME
OF
BASIC HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS
(4)
Askey-Wilson
q-Racah
(3)
Continuous dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
(2)
Al-Salam - Chihara
q-Meixner - Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
(1)
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam - Carlitz I
Al-Salam - Carlitz II
(0)
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II

q-calculus

q-Pochhammer symbol

$$(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1}), \quad (a;q)_0 := 1$$

Notation

$$(a_1,\dots,a_r;q)_n := (a_1;q)_n \dots (a_r;q)_n$$

Basic hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix};q,z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s,q;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n$$

$${}_{s+1}\phi_s\left(\begin{matrix} a_1,\dots,a_{s+1} \\ b_1,\dots,b_s \end{matrix};q,z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{s+1};q)_n}{(b_1,\dots,b_s;q)_n} \frac{z^n}{(q;q)_n}$$

Connection relations and coefficients

$$P_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_{n,k}(\alpha;\beta) P_k^{(\beta)}(x)$$

What are the $c_{n,k}$? This is a **problem** in **orthogonal polynomials.** In general, one can compute connection relations by using **orthogonality**

$$\int_a^b P_k^{(\alpha)}(x) P_{k'}^{(\alpha)}(x) w(x) dx = d_k(\alpha)\delta_{k,k'}.$$

Therefore

$$c_{n,k}(\alpha,\beta) = \frac{1}{d_k(\beta)} \int_a^b P_n^{(\alpha)}(x) P_k^{(\beta)}(x) w(x) dx.$$

Generating functions

$$f(x,\rho;\alpha)=\sum_{n=0}^{\infty}c_n(\alpha)\rho^n P_n^{(\alpha)}(x)$$

Examples:

- **Hermite polynomials**

$$\exp(2x\rho-\rho^2)=\sum_{n=0}^{\infty}\frac{1}{n!}\rho^n H_n(x)$$

- **Gegenbauer polynomials**

$$\frac{1}{(1+\rho^2-2\rho x)^{\nu}}=\sum_{n=0}^{\infty}\rho^n C_n^{\nu}(x)$$

- **Jacobi polynomials**

$$2^{\alpha+\beta}\mathsf{R}^{-1}(1-\rho+\mathsf{R})^{-\alpha}(1+\rho+\mathsf{R})^{-\beta}=\sum_{n=0}^{\infty}\rho^n P_n^{(\alpha,\beta)}(x),$$

where $\mathsf{R}=\sqrt{1+\rho^2-2\rho x}$.

Ex: Laguerre polynomial connection rel. via generating fn.

Definition (orthogonal, monic):

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(n+\alpha)_{n-k}}{(n-k)!} \frac{(-x)^k}{k!}$$

Generating function

$$(1-\rho)^{-\alpha-1} \exp\left(\frac{x\rho}{\rho-1}\right) = \sum_{n=0}^{\infty} \rho^n L_n^{(\alpha)}(x)$$

$$\frac{(1-\rho)^{-\alpha-1}}{(1-\rho)^{-\beta-1}} (1-\rho)^{-\beta-1} \exp\left(\frac{x\rho}{\rho-1}\right) = (1-\rho)^{\beta-\alpha} \sum_{n=0}^{\infty} \rho^n L_n^{(\beta)}(x)$$

$$(1-\rho)^{-r} = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} x^{r-k} y^k \quad \Longrightarrow \quad (1-\rho)^{\beta-\alpha} = \sum_{j=0}^{\infty} \frac{(\alpha-\beta)_j}{j!} \rho^j$$

Example: Laguerre polynomial (cont.)

$$\sum_{j=0}^{\infty}\frac{(\alpha-\beta)_j}{j!}\rho^j\sum_{k=0}^{\infty}\rho^k L_k^{(\beta)}(x)=\sum_{n=0}^{\infty}\rho^n L_n^{(\alpha)}(x)$$

$$j+k=n \quad \Longrightarrow \quad j=n-k$$

$$\sum_{n=0}^{\infty}\left\{L_n^{\alpha}(x)-\sum_{k=0}^{n}\frac{(\alpha-\beta)_{n-k}}{(n-k)!}L_k^{(\beta)}(x)\right\}=0$$

Connection relation (1 free parameter)

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}c_{n,k}(\alpha;\beta)\,L_k^{(\beta)}(x),$$

where

$$c_{n,k}(\alpha;\beta)=\frac{(\alpha-\beta)_{n-k}}{(n-k)!}$$

Generalizations of generating function families

Generalized hypergeometric orthogonal polynomials

- Laguerre polynomials
- Jacobi, Gegenbauer, Chebyshev and Legendre polynomials
- Wilson polynomials
- Continuous Hahn polynomials
- Continuous dual Hahn polynomials
- Meixner-Pollaczek polynomials

Basic hypergeometric orthogonal polynomials

- q-ultraspherical /Rogers polynomials
- q-Laguerre polynomials
- big q-Laguerre
- Al-Salam-Carlitz II
- big/little q-Jacobi polynomials (ongoing)
- Stieltjes-Wigert polynomials (ongoing)
- Askey-Wilson (ongoing)

Example: Al-Salam-Carlitz II generating function

Definition (orthogonal, monic):

$$V_n^{(\alpha)}(x;q) = (-\alpha)^n q^{-\binom{n}{2}} \, {}_2\phi_0\left(\begin{matrix} q^{-n}, x \\ - \end{matrix}; q, \frac{q^n}{\alpha}\right), \quad \alpha < 0$$

Connection relation (1 free parameter)

$$V_n^{(\alpha)}(x;q) = \sum_{k=0}^{n} c_{n,k}(\alpha;\beta)\, V_k^{(\beta)}(x;q).$$

where

$$c_{n,k}(\alpha;\beta) := \frac{\beta^{n-k} q^{k(k-n)} (q^{k-n+1}\alpha/\beta;q)_{n-k} (q^{n-k+1};q)_k}{(q;q)_k}$$

Generating function:

$$(\alpha\rho;q)_\infty \, {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}\rho^n}{(q;q)_n} V_n^{(\alpha)}(x;q)$$

Example: Al-Salam-Carlitz II (cont.)

Therefore:

$$(\alpha\rho;q)_\infty\ {}_1\phi_1\left(\begin{matrix}x\\ \alpha\rho\end{matrix};q,\rho\right)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}\rho^n}{(q;q)_n}$$

$$\times\sum_{k=0}^{n}\frac{\beta^{n-k}q^{k(k-n)}(q^{k-n+1}\alpha/\beta;q)_{n-k}(q^{n-k+1};q)_k}{(q;q)_k}V_k^{(\beta)}(x)$$

Reverse the **order** of **summation** and let $n\mapsto n+k$

$$(\alpha\rho;q)_\infty\ {}_1\phi_1\left(\begin{matrix}x\\ \alpha\rho\end{matrix};q,\rho\right)=\sum_{k=0}^{\infty}\frac{\rho^kV_k^{(\beta)}(x,q)}{(q;q)_k}$$

$$\times\sum_{n=0}^{\infty}\frac{\beta^nq^{-nk}(\alpha q^{1-n}/\beta;q)_n(q^{n+1};q)_kq^{(n+k)(n+k-1)}\rho^n}{(q;q)_{n+k}}$$

Example: Al-Salam-Carlitz II (cont.)

Using the **properties** of q-**Pochhammer** symbols:

$$(a;q)_{n+k} = (a;q)_n (aq^n, q)_k,$$

$$(aq^{-n};q)_n = (q/a;q)_n (-a)^n q^{-n-\binom{n}{2}}, \quad a \neq 0,$$

one obtains after **cancellation**

$$(\alpha\rho;q)_\infty \; {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{k=0}^{\infty} \frac{\rho^k q^{k(k-1)} V_k^{(\beta)}(x,q)}{(q;q)_k} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\frac{\beta}{\alpha};q)_n (-\alpha\rho q^k)^n}{(q;q)_n}$$

therefore with $\alpha, \beta < 0$, one has

$$(\alpha\rho;q)_\infty \; {}_1\phi_1\left(\begin{matrix} x \\ \alpha\rho \end{matrix}; q, \rho\right) = \sum_{n=0}^{\infty} \frac{\rho^n q^{n(n-1)} V_n^{(\beta)}(x,q)}{(q;q)_n} \; {}_1\phi_1\left(\begin{matrix} \beta/\alpha \\ 0 \end{matrix}; q, \alpha\rho q^n\right)$$

Classical expansions for orthogonal polynomials

Gegenbauer generating function (1874)

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^\infty \rho^n C_n^\nu(x)$$

Legendre generating function (1783)

$$\frac{1}{\sqrt{1+\rho^2-2\rho x}}=\sum_{n=0}^\infty \rho^n P_n(x)$$

Heine's reciprocal square root identity (1881)

$$\frac{1}{\sqrt{z-x}}=\frac{\sqrt{2}}{\pi}\sum_{n=0}^\infty \epsilon_n Q_{n-1/2}(z)T_n(x), \qquad \epsilon_n := \begin{cases} 1 & \text{if } n=0, \\ 2 & \text{if } n=1,2,3,\ldots \end{cases}$$

$$\frac{1}{\sqrt{1+\rho^2-2\rho x}}=\frac{1}{\pi\sqrt{\rho}}\sum_{n=0}^\infty \epsilon_n Q_{n-1/2}\left(\frac{1+\rho^2}{2\rho}\right)T_n(x)$$

Classical Gauss hypergeometric orthogonal polynomials

- **Jacobi polynomials**, $\alpha,\beta>-1$, $P_n^{(\alpha,\beta)}:\mathbf{C}\to\mathbf{C}$, defined as

$$P_n^{(\alpha,\beta)}(z):=\frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c}-n,n+\alpha+\beta+1\\ \alpha+1\end{array};\frac{1-z}{2}\right)$$

$$\int_{-1}^{1}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}\delta_{m,n}$$

- **Gegenbauer polynomials**, $\mu\in(-\frac12,\infty)\setminus\{0\}$, $C_n^\mu:\mathbf{C}\to\mathbf{C}$, defined as

$$C_n^\mu(z):=\frac{(2\mu)_n}{n!}\,{}_2F_1\left(\begin{array}{c}-n,n+2\mu\\ \mu+\frac12\end{array};\frac{1-z}{2}\right)=\frac{(2\mu)_n}{(\mu+\frac12)_n}P_n^{(\mu-1/2,\mu-1/2)}$$

Classical Gauss hypergeometric orthogonal polynomials

- **Legendre polynomials**:

$$P_n(z) = C_n^{1/2}(z)$$

- **Chebyshev polynomials of the 1st kind**:

$$T_n(\cos\theta) = \cos(n\theta)$$

$$T_n(z) = \frac{1}{\epsilon_n}\lim_{\mu\to 0}\frac{n+\mu}{\mu}C_n^{\mu}(z)$$

- **Chebyshev polynomials of the 2nd kind**:

$$U_n(z) = C_n^{1}(z)$$

Special functions: associated Legendre functions

- **Ferrers function of the first kind:** $\mathrm{P}_\nu^\mu : (-1,1) \to \mathbf{C}$ **(associated Legendre function of the first kind on the cut)**

$$\mathrm{P}_\nu^\mu(x) := \frac{1}{\Gamma(1-\mu)} \left[\frac{1+x}{1-x}\right]^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$

- **Legendre function of the first kind**: $P_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1}\right]^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$

- **Legendre function of the second kind**, $Q_\nu^\mu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi} e^{i\pi\mu} \Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) z^{\nu+\mu+1}}$$

$$\times {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

The alg. functions $\sqrt{1+\rho^2-2\rho x}$, $\sqrt{z-x}$ from geometry

The **distance between two points** $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$ in a **polyspherical coordinate system** on $\mathbf{R}^d$ is given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{r^2 + r'^2 - 2rr'\cos\gamma}, \tag{1}$$

where $r = \|\mathbf{x}\|$, $r' = \|\mathbf{x}'\|$, and $\cos\gamma = \dfrac{\mathbf{x}\cdot\mathbf{x}'}{rr'}$. If you define $r_{\lessgtr} := {\min \atop \max}\{r, r'\}$, then you can rewrite (1) as

$$\|\mathbf{x}-\mathbf{x}'\| = r_> \sqrt{1 + \left(\frac{r_<}{r_>}\right)^2 - 2\frac{r_<}{r_>}\cos\gamma},$$

or with $\rho := \dfrac{r_<}{r_>}$, and $x := \cos\gamma$ we have

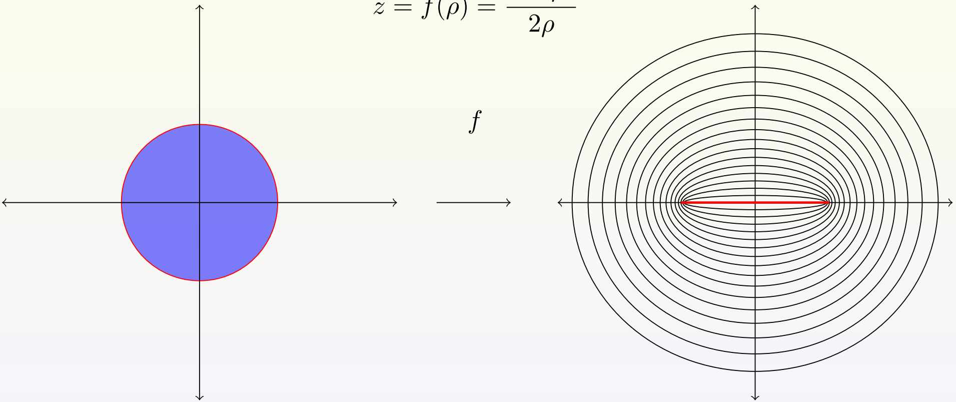
$$\|\mathbf{x}-\mathbf{x}'\| = r_> \sqrt{1+\rho^2-2\rho x},$$

where $\rho \in (0,1)$. The other option is:

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2rr'}\sqrt{z-x}, \quad \text{where} \quad z = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right) = \frac{1+\rho^2}{2\rho} \in (1,\infty)$$

Szegő transformation

$$z = f(\rho) = \frac{1+\rho^2}{2\rho}$$



Expansion of an analytic function in orthogonal polynomials

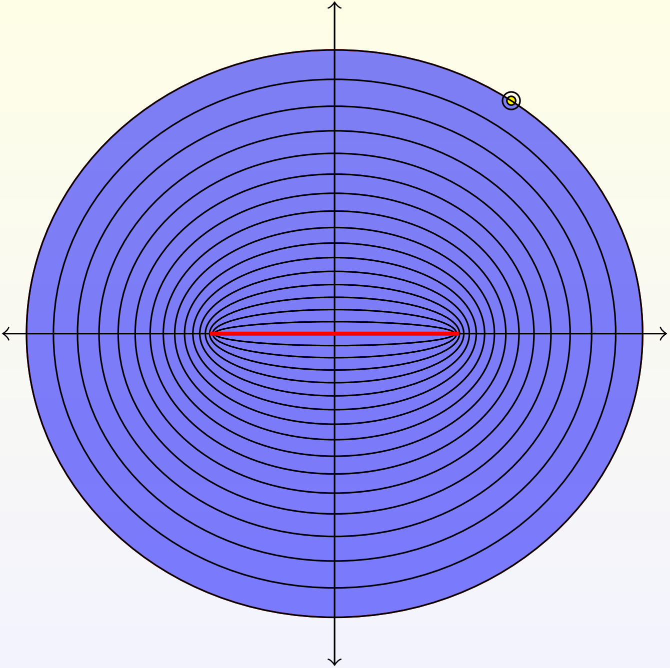
Theorem (Szegő): Let the **weight function** $w(x)$ on $[-1,1]$ have a **geometric mean,** and let $\{p_n(x)\}$ be the associated **orthonormal systems of polynomials**. Let f be an **analytic function** on $[-1,1]$ and let

$$f(x) = \sum_{n=0}^{\infty} f_n p_n(x) \tag{2}$$

and

$$f_n = \int_{-1}^{1} f(x) p_n(x) w(x) dx$$

be its **orthogonal polynomial expansion**. Let $\mathcal{E}$ be the **largest ellipse** with **foci** at ± 1 in the **interior** of which f is **regular.** Then the **orthogonal polynomial expansion** (2) is **convergent** with the sum $f(x)$ in the **interior** of $\mathcal{E}$ and **divergent** in the **exterior** of $\mathcal{E}$. The convergence is uniform on every closed set lying in the interior of the ellipse.



Obtaining generalizations: Gegenbauer generating function

Gegenbauer generating function:

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^{\infty}\rho^n C_n^\nu(x)$$

Connection relation:

$$C_n^\nu(x)=\frac{1}{\mu}\sum_{k=0}^{\lfloor n/2\rfloor}(\mu+n-2k)\frac{(\nu-\mu)_k\,(\nu)_{n-k}}{k!(\mu+1)_{n-k}}C^\mu_{n-2k}(x).$$

Gegenbauer expansion:

$$\frac{1}{(z-x)^\nu}=\frac{2^{\mu+\frac12}\Gamma(\mu)e^{i\pi(\mu-\nu+\frac12)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{\frac{\nu-\mu}{2}-\frac14}}\sum_{n=0}^{\infty}(n+\mu)Q^{\nu-\mu-\frac12}_{n+\mu-\frac12}(z)C^\mu_n(x) \qquad (3)$$

Implies **Chebyshev 1st kind expansion** (see Cohl & Dominici (2011)):

$$\frac{1}{(z-x)^\nu}=\sqrt{\frac{2}{\pi}}\frac{e^{i\pi(\frac12-\nu)}}{\Gamma(\nu)(z^2-1)^{\frac{\nu}{2}-\frac14}}\sum_{n=0}^{\infty}\epsilon_n Q^{\nu-\frac12}_{n-\frac12}(z)T_n(x),$$

Jacobi generalizations of Gegenbauer generating function

Gegenbauer→Jacobi generating function:

$$\frac{1}{(1+\rho^2-2\rho x)^\nu}=\sum_{n=0}^\infty \rho^n \frac{(2\nu)_n}{(\nu+\frac12)_n} P_n^{(\nu-\frac12,\nu-\frac12)}(x)$$

Jacobi connection relation (2 free parameters) Ismail (2005)

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^n c_{n,k}(\alpha,\beta;\gamma,\delta)P_k^{(\gamma,\delta)}(x),$$

$$c_{n,k}(\alpha,\beta;\gamma,\delta):=\frac{(\alpha+k+1)_{n-k}(n+\alpha+\beta+1)_k\Gamma(\gamma+\delta+k+1)}{(n-k)!\Gamma(\gamma+\delta+2k+1)}$$
$$\times\,{}_3F_2\left(\begin{array}{c}-n+k,n+k+\alpha+\beta+1,\gamma+k+1\\ \alpha+k+1,\gamma+\delta+2k+2\end{array};1\right)$$

Jacobi generalization of Gegenbauer generating function

Theorem

Let $\alpha,\beta>-1$, $z\in\mathbf{C}\setminus(-\infty,1]\to\mathbf{C}$, *on any ellipse with foci at* ±1 *and* x *in the interior of that ellipse. Then*

$$\frac{1}{(z-x)^\nu}=\frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}}$$

$$\times\sum_{n=0}^\infty\frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z)P_n^{(\alpha,\beta)}(x)$$

Jacobi function of the second kind

Jacobi function of the 2nd kind $Q_\gamma^{(\alpha,\beta)}:\mathbf{C}\setminus(-\infty,1]\to\mathbf{C}$ defined by

$$Q_\gamma^{(\alpha,\beta)}(z):=\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+2\gamma+2)(z-1)^{\alpha+\gamma+1}(z+1)^{\beta}}$$
$$\times\,{}_2F_1\left(\begin{array}{c}\gamma+1,\alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{array};\frac{2}{1-z}\right),$$

where $\alpha+\gamma,\beta+\gamma\notin-\mathbf{N}$.

Lemma. Let $n\in\mathbf{N}_0$, $\mu\in\mathbf{C}\setminus\{-\frac12,-\frac32,-\frac52,\ldots\}$, $\nu\in\mathbf{C}\setminus-\mathbf{N}_0$, $z\in\mathbf{C}\setminus(-\infty,1]$,

$$Q_{n+\nu-1}^{(\mu-\nu+\frac12,\mu-\nu+\frac12)}(z)=\frac{2^{\mu-\nu+\frac12}\Gamma\left(\mu+n+\frac12\right)e^{i\pi(\mu-\nu+\frac12)}}{\Gamma(\nu+n)(z^2-1)^{(\mu-\nu)/2+1/4}}Q_{n+\mu-\frac12}^{\nu-\mu-\frac12}(z).$$

Tom Koornwinder mention

The above **expansion** is consistent with the **Jacobi binomial expansion** given in Koekoek & Koekoek (2007), namely for $n \in \mathbf{N}_0$,

$$(z-x)^n = (-1)^n 2^n n! \Gamma(\alpha+\beta+1) \times \sum_{k=0}^{n} \frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{\Gamma(\alpha+\beta+n+k+2)} P_{n-k}^{(-\alpha-n-1,-\beta-n-1)}(z) P_k^{(\alpha,\beta)}(x)$$

The **consistency** is established through the formula

$$P_{n-k}^{(-\alpha-n-1,-\beta-n-1)}(z) = \frac{(-1)^{n+k}\Gamma(\alpha+\beta+n+k+2)}{2^{\alpha+\beta+2n+1}(n-k)!\,\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times (z-1)^{\alpha+n+1}(z+1)^{\beta+n+1}\, Q_{k-n-1}^{(\alpha+n+1,\beta+n+1)}(z)$$

Jacobi polynomials: $P_n^{(\alpha,\beta)} : \mathbf{C} \to \mathbf{C}$

DLMF (18.12.3) **generating function**

$$\frac{1}{(1+\rho)^{\alpha+\beta+1}}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}\\ \beta+1\end{matrix};\frac{2\rho(1+x)}{(1+\rho)^2}\right)$$
$$= \left(\frac{2}{\rho(1+x)}\right)^{\beta/2}\frac{\Gamma(\beta+1)}{\mathsf{R}^{\alpha+1}}P_\alpha^{-\beta}\left(\frac{1+\rho}{\mathsf{R}}\right)$$
$$= \sum_{n=0}^\infty \frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^n P_n^{(\alpha,\beta)}(x),$$

Ismail (2005) (4.3.2) **generating function**

$$\frac{(\alpha+\beta+1)(1+\rho)}{(1-\rho)^{\alpha+\beta+2}}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}\\ \alpha+1\end{matrix};\frac{-2\rho(1-x)}{(1-\rho)^2}\right)$$
$$= \left(\frac{2}{\rho(1-x)}\right)^{\alpha/2}\frac{(\alpha+\beta+1)(1+\rho)\Gamma(\alpha+1)}{\mathsf{R}^{\beta+2}}\mathrm{P}_{\beta+1}^{-\alpha}\left(\frac{1-\rho}{\mathsf{R}}\right)$$
$$= \sum_{n=0}^\infty (2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^n P_n^{(\alpha,\beta)}(x),$$

Generalized expansions for Jacobi polynomials

Theorem

Let $m \in \mathbf{N}_0$, $\alpha, \beta > -1$, $x \in [-1,1]$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1,0]$. *Then*

$$\frac{(1+x)^{-\beta/2}}{\mathsf{R}^{\alpha+m+1}} P^{-\beta}_{\alpha+m}\left(\frac{1+\rho}{\mathsf{R}}\right)$$

$$= \frac{\rho^{-(\alpha+1)/2}}{2^{\beta/2}(1-\rho)^m} \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{\Gamma(\beta+n+1)} \times P^{-\alpha-\beta-2n-1}_{-m}\left(\frac{1+\rho}{1-\rho}\right) P^{(\alpha,\beta)}_n(x).$$

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, γ, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\} \setminus (-1, 0]$, $x \in [-1, 1]$. *Then*

$$\frac{(1+x)^{-\beta/2}}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}} P_{\alpha}^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right)$$

$$= \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2}\Gamma(\beta+1)(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}}$$

$$\times \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k\,(\alpha+\beta+1)_{2k}}{(\beta+1)_k}$$

$$\times P_{\gamma-\alpha}^{-\gamma-\beta-2k-1}\left(\frac{1+\rho}{1-\rho}\right) P_k^{(\gamma,\beta)}(x).$$

$$\frac{(1+x)^{-\beta/2}}{(1+\rho^2-2\rho x)^{(\alpha+1)/2}} P_\alpha^{-\beta}\left(\frac{1+\rho}{\sqrt{1+\rho^2-2\rho x}}\right) = \frac{\Gamma(\gamma+\beta+1)}{2^{\beta/2}\Gamma(\beta+1)(1-\rho)^{\alpha-\gamma}\rho^{(\gamma+1)/2}} \sum_{n=0}^\infty \frac{(2n+\gamma+\beta+1)(\gamma+\beta+1)_n(\alpha+\beta+1)_{2n}}{(\beta+1)_n} P_{\gamma-\alpha}^{-\gamma-\beta-2n-1}\left(\frac{1+\rho}{1-\rho}\right) P_n^{(\gamma,\beta)}(x)$$

Theorem 1 in Cohl & MacKenzie (2013) *JCA* and Theorem 4 in Cohl, MacKenzie & Volkmer (2013) *JMAA*

$$\frac{1}{z-x} = \sum_{n=0}^\infty (2n+1) Q_n(z) P_n(x)$$

Heine (1851) Heine's formula

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \sum_{n=0}^\infty \rho^n C_n^\nu(x)$$

Gegenbauer (1874) generating function

$$\frac{1}{\sqrt{z-x}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^\infty \epsilon_n Q_{n-1/2}(z) T_n(x)$$

Heine (1881)

$$\frac{1}{(z-x)^\nu} = \frac{(z^2-1)^{(1-\nu)/2}}{e^{i\pi(\nu-1)}\Gamma(\nu)} \sum_{n=0}^\infty (2n+1) Q_n^{\nu-1}(z) P_n(x)$$

(13) in Cohl (2013) *ITSF*

$$\frac{1}{(z-x)^\nu} = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{-\nu/2+1/4}}{e^{i\pi(\nu-1/2)}\Gamma(\nu)} \times \sum_{n=0}^\infty \epsilon_n Q_{n-1/2}^{\nu-1/2}(z) T_n(x)$$

(3.10) in Cohl & Dominici (2011) *PRA*

$$\frac{1}{z-x} = \frac{2^{\mu+1/2}\Gamma(\mu) e^{i\pi(\mu-1/2)}}{\sqrt{\pi}(z^2-1)^{-\mu/2+1/4}} \sum_{n=0}^\infty (n+\mu) Q_{n+\mu-1/2}^{-\mu+1/2}(z) C_n^\mu(x)$$

(7.2) in Durand, Fishbane & Simmons (1976)

$$\frac{1}{(z-x)^\nu} = \frac{2^{\mu+1/2}\Gamma(\mu) e^{i\pi(\mu-\nu+1/2)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{(\nu-\mu)/2-1/4}} \times \sum_{n=0}^\infty (n+\mu) Q_{n+\mu-1/2}^{\nu-\mu-1/2}(z) C_n^\mu(x)$$

Theorem 2.1 in Cohl (2013) *ITSF*

$$\frac{1}{z-x} = \frac{(z-1)^\alpha (z+1)^\beta}{2^{\alpha+\beta}} \sum_{n=0}^\infty \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} Q_n^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(x)$$

(9.2.1) in Szegő (1959)

$$\frac{1}{(z-x)^\nu} = \frac{(z-1)^{\alpha+1-\nu}(z+1)^{\beta+1-\nu}}{2^{\alpha+\beta+1-\nu}} \sum_{n=0}^\infty \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\nu)_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} Q_{n+\nu-1}^{(\alpha+1-\nu,\beta+1-\nu)}(z) P_n^{(\alpha,\beta)}(x)$$

Theorem 1 in Cohl (2013) *SIGMA*

Other Jacobi generating functions

Connection relation (1 free parameter)

$$P_n^{(\alpha,\beta)}(x) = \frac{(\beta+1)_n}{(\gamma+\beta+1)(\gamma+\beta+2)_n}$$

$$\times \sum_{k=0}^{n} \frac{(\gamma+\beta+2k+1)(\gamma+\beta+1)_k\,(n+\beta+\alpha+1)_k(\alpha-\gamma)_{n-k}}{(\beta+1)_k\,(n+\gamma+\beta+2)_k(n-k)!} P_k^{(\gamma,\beta)}(x).$$

Generating function for Jacobi polynomials: DLMF (18.12.1)

$$\frac{2^{\alpha+\beta}}{\mathsf{R}\,(1+\mathsf{R}-\rho)^{\alpha}\,(1+\mathsf{R}+\rho)^{\beta}} = \sum_{n=0}^{\infty} \rho^n P_n^{(\alpha,\beta)}(x),$$

where $\mathsf{R} := \sqrt{1+\rho^2-2\rho x}$.

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, γ, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1,1]$. *Then*

$$\frac{2^{\alpha+\beta}}{\mathsf{R}\left(1+\mathsf{R}-\rho\right)^{\alpha}\left(1+\mathsf{R}+\rho\right)^{\beta}}$$

$$= \frac{1}{\gamma+\beta+1}\sum_{k=0}^{\infty}\frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k \left(\frac{\alpha+\beta+1}{2}\right)_k \left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+\beta+1)_k \left(\frac{\gamma+\beta+2}{2}\right)_k \left(\frac{\gamma+\beta+3}{2}\right)_k}$$

$$\times {}_3F_2\left(\begin{array}{c}\beta+k+1, \alpha+\beta+2k+1, \alpha-\gamma \\ \alpha+\beta+k+1, \gamma+\beta+2k+2\end{array}; \rho\right)\rho^k P_k^{(\gamma,\beta)}(x).$$

Generalized expansions for Jacobi polynomials

Theorem

Let $\alpha \in \mathbf{C}$, γ, $\beta > -1$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1,1]$. *Then*

$$\left(\frac{2}{(1-x)\rho}\right)^{\alpha/2} \left(\frac{2}{(1+x)\rho}\right)^{\beta/2} J_\alpha\left(\sqrt{2(1-x)\rho}\right) I_\beta\left(\sqrt{2(1+x)\rho}\right)$$

$$= \frac{1}{(\gamma+\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)}$$

$$\times \sum_{k=0}^{\infty} \frac{(2k+\gamma+\beta+1)(\gamma+\beta+1)_k \left(\frac{\alpha+\beta+1}{2}\right)_k \left(\frac{\alpha+\beta+2}{2}\right)_k}{(\alpha+1)_k\,(\beta+1)_k\,(\alpha+\beta+1)_k \left(\frac{\gamma+\beta+2}{2}\right)_k \left(\frac{\gamma+\beta+3}{2}\right)_k}$$

$$\times {}_2F_3\left(\begin{array}{c} 2k+\alpha+\beta+1, \alpha-\gamma \\ \alpha+\beta+k+1, \gamma+\beta+2k+2, \alpha+k+1 \end{array} ; \rho\right) \rho^k P_k^{(\gamma,\beta)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{\rho^n}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} P_n^{(\alpha,\beta)}(x).$$

Gegenbauer polynomials: $C_n^\mu : \mathbf{C} \to \mathbf{C}$

Generating function from (9.8.32) of Koekoek, Lesky & Swarttouw (2010)

$$
{}_2F_1\left(\begin{array}{c}\lambda, 2\alpha-\lambda\\ \alpha+\frac{1}{2}\end{array}; \frac{1-R-\rho}{2}\right) {}_2F_1\left(\begin{array}{c}\lambda, 2\alpha-\lambda\\ \alpha+\frac{1}{2}\end{array}; \frac{1-R+\rho}{2}\right)
$$
$$
= \frac{\Gamma^2(\alpha+\frac{1}{2})}{2^{1/2-\alpha}}(1-x^2)^{1/4-\alpha/2}\rho^{1/2-\alpha}
$$
$$
\times P^{1/2-\alpha}_{\alpha-\lambda-1/2}\left(\sqrt{1+\rho^2-2\rho x}+\rho\right)\mathrm{P}^{1/2-\alpha}_{\alpha-\lambda-1/2}\left(\sqrt{1+\rho^2-2\rho x}-\rho\right)
$$
$$
= \sum_{n=0}^{\infty}\frac{(\lambda)_n\,(2\alpha-\lambda)_n}{(2\alpha)_n\,(\alpha+\frac{1}{2})_n}\rho^n C_n^\alpha(x).
$$

Generalized expansions for Gegenbauer polynomials

Theorem

Let $\alpha, \mu \in \mathbf{C}$, $\nu \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in [-1, 1]$. *Then*

$$(1 - x^2)^{1/4-\mu/2}$$

$$\times P^{1/2-\mu}_{\mu-\alpha-1/2}\left(\sqrt{1+\rho^2-2\rho x}+\rho\right) \mathrm{P}^{1/2-\mu}_{\mu-\alpha-1/2}\left(\sqrt{1+\rho^2-2\rho x}-\rho\right)$$

$$= \frac{2^{1/2-\mu}}{\Gamma(\mu+1/2)} \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(2\mu-\alpha)_n\,(\mu)_n}{(2\mu)_n\,(\nu)_n\,\Gamma(\mu+n+1/2)} \rho^{\mu+n-1/2}$$

$$\times {}_6F_5\left(\begin{array}{c} \frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}, \frac{2\mu-\alpha+n}{2}, \frac{2\mu-\alpha+n+1}{2}, \mu-\nu, \mu+n \\ \frac{2\mu+n}{2}, \frac{2\mu+n+1}{2}, \frac{\mu+n+\frac{1}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}, \nu+1+n \end{array} ; \rho^2\right) C^{\nu}_n(x).$$

Associated Legendre function, Ferrers functions, and Gegenbauer polynomials

If $n \in \mathbf{N}_0$, then through DLMF (14.3.22)

$$P^{1/2-\mu}_{n+\mu-1/2}(z) = \frac{2^{\mu-1/2}\Gamma(\mu)n!}{\sqrt{\pi}\,\Gamma(2\mu+n)}(z^2-1)^{\mu/2-1/4}\,C^{\mu}_n(z),$$

and from DLMF (14.3.21), one has

$$\mathrm{P}^{1/2-\mu}_{n+\mu-1/2}(x) = \frac{2^{\mu-1/2}\Gamma(\mu)n!}{\sqrt{\pi}\,\Gamma(2\mu+n)}(1-x^2)^{\mu/2-1/4}\,C^{\mu}_n(x).$$

Gegenbauer expansions

For specific values can be this **nice expansion**

$$C_m^\mu(\mathsf{R}+\rho)C_m^\mu(\mathsf{R}-\rho) = \frac{(2\mu)_m^2}{(m!)^2}\sum_{n=0}^{m}\frac{(-m)_n(2\mu+m)_n}{(2\mu)_n(\mu+\frac{1}{2})_n}\rho^n C_n^\mu(x),$$

and from the above **generalized result** we have

$$C_m^\mu(\mathsf{R}+\rho)C_m^\mu(\mathsf{R}-\rho) = \frac{(2\mu)_m^2}{\nu(m!)^2}\sum_{n=0}^{m}\frac{(\nu+n)\,(-m)_n\,(2\mu+m)_n\,(\mu)_n}{(2\mu)_n\,(\mu+1/2)_n\,(\nu+1)_n}$$

$$\times\,{}_6F_5\left(\begin{array}{c}\frac{-m+n}{2},\frac{-m+n+1}{2},\frac{2\mu+m+n}{2},\frac{2\mu+m+n+1}{2},\mu-\nu,\mu+n\\ \frac{2\mu+n}{2},\frac{2\mu+n+1}{2},\frac{\mu+n+\frac{1}{2}}{2},\frac{\mu+n+\frac{3}{2}}{2},\nu+1+n\end{array};\rho^2\right)\rho^n C_n^\nu(x),$$

which reduces to **above formula** when $\nu=\mu$.

Wilson polynomials

$$W_n(x^2;a,b,c,d) := (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3\left(\begin{array}{c} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{array}; 1\right).$$

Connection relation with **one free parameter** for the **Wilson polynomials**

$$W_n\left(x^2;a,b,c,d\right) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} W_k\left(x^2;a,b,c,h\right) \times \frac{(n+a+b+c+d-1)_k\,(d-h)_{n-k}\,(k+a+b)_{n-k}\,(k+a+c)_{n-k}}{(k+b+c)_{n-k}^{-1}(k+a+b+c+h-1)_k\,(2k+a+b+c+h)_{n-k}}.$$

Generalized generating function for Wilson polynomials

Theorem

Let $\rho \in \{z \in \mathbf{C} : |z| < 1\}$, $x \in (0, \infty)$, $\Re a, \Re b, \Re c, \Re d, \Re h > 0$ *and non-real parameters* a, b, c, d, h *occurring in conjugate pairs. Then*

$$
{}_2F_1\left(\begin{array}{c} a+ix,\, b+ix \\ a+b \end{array};\rho\right){}_2F_1\left(\begin{array}{c} c-ix,\, d-ix \\ c+d \end{array};\rho\right)
$$

$$
=\sum_{k=0}^{\infty}\frac{(k+a+b+c+d-1)_k}{(k+a+b+c+h-1)_k\,(a+b)_k\,(c+d)_k\,k!}
$$

$$
\times\,{}_4F_3\left(\begin{array}{c} d-h,\, 2k+a+b+c+d-1,\, k+a+c,\, k+b+c \\ k+a+b+c+d-1,\, 2k+a+b+c+h,\, k+c+d \end{array};\rho\right)
$$

$$
\times\,\rho^k\,W_k\left(x^2;a,b,c,h\right).
$$

Generalized generating function for Wilson polynomials

Theorem

Let $\rho \in \mathbf{C}$, $|\rho| < 1$, $x \in (0, \infty)$, and a, b, c, d, h complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among a, b, c, d and a, b, c, h. Then

$$(1-\rho)^{1-a-b-c-d}$$

$$\times {}_4F_3\left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; -\frac{4\rho}{(1-\rho)^2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(k+a+b+c+d-1)_k (a+b+c+d-1)_k}{(k+a+b+c+h-1)_k (a+b)_k (a+c)_k (a+d)_k k!} \rho^k$$

$$\times {}_3F_2\left(\begin{matrix} 2k+a+b+c+d-1, d-h, k+b+c \\ 2k+a+b+c+h, a+d+k \end{matrix} ; \rho\right) W_k(x^2; a, b, c, h)$$

Continuous dual Hahn polynomials

$$S_n(x^2;a,b,c) := (a+b)_n(a+c)_n\,{}_3F_2\left(\begin{matrix}-n, a+ix, a-ix\\ a+b, a+c\end{matrix};1\right),$$

where $a,b,c>0$, except for possibly a pair of complex conjugates with positive real parts. **Connection coefficient** for the **continuous dual Hahn polynomials** with **two free parameters**.

Lemma

Let $x\in(0,\infty)$, and $a,b,c,f,g\in\mathbf{C}$ with positive real parts and non-real values appearing in conjugate pairs among a,b,c and a,f,g. Then

$$S_n(x^2;a,b,c)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(k+a+b)_{n-k}(k+a+c)_{n-k}S_k(x^2;a,f,g)$$

$$\times\,{}_3F_2\left(\begin{matrix}k-n, k+a+f, k+a+g\\ k+a+b, k+a+c\end{matrix};1\right).$$

Generalized gen. fn. for continuous dual-Hahn polynomials

Theorem

Let $\rho \in \mathbf{C}$ *with* $|\rho| < 1$, $x \in (0, \infty)$ *and* $a, b, c, d, f > 0$ *except for possibly pairs of complex conjugates with positive real parts among* a, b, c *and* a, d, f. *Then*

$$(1-\rho)^{-d+ix}\,{}_2F_1\left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}; \rho\right) = \sum_{k=0}^{\infty} \frac{S_k(x^2; a, d, f)\rho^k}{(a+b)_k k!}\,{}_2F_1\left(\begin{matrix} b-f, k+a+d \\ k+a+b \end{matrix}; \rho\right).$$

Theorem

Let $\rho \in \mathbf{C}$, $x \in (0, \infty)$, *and* $a, b, c, d > 0$ *except for possibly a pair of complex conjugates with positive real parts among* a, b, c *and* a, b, d. *Then*

$$e^{\rho}\,{}_2F_2\left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix}; -\rho\right) = \sum_{k=0}^{\infty} \frac{\rho^k S_k(x^2; a, b, d)}{(a+b)_k (a+c)_k k!}\,{}_1F_1\left(\begin{matrix} c-d \\ k+a+c \end{matrix}; \rho\right).$$

Connection relation for Meixner-Pollaczek polynomials

If $\lambda > 0$ and $\phi \in (0, \pi)$, then the **Meixner-Pollaczek** polynomials are orthogonal

$$P_n^{(\lambda)}(x;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left({-n, \lambda + ix \atop 2\lambda} ; 1 - e^{-2i\phi} \right).$$

Connection relation with one free parameter:

Lemma

Let $\lambda > 0, \phi, \psi \in (0, \pi)$. *Then*

$$P_n^{(\lambda)}(x;\phi) = \frac{1}{\sin^n \psi} \sum_{k=0}^{n} \frac{(2\lambda + k)_{n-k}}{(n-k)!} \sin^k \phi \sin^{n-k}(\psi - \phi) P_k^{(\lambda)}(x;\psi).$$

Expansions for Meixner-Pollaczek polynomials

Theorem

Let $\lambda > 0$, $\psi, \phi \in (0, \pi)$, $x \in \mathbf{R}$, *and* $\rho \in \mathbf{C}$ *such that*

$$|\rho|(\sin\phi + |\sin(\psi - \phi)|) < \sin\psi.$$

Then

$$(1 - e^{i\phi}\rho)^{-\lambda+ix}(1 - e^{-i\phi}\rho)^{-\lambda-ix} = \left(1 - \rho\frac{\sin(\psi - \phi)}{\sin\psi}\right)^{-2\lambda} \sum_{k=0}^{\infty} P_k^{(\lambda)}(x;\psi)\tilde{\rho}^k,$$

where

$$\tilde{\rho} = \frac{\rho\sin\phi}{\sin\psi - \rho\sin(\psi - \phi)}.$$

etc. for continuous Hahn polynomials

Some results for q-Pochhammer symbols

$$(a;q)_{n+k} = (a;q)_n (aq^n;q)_k,$$

$$(a;q)_{2n} = (a,aq;q^2)_n,$$

$$(aq^{-n};q)_n = (a^{-1}q;q)_n (-a)^n q^{-n-\binom{n}{2}}, \quad a \neq 0,$$

$$(aq^n;q)_k = \frac{(a;q)_k}{(a;q)_n}(aq^k;q)_n,$$

$$(a^2;q^2)_n = (a,-a;q)_n,$$

$$(-a^2;q^2)_n = (ia,-ia;q)_n.$$

The q-ultraspherical/Rogers polynomials

Definition **(orthogonal)** for $|\beta| < 1$

$$C_n(x;\beta|q) := \frac{(\beta^2;q)_n\beta^{-n/2}}{(q;q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n},\beta^2q^n,\beta^{1/2}e^{i\theta},\beta^{1/2}e^{-i\theta} \\ \beta q^{1/2},-\beta,-\beta q^{1/2}\end{matrix};q,q\right)$$

$$= \frac{(\beta^2;q)_n\beta^{-n}e^{-in\theta}}{(q;q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},\beta,\beta e^{2i\theta} \\ \beta^2,0\end{matrix};q,q\right)$$

$$= \frac{(\beta;q)_n e^{in\theta}}{(q;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n},\beta \\ \beta^{-1}q^{1-n}\end{matrix};q,q\beta^{-1}e^{-2i\theta}\right),$$

where $x = \cos\theta$.

Theorem

Let $x \in [-1,1]$, $\beta \in (-1,1) \setminus \{0\}$, $q \in (0,1)$. *Then*

$$
{}_2\phi_1\left(\begin{array}{c} \beta^{\frac12}e^{i\theta}, (\beta q)^{\frac12}e^{i\theta} \\ \beta q^{\frac12} \end{array}; q, e^{-i\theta}t\right) {}_2\phi_1\left(\begin{array}{c} -\beta^{\frac12}e^{-i\theta}, -\beta^{\frac12}q^{\frac12}e^{-i\theta} \\ \beta q^{\frac12} \end{array}; q, e^{i\theta}t\right)
$$

$$
= \frac{1}{1-\gamma}\sum_{n=0}^{\infty} \frac{(\beta, -\beta, -\beta q^{\frac12}; q)_n (1-\gamma q^n) t^n}{(\beta^2, \beta q^{\frac12}, q\gamma; q)_n} C_n(x;\gamma|q)
$$

$$
\times\, {}_{10}\phi_9\left(\begin{array}{c} \beta/\gamma, \beta q^n, i(\beta q^n)^{\frac12}, -i(\beta q^n)^{\frac12}, i(\beta q^{n+1})^{\frac12}, -i(\beta q^{n+1})^{\frac12}, \\ \gamma q^{n+1}, \beta q^{n/2}, -\beta q^{n/2}, \beta q^{(n+1)/2}, -\beta q^{(n+1)/2}, (\beta q^{n+\frac12})^{\frac12}, \\ i(\beta q^{n+\frac12})^{\frac12}, -i(\beta q^{n+\frac12})^{\frac12}, i(\beta q^{n+3/2})^{\frac12}, -i(\beta q^{n+3/2})^{\frac12} \\ -(\beta q^{n+\frac12})^{\frac12}, (\beta q^{n+3/2})^{\frac12}, -(\beta q^{n+3/2})^{\frac12} \end{array}; q, \gamma t^2\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{(-\beta, -\beta q^{1/2}; q)_n}{(\beta^2, \beta q^{1/2}; q)_n} C_n(x;\beta|q) t^n.
$$

Theorem

Let $x \in [-1,1]$, $\beta \in (-1,1) \setminus \{0\}$, $q \in (0,1)$. *Then*

$$\frac{(\gamma e^{i\theta}t;q)_\infty}{(e^{i\theta}t;q)_\infty}\,{}_3\phi_2\left(\begin{array}{c}\gamma,\beta,\beta e^{2i\theta}\\ \beta^2,\gamma e^{i\theta}t\end{array};q,e^{-i\theta}t\right)$$

$$=\frac{1}{1-\gamma}\sum_{n=0}^{\infty}\frac{(\beta,\gamma;q)_n(1-\gamma q^n)t^n}{(\beta^2,q\gamma;q)_n}C_n(x;\gamma|q)$$

$$\times\,{}_6\phi_5\left(\begin{array}{c}\beta/\gamma,\beta q^n,(\gamma q^n)^{\frac{1}{2}},-(\gamma q^n)^{\frac{1}{2}},(\gamma q^{n+1})^{\frac{1}{2}},-(\gamma q^{n+1})^{\frac{1}{2}}\\ \beta q^{n/2},-\beta q^{n/2},\beta q^{(n+1)/2},-\beta q^{(n+1)/2},\gamma q^{n+1}\end{array};q,\gamma t^2\right)$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma;q)_n}{(\beta^2;q)_n}C_n(x;\beta|q)t^n,$$

etc. for q-ultraspherical/Rogers polynomials.

The q-Laguerre polynomials

Definition **(orthogonal)** $\alpha > -1$

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array};q,-q^{n+\alpha+1}x\right)$$

$$= \frac{1}{(q;q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},-x \\ 0 \end{array};q,q^{n+\alpha+1}\right).$$

Connection relation

$$L_n^{(\alpha)}(x;q)$$

$$= \frac{q^{n(\alpha-\beta)}}{(q;q)_n}\sum_{j=0}^{n}(-1)^{n-j}q^{\binom{n-j}{2}}(q^{n-j+1};q)_j(q^{j-n+\beta-\alpha+1};q)_{n-j}\,L_j^{(\beta)}(x;q)$$

Generalized generating function for q-Laguerre

Theorem

Let $\alpha > -1$. *Then*

$$\frac{1}{(t;q)_\infty}\,{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-xtq^{\alpha+1}\right)$$

$$=\sum_{n=0}^\infty \frac{(q^{(\alpha-\beta)}t)^n L_n^{(\beta)}(x;q)}{(q^{\alpha+1};q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{\alpha-\beta},0\\ q^{\alpha+n+1}\end{matrix};q,t\right),$$

$$=\sum_{n=0}^\infty \frac{L_n^{(\alpha)}(x;q)}{(q^{\alpha+1};q)_n}t^n.$$

Generalized generating function for q-Laguerre (cont.)

Theorem

Let $\alpha > -1$. *Then*

$$(t;q)_\infty \, {}_0\phi_2\left(\begin{matrix} - \\ q^{\alpha+1}, t \end{matrix} ; q, -q^{\alpha+1}xt \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-tq^{\alpha-\beta})^n q^{\binom{n}{2}} L_n^{(\beta)}(x;q)}{(q^{\alpha+1};q)_n} \, {}_1\phi_1\left(\begin{matrix} q^{\alpha-\beta} \\ q^{\alpha+n+1} \end{matrix} ; q, q^n t \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q^{\alpha+1};q)_n} L_n^{(\alpha)}(x;q) t^n.$$

etc. for q-Laguerre and etc. for little q-Laguerre.

Final Remarks & see publications

- **Definite integrals** from **orthogonality** for the **basic case**
- **Discrete hypergeometric orthogonal polynomials**
- **Big/little q-Jacobi**, **continuous q-Jacobi**, **Askey-Wilson**

- "On a generalization of the generating function for Gegenbauer polynomials," H. S. Cohl, 2013, *Integral Transforms and Special Functions*, **24**, 10, 807–816, 10 pp.
- "Generalizations and specializations of generating functions for Jacobi, Gegenbauer, Chebyshev and Legendre polynomials with definite integrals," H. S. Cohl and Connor MacKenzie, 2013, *Journal of Classical Analysis*, **3**, 1, 17-33, 17 pp.
- "Fourier, Gegenbauer and Jacobi expansions for a power-law fundamental solution of the polyharmonic equation and polyspherical addition theorems," H. S. Cohl, 2013, *Symmetry, Integrability and Geometry: Methods and Applications*, **9**, 042, 26 pp.
- "Generalizations of generating functions for hypergeometric orthogonal polynomials with definite integrals," H. S. Cohl, Connor MacKenzie, and H. Volkmer, 2013, *Journal of Mathematical Analysis and Applications*, **407**, 2, 211-225, 15 pp.
- "Generalized generating functions for higher continuous orthogonal polynomials in the Askey scheme," M. A. Baeder, H. S. Cohl, H. Volkmer (submitted).