Fourier Transformation of the Nonlinear VOR Model to Approximate Linear Form

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MODEL TO APPROXIMATE LINEAR FORM

by

Dominic F. Vecchia

This technical note describes a method for transforming a particular nonlinear regression model to a form which is approximately linear in the unknown parameters. The technique involves computation of the Fourier coefficients for a set of sample data and uses phase variables to estimate the parameters. The phase spectrum transformation is employed to obtain bearing angle estimates for a model associated with the Very-High-Frequency Omni-Directional Range (VOR) aircraft navigation system. The transformation provides a model linear in relevant phase parameters. Thus, estimation of VOR bearing angle utilizes existing statistical theory. Finally, it is shown that certain generalizations of the VOR model also are reduced to approximate linear form by the phase spectrum transformation.

Key Words: Fourier coefficients; linear model; nonlinear model; phase spectrum transformation; spectrum; VOR aircraft navigation system; white noise.

1. INTRODUCTION

There are many different reasons for making a transformation of variables in the statistical analysis of data. This technical note discusses an unusual type of transformation useful in connection with a particular nonlinear regression model for audiofrequency signals from the Very-High-Frequency Omni-Directional Range (VOR) air navigation system. The model, which is considered in a more general form than the VOR requires, is intrinsically nonlinear in the unknown parameters. By intrinsically nonlinear we mean that a single observation cannot be transformed into linear form. For example, consider the two models

$$Y = \exp(\beta_1 + \beta_2 x + \varepsilon)$$

$$Y = [\beta_1/(\beta_1-\beta_2)]\,[\exp(-\beta_2 x) - \exp(-\beta_1 x)] + \varepsilon \quad ,$$

where β_1 and β_2 are unknown parameters, x is an independent variable, and ε is a random error term. Both models are nonlinear in β_1 and β_2, but the first is <u>intrinsically</u> linear because the transformed variable lnY is linear in β_1 and β_2. However, the second model is intrinsically nonlinear because it is impossible to convert the model into a form linear in the parameters. For a discussion of these concepts see reference [1].

Usually, it is not useful to transform a model of the second type because it remains nonlinear whatever transformation is applied. The transformation introduced in this paper is unusual because it involves computation of the

Fourier coefficients of the sample data and uses phase variables to estimate parameters. For this reason the procedure to be described is called the phase spectrum transformation. The method will be demonstrated for the model specific to the VOR air navigation system.

The VOR is a fundamental component of the present-day air navigation system. A feature of the VOR system which provides much versatility for defining controlled airways is that the facility emits an infinite number of radial courses providing aircraft bearing information. This information is contained in the phase angles of two 30 Hz audiofrequency signals. The first has a constant phase at all points around a VOR station and is called the reference signal. The other, called the variable signal, has a phase equal to the bearing angle to (or from) the VOR transmitter. In the aircraft, bearing information is determined by measuring the phase difference between the two component signals.

The importance of the accuracy of bearing angle estimation devices cannot, of course, be overstated. At present, measurement accuracy for VOR test instruments depends upon calibration with commercial equipment designed for that purpose. As system requirements become more severe because of increasing traffic in the air lanes, it is clear that both the accuracy and precision of present VOR calibration equipment will require additional scrutiny. Hopefully, this will increase the safety and efficiency of aircraft operations. For a general discussion of the VOR system, see [2].

This paper presents a statistical technique for estimation of VOR bearing angle and gives the corresponding precision of the estimated angle. The general method is based on regression analysis of samples taken by a sample-and-hold amplifier and an analog-to-digital converter. The method provides a linear model in relevant phase parameters. Thus, the bearing angle estimation utilizes existing statistical theory.

In section 3 of this technical note, the nonlinear regression model is represented in continuous time. Fourier coefficients are obtained for the noise-free signal, and results for the special case of the VOR signal are stated. The results for the VOR application are extended without proof to the discrete time sample model in section 4. The spectrum for the VOR model with noise is derived and the phase spectrum transformation is defined. In section 5 the approximate linear model for transformed variables is used to estimate unknown parameters and the usefulness of the transformed model is discussed. Section 6 is a limited discussion concerning the properties of estimators if some assumptions specific to the VOR model are invalid.

2. NOTATION

For X, a random variable with probability density f(x), we denote the mean and variance of X by E[X] and Var[X], and the covariance between X and a random variable Y is denoted by Cov[X,Y].

Vectors and matrices are denoted by underlined letters, for example, $\underline{\theta}$ and $\underline{V}$. If $\underline{\theta}$ denotes a vector, then $\underline{\theta}'$ will denote the transpose of $\underline{\theta}$. An estimator of $\underline{\theta}$ will be denoted by $\hat{\underline{\theta}}$.

3. CONTINUOUS TIME NONLINEAR MODEL

The nonlinear model considered in this report will be represented, initially, as a continuous function of time. In a later section the mathematical results obtained for the continuous time model are extended to the case where the data are equally spaced observations from the continuous signal.

To represent the deterministic component of the model requires two periodic functions described below in (3.0.1). These functions are added to obtain the expected (ideal) value of the output signal in a nonlinear regression model. The two component functions are

$$v(t;\delta,\underline{\phi}) = \alpha_1 \cos[2\pi f_1(t+\delta) + \phi_1]$$

and (3.0.1)

$$s(t;\delta,\underline{\phi}) = \alpha_2 \cos\left[2\pi f_2(t+\delta) + \phi_2 + \beta \sin[2\pi f_1(t+\delta) + \phi_3]\right]$$

The waveform generated by the sum of $v(t;\delta,\underline{\phi})$ and $s(t;\delta,\underline{\phi})$, with some parameters assumed known, may be used to represent the ideal audiofrequency signal for the VOR aircraft navigation system. In this context, $v(t;\delta,\underline{\phi})$ is called the <u>variable phase signal</u>, and $s(t;\delta,\underline{\phi})$ is the frequency modulated <u>subcarrier signal</u>. The frequency modulating sinusoid, contained in the argument of $s(t;\delta,\underline{\phi})$, is called the <u>reference phase signal</u>. Equations for the component signals have been presented in a more general form than the VOR application requires. However, the above terminology is used throughout this paper. Following are descriptions of the model parameters:

α_1 = amplitude of variable phase signal;

α_2 = amplitude of subcarrier signal;

f_1 = variable (and reference) signal frequency;

f_2 = subcarrier frequency;

β = modulation index;

δ = arbitrary fixed time offset;

ϕ_1 = phase angle of variable phase signal;

ϕ_2 = phase angle of subcarrier signal;

ϕ_3 = phase angle of reference phase signal.

The fixed time offset, δ, is included in (3.0.1) because the output signal will be observed and sampled from an unknown starting point in the waveform. We cannot, in general, be assured that observation of the signal begins at a zero crossing on the time axis.

Realistically, measurement of the composite signal involves random measurement error in some form. In this paper the random error process, $e(t)$, is assumed to be additive white noise [3], and the resulting process $Y(t)$ can be represented by

$$Y(t) = \mu + v(t;\delta,\underline{\phi}) + s(t;\delta,\underline{\phi}) + e(t) \ , \qquad (3.0.2)$$

where μ is a fixed but unknown offset. Unless otherwise specified, $e(t)$ is not assumed to be Gaussian white noise. However, for the distributional result obtained in the appendix, we require normality and independence of the discrete time error series to be as described in subsection 4.1.

In subsection 3.2, a form of the model specific to VOR navigation system is discussed. For this application some of the parameters in the general form of the model are assumed to be fixed, known values. On this basis a statistical method for VOR bearing angle estimation will be derived. Because the relevant angle for the VOR application is $(\phi_1-\phi_3)$, it is sufficient to consider a reparameterized form of the model where we define new parameters $\underline{\theta}$ by

$$\theta_1 = \phi_1 - \phi_3$$

$$\theta_2 = 2\pi f_1 \delta + \phi_3$$

$$\theta_3 = 2\pi f_2 \delta + \phi_2 \ .$$

For this parameterization the general model becomes

$$Y(t) = \mu + v(t;\underline{\theta}) + s(t;\underline{\theta}) + e(t) \qquad (3.0.3)$$

where

$$v(t;\underline{\theta}) = \alpha_1 \cos[2\pi f_1 t + \theta_1 + \theta_2]$$

and

$$s(t;\underline{\theta}) = \alpha_2 \cos\left[2\pi f_2 t + \theta_3 + \beta \sin[2\pi f_1 t + \theta_2]\right] .$$

In the following section, we obtain the Fourier sine and cosine transforms of $E[Y(t)]=\mu+v(t;\underline{\theta})+s(t;\underline{\theta})$ under the assumption that $f_2=mf_1$. Hence, $E[Y(t)]$ is periodic with period $(1/f_1)$. Utilizing the general result, the specific transform for the VOR signal is determined in subsection 3.2.

3.1 Fourier Representation of General Model

Let $y(t;\underline{\theta})$ denote the expected value of $Y(t)$. The deterministic function $y(t;\underline{\theta})$ is given by

$$y(t;\underline{\theta}) = \mu + v(t;\underline{\theta}) + s(t;\underline{\theta}).$$

Suppose that the frequencies f_1 and f_2 in the definition of $v(t;\underline{\theta})$ and $s(t;\underline{\theta})$ are such that $f_2=mf_1$ for some positive integer m. Thus, $y(t;\underline{\theta})$ is periodic with period $(1/f_1)$. Under this assumption, the Fourier coefficients of $y(t;\underline{\theta})$ can be obtained from the real and imaginary parts of the complex integral

$$s_k = 2f_1 \int_{-1/2f_1}^{1/2f_1} y(t;\underline{\theta}) \exp[i2\pi f_1 kt]\, dt, \quad k=0,\pm 1,\pm 2, \ldots$$

Let a_k and b_k denote the real and imaginary parts of s_k, so that $s_k = a_k + ib_k$. That is, a_k and b_k are the Fourier cosine and sine transforms, respectively, of $y(t;\underline{\theta})$. In the appendix, it is shown that the Fourier coefficients a_k and b_k for $\overline{k}=0,1,\ldots$, are

$$a_k = \begin{cases} \mu & , k=0 \\ \alpha_1 \cos[\theta_1+\theta_2] + \alpha_2\, a_k(s) & , k=1 \\ \alpha_2\, a_k(s) & , k\geq 2 \end{cases}$$

and (3.1.1)

$$b_k = \begin{cases} 0 & , k=0 \\ -\alpha_1 \sin[\theta_1+\theta_2] - \alpha_2\, b_k(s) & , k=1 \\ -\alpha_2\, b_k(s) & , k\geq 2 \end{cases}$$

where, for $k\geq 1$,

$$a_k(s) = J_{k-m}(\beta)\cos[\theta_3 + (k-m)\theta_2] + J_{-k-m}(\beta)\cos[\theta_3 - (k+m)\theta_2]$$

and

$$b_k(s) = J_{k-m}(\beta)\sin[\theta_3 + (k-m)\theta_2] + J_{-k-m}(\beta)\sin[\theta_3 - (k+m)\theta_2] .$$

The notation $J_n(z)$ denotes a Bessel function of the first kind [4].

Equations (3.1.1) represent the Fourier coefficients for the mean value function of the general continuous time model where the subcarrier frequency f_2 is an integer multiple of f_1, the frequency of the variable phase signal. For the VOR model, where some parameters in the general model are assumed known, we will see that the Fourier coefficients can be greatly simplified. The form of the coefficients for this special case will suggest a method for estimating the unknown parameters.

3.2 Fourier Representation of VOR Signal

The VOR audio frequency waveform consists of a 30 Hz variable phase signal linearly added to a frequency modulated 9960 Hz subcarrier signal. The modulation index for the reference phase signal is assumed to be fixed and known, as are the amplitudes of each signal. Specifically, parameter values assumed to be known are:

$$\mu = 0$$
$$\alpha_1 = 2^{1/2}$$
$$\alpha_2 = 2^{1/2}$$
$$f_1 = 30$$
$$f_2 = 9960 \text{ (so } m = f_2/f_1 = 332)$$
$$\beta = 16 \quad .$$

These specifications define the VOR model

$$Y(t) = y(t;\underline{\theta}) + e(t)$$

where

$$y(t;\underline{\theta}) = 2^{1/2}\{\cos[2\pi 30t + \theta_1 + \theta_2] + \cos[2\pi 9960t + \theta_3 + 16\sin[2\pi 30t + \theta_2]]\}.$$

Noting that $y(t;\underline{\theta})$ is nonlinear in the unknown parameters $\underline{\theta}'=[\theta_1,\theta_2,\theta_3]$, we will show in section 4 that the spectrum of $Y(t)$ can be used to transform a set of sample data to new observations satisfying a model approximately linear in $\underline{\theta}$. The transformation to linearity will depend on the simplified form of the Fourier coefficients for $y(t;\underline{\theta})$ when known values of parameters in the VOR model are substituted in the general equations (3.1.1).

Substituting known values in the expressions for $a_k(s)$ and $b_k(s)$, we obtain for $k>1$,

$$a_k(s) = J_{k-332}(16)\cos[\theta_3+(k-332)\theta_2] + J_{-k-332}(16)\cos[\theta_3-(k+332)\theta_2]$$

and

$$b_k(s) = J_{k-332}(16)\sin[\theta_3+(k-332)\theta_2] + J_{-k-332}(16)\cos[\theta_3-(k+332)\theta_2].$$

We need the following results for Bessel functions of the first kind.

Lemma 3.1: [4, page 358] For integer n, $J_n(z)$ satisfies

$$J_{-n}(z) = (-1)^n J_n(z) \; .$$

Lemma 3.2: [4, page 365] For fixed z, as $n\to\infty$ through real positive values,

$$J_n(z) \cong (2\pi n)^{-1/2}(ez/2n)^n \; .$$

From lemmas 3.1 and 3.2, it follows that

$$
\begin{aligned}
J_{-k-332}(16) &= (-1)^{k+332} J_{k+332}(16) \\
&\cong (-1)^{k+332}[2\pi(k+332)]^{-1/2}[16e/2(k+332)]^{k+332} \\
&\cong 0
\end{aligned}
$$

Clearly, the value of $J_{k+332}(16)$ is immeasurably small. Note, for example, that if k=1, $J_{k+332}(16) \cong .02(.065)^{333}$. Similarly, for small k, we have that $J_{k-332} \cong 0$.

Using the above results, we get

$$
a_k(s) \cong \begin{cases} 0 & , k=1 \\ J_{k-332}(16)\cos[\theta_3 + (k-332)\theta_2] & , k \geq 2 \end{cases} .
$$

and

$$
b_k(s) \cong \begin{cases} 0 & , k=1 \\ J_{k-332}(16)\sin[\theta_3 + (k-332)\theta_2] & , k \geq 2 \end{cases} .
$$

Substituting in the general expressions (3.1.1), the approximate Fourier coefficients for $y(t;\underline{\theta})$ in the VOR model become

$$
a_k \cong \begin{cases} 0 & , k=0 \\ 2^{1/2} \cos[\theta_1 + \theta_2] & , k=1 \\ 2^{1/2} J_{k-332}(16)\cos[\theta_3 + (k-332)\theta_2] & , k \geq 2 \end{cases}
$$

and

$$
b_k \cong \begin{cases} 0 & , k=0 \\ -2^{1/2} \sin[\theta_1 + \theta_2] & , k=1 \\ -2^{1/2} J_{k-332}(16)\sin[\theta_3 + (k-332)\theta_2] & , k \geq 2. \end{cases}
$$

Because omitted terms are negligible, in the following sections we consider the Fourier coefficients to be exact.

4. DISCRETE TIME NONLINEAR MODEL

The general model and corresponding Fourier transforms introduced in section 3 will facilitate a later discussion about errors in assumptions for the VOR model. Because the approximations discussed in the previous section depend on the particular values of some parameters in the general model, the phase spectrum transformation will be developed only for the VOR specifications. In subsection 4.1 we consider the discrete time analogs of the VOR model and corresponding Fourier coefficients, since a digital phase estimation technique is desired.

4.1 Fourier Coefficients of VOR Signal

Let Y_j, $j=1,\ldots,N$, be N equally spaced observations from one period of the continuous time VOR series. For convenience, N is assumed to be even in the results to follow. The sample model for VOR applications can be written

$$\left.\begin{array}{c} Y_j = y_j(\underline{\theta}) + e_j \\ E[e_j]=0;\ \mathrm{Var}[e_j]=\sigma^2;\ E[e_j e_k]=0 \text{ if } j\neq k \end{array}\right\}\ j,k=1,\ldots,N$$

where

$$y_j(\underline{\theta}) = 2^{1/2}\{\cos[2\pi(j-1)/N + \theta_1 + \theta_2] + \cos[2\pi 332(j-1)/N + \theta_3 + 16\sin[2\pi(j-1)/N + \theta_2]]\}.$$

The e_j's denote uncorrelated random error terms with unknown variance σ^2.

The Fourier coefficients ($s_k = a_k + ib_k$) for $y_j(\underline{\theta})$ are given by

$$s_k = (2/N)\sum_{j=1}^{N} y_j(\underline{\theta})\exp[i2\pi k(j-1)/N] .$$

The a_k and b_k follow directly from the continuous time model. Excluding the coefficients for k=0, which are not useful to estimate $\underline{\theta}$, equations for a_k and b_k are

$$a_k = \begin{cases} 2^{1/2}\cos[\theta_1+\theta_2] & ,\ k=1 \\ 2^{1/2} J_{k-332}(16)\cos[\theta_3 + (k-332)\theta_2], & k=2,3,\ldots,N/2, \end{cases}$$

and (4.1.1)

$$b_k = \begin{cases} -2^{1/2}\sin[\theta_1+\theta_2] & ,\ k=1 \\ -2^{1/2} J_{k-332}(16)\sin[\theta_3 + (k-332)\theta_2], & k=2,3,\ldots,N/2-1 . \end{cases}$$

Recall that these coefficients can reasonably be considered exact expressions because omitted terms are negligible. The interesting feature of the equations for a_k and b_k is the form of the phase of the k^{th} harmonic. If we let q represent the phase at a chosen harmonic, then the basic equation for q is tan(q)=-b/a. For convenience, we have dropped the subscripts on q, b, and a. Because Arctan(-b/a) gives the same value for -b and -a as for b and a, the full solution for q in the interval $(-\pi,\pi]$ is the following [5, page 12]:

$$q^* = \begin{cases} \text{Arctan}(-b/a), & a>0 \\ \text{Arctan}(-b/a)-\pi, & a<0,\ b>0 \\ \text{Arctan}(-b/a)+\pi, & a<0,\ b<0 \\ -\pi/2, & a=0,\ b>0 \\ \pi/2, & a=0,\ b<0 \\ \text{arbitrary}, & a=0,\ b=0. \end{cases} \tag{4.1.2}$$

The notation Arctan(x) is used to denote the principal value, so that $-\pi/2<\text{Arctan}(x)<\pi/2$. If arctan(x) denotes <u>any</u> angle whose tangent is x, then it follows that

$$\arctan(-b/a) = q^* + j2\pi \tag{4.1.3}$$

where j is an arbitrary integer. From (4.1.3) and the expressions for a_k and b_k, it then follows that there exist integers γ_k such that $q_k(\underline{\theta})=q_k{}^*+\gamma_k 2\pi$ is given by

$$q_k(\underline{\theta}) = \begin{cases} \theta_1 + \theta_2, & k=1 \\ (k-332)\theta_2 + \theta_3, & |k-332|<K, \end{cases} \tag{4.1.4}$$

where K is a constant chosen to assure that $J_{k-332}(16)$ is non-negligible. A zero value for the Bessel function leads to an arbitrary phase because $a_k=b_k=0$. Table 1 lists the values of $J_n(16)$ for n=0,...,24. In a later section it will be shown that a value of K≈10 is sufficient for the proposed estimation of θ.

Table 1. Bessel Functions

n	$J_n(16)$	n	$J_n(16)$
0	-.1748990739		
1	.0903971756	13	.2368225047
2	.1861987209	14	.2724363352
3	-.0438474954	15	.2399410820
4	-.2026415317	16	.1774531934
5	-.0574732704	17	.1149653049
6	.1667207377	18	.0668480795
7	.1825138237	19	.0354428740
8	-.0070211419	20	.0173287462
9	-.1895349656	21	.0078789915
10	-.2062056944	22	.0033536066
11	-.0682221523	23	.0013434266
12	.1124002349	24	.0005087450

The multiples of 2π indexed by γ in the expression for $q_k(\underline{\theta})$ are necessary to adjust the q_k^* from the interval $(-\pi,\pi]$ to the interval $(-\infty,\infty)$. Because the θ's will represent unknown parameters, the γ's are not known in general. If we assume, however, that $\theta_2 \geq 0$, it is clear from (4.1.4) that the γ's must satisfy

$$q_j^* + \gamma_j 2\pi \leq q_i^* + \gamma_i 2\pi \ , \ j<i \ ; \ i\neq 1\neq j \quad ,$$

so (4.1.5)

$$(\gamma_j - \gamma_i)2\pi \leq q_i^* - q_j^* \ , \ j<i \ ; \ i\neq 1\neq j \quad .$$

This implies that for $|k-332|\leq K$, the γ_k's can be uniquely determined if, for example, the following conditions are assumed:

$$-\pi<\theta_1+\theta_2\leq\pi$$

$$\theta_2 \geq 0$$

$$-\pi<\theta_3\leq\pi \quad .$$

With these constraints $\gamma_1=\gamma_{332}=0$ and successive values of γ_k near k=332 are determined from (4.1.5). For the extension of the results to the VOR signal with error we will assume that the γ's are known.

4.2 Spectrum of VOR Noise Model

In the previous subsection Fourier coefficients were obtained for the deterministic component of the VOR process. For the nonlinear VOR regression model:

$$\left.\begin{array}{c} Y_j = y_j(\underline{\theta}) + e_j \\ E[e_j]=0;\ Var[e_j]=\sigma^2;\ E[e_j e_k]=0 \text{ if } j\neq k \end{array}\right\} \ j,k=1,\ldots,N$$

the Fourier coefficients of $y_j(\underline{\theta})$, represented by $s_k=a_k+ib_k$, were shown to have phase values linear in $\underline{\theta}$. In this section we prove that random phase variables derived from Fourier transformation of Y_j, j=1,...,N are appropriately represented by a regression model linear in $\underline{\theta}$.

Letting the random variables $S_k=A_k+iB_k$, k=1,...,N/2-1, represent the Fourier coefficients of Y_j, we have

$$S_k = (2/N)\sum_{j=1}^{N} Y_j \exp\left[i2\pi k(j-1)/N\right]$$

$$= (2/N)\sum_{j=1}^{N} \left(y_j(\underline{\theta})+e_j\right)\exp\left[i2\pi k(j-1)/N\right]$$

$$= s_k + (2/N)\sum_{j=1}^{N} e_j \exp\left[i2\pi k(j-1)/N\right]$$

where $s_k=a_k+ib_k$ are the Fourier coefficients for $y_j(\underline{\theta})$ given in (4.1.1). If we let the transform of the random error sequence be denoted by

$$g_k + ih_k = (2/N) \sum_{j=1}^{N} e_j \exp\left[i2\pi k(j-1)/N\right] \quad ,$$

it is well known [6, page 110] that the random variables g_k and h_k, $k=1,\ldots,N/2-1$ are mutually uncorrelated and

$$\left.\begin{aligned} E[g_k] &= E[h_k] = 0 \\ Var[g_k] &= Var[h_k] = (2/N)\sigma^2 \end{aligned}\right\} \quad k=1,\ldots,N/2-1$$

It therefore follows that the A_k and B_k are uncorrelated and satisfy regression models

$$\left.\begin{aligned} A_k &= a_k + g_k \\ B_k &= b_k + h_k \end{aligned}\right\} \quad k=1,\ldots,N/2-1$$

Linearity of $\arctan(-b_k/a_k)$ in $\underline{\theta}$ suggests that we consider the phase spectrum of Y_j to estimate $\underline{\theta}$. In the next section it is shown that the expected values of phase variables are approximately linear in $\underline{\theta}$.

4.3 Phase Spectrum Transformation

The definition of phase random variables parallels the description of the phase $q_k(\underline{\theta})$ for the deterministic component $y_j(\underline{\theta})$. Phase variables will be denoted by Q_k and initially are defined using principal values in the interval $(-\pi,\pi]$. Define

$$Q_k{*} = \text{Arctan}\left[-B_k/A_k\right] \;,\; k=1,\ldots,N/2-1 \;. \qquad (4.3.1)$$

In the appendix the distribution of the $Q_k{*}$, $k=1,\ldots,N/2-1$ is determined when the errors are Gaussian. The expected value of $Q_k{*}$ is not obtained, but the complexity of the distribution illustrates the usefulness of approximate moments of $Q_k{*}$ which result from a suitable propagation of errors formula.

To conclude this subsection we obtain approximate formulas for the mean and variance of $Q_k{*}$. These results are the basis for the linear models used to estimate θ in Section 5. As defined, A_k and B_k appearing in (4.3.1) satisfy:

$$\left.\begin{aligned} &E[A_k] = a_k \;;\; E[B_k] = b_k \\ &Var[A_k] = Var[B_k] = (2/N)\sigma^2 \\ &Cov[A_k,B_j] = 0 \quad \text{all } j,k \end{aligned}\right\} \quad k,j=1,\ldots,N/2-1$$

For values of k such that $a_k \neq 0 \neq b_k$, it can be shown [7, page 333] that, to order N^{-2}, Q_k^* has approximate mean and variance given by:

$$E[Q_k^*] \cong \text{Arctan}[-b_k/a_k]$$
$$\text{Var}[Q_k^*] \cong (2/N) r_k^{-2} \sigma^2 \qquad (4.3.2)$$

where

$$r_k^2 = a_k^2 + b_k^2 = \begin{cases} 2 & , \quad k=1 \\ 2J^2_{k-332}(16), & |k-332| < K \ . \end{cases}$$

The value of K is chosen to assure that $J_{k-332}(16)$ is nonzero. Note that $E[Q_k^*] \cong q_k^*$, where the solution for q_k^* in the interval $(-\pi, \pi]$ was given in (4.1.2). Based on the discussion following (4.1.2) we can define $Q_k = Q_k^* + \gamma_k 2\pi$ where the γ_k's are (generally) unknown integers such that

$$E[Q_k] \cong q_k(\underline{\theta})$$
$$\text{Var}[Q_k] \cong (2/N) r_k^{-2} \sigma^2$$

where

$$q_k(\underline{\theta}) = \begin{cases} \theta_1 + \theta_2 & , \quad k=1 \\ (k-332)\theta_2 + \theta_3 & , \quad |k-332| < K \ . \end{cases} \qquad (4.3.3)$$

The method of estimating $\underline{\theta}$ to be outlined in Section 5 is based on the above results. To the chosen degree of approximation, the important features are:

1. $E[Q_k]$ is linear in $\underline{\theta}$ for all permissible k.

2. $\text{Var}[Q_k]$ is proportional to σ^2 with known constant of proportionality.

3. For all permissible k, the Q_k's are mutually uncorrelated (assuming white noise errors in the original model).

Because the linear model of Section 5 requires that Q_k be an observable random variable, it will be necessary to assume that the γ_k's are known integers. The γ_k's are used to adjust the computed values of the arctan function to satisfy inequalities implied by (4.3.3). Though it is not obvious that this correction can be accomplished with the Q_k, which are subject to error, computer simulations indicate that the adjustment is possible if the measurement error variance, σ^2, is small. Values used for σ^2 are thought to be representative of measurement precision for a new system designed to obtain sample values Y_j, $j=1,\ldots,N$.

5. LINEAR MODEL FOR PHASE SPECTRUM

In the previous section approximate formulas for the means of phase random variables for the nonlinear VOR model were shown to be linear in the unknown parameters $\underline{\theta}$. Corresponding variance approximations are unequal at the harmonics and proportional to σ^2, but do not depend on other unknown parameters. The additional observation that phase random variables are uncorrelated suggests that $\underline{\theta}$ and σ^2 may be estimated using a linear model in $\underline{\theta}$ with known error covariances. For the description to follow the reader is reminded that expressions for the mean and variance of Q_k are <u>not</u> stated as approximations.

Consider the n=2+2K equations

$$Q_k = \begin{cases} \theta_1 + \theta_2 + \varepsilon_1 & , k=1 \\ (k-332)\theta_2 + \theta_3 + \varepsilon_k & , k=332-K,\ldots,331,332,333,\ldots,332+K \end{cases}$$

where ε_k represents a random error term such that

$$E[\varepsilon_k]=0 \;;\; Var[\varepsilon_k]=(2/N)r_k^{-2}\sigma^2;$$

$$Cov[\varepsilon_k,\varepsilon_j]=0 \text{ if } k\neq j \;;$$

and where

(1) the Q_k are <u>observable</u> random variables;

(2) the r_k^2 are known constants.

(3) θ_1, θ_2, θ_3, and σ^2 are unknown parameters.

The model can be represented as a single matrix equation

$$\underline{Q} = \underline{X}\underline{\theta} + \underline{\varepsilon} \qquad E[\underline{\varepsilon}] = \underline{0} \qquad Cov[\underline{\varepsilon}]=\sigma^2\underline{V} \tag{5.0.1}$$

where the vectors and matrices are

$$\underline{Q} = \begin{bmatrix} Q_1 \\ Q_{332-K} \\ \cdot \\ \cdot \\ \cdot \\ Q_{331} \\ Q_{332} \\ Q_{333} \\ \cdot \\ \cdot \\ \cdot \\ Q_{332+K} \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -K & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & K & 1 \end{bmatrix} \qquad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_{332-K} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{331} \\ \varepsilon_{332} \\ \varepsilon_{333} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{332+K} \end{bmatrix} \qquad \underline{V} = (1/N) \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & [J_{-K}(16)]^{-2} & & & \\ & & \ddots & & \\ \vdots & & & [J_0(16)]^{-2} & \vdots \\ & & & \ddots & \\ 0 & & \cdots & & [J_K(16)]^{-2} \end{bmatrix}$$

For these specifications of the linear model, unbiased estimators of $\underline{\theta}$ and σ^2 are given by [8, page 207]:

$$\begin{aligned} \hat{\underline{\theta}} &= (\underline{X}'\underline{V}^{-1}\underline{X})^{-1}\underline{X}'\underline{V}^{-1}\underline{Q} \\ \hat{\sigma}^2 &= \frac{1}{n-3}\,\underline{Q}'[\underline{V}^{-1}-\underline{V}^{-1}\underline{X}(\underline{X}'\underline{V}^{-1}\underline{X})^{-1}\underline{X}'\underline{V}^{-1}]\underline{Q} \ . \end{aligned} \qquad (5.0.2)$$

Variances and covariances of the $\hat{\theta}$'s can be estimated by substituting the estimator of σ^2 in

$$\mathrm{Cov}[\hat{\underline{\theta}}] = \sigma^2(\underline{X}'\underline{V}^{-1}\underline{X})^{-1} \qquad (5.0.3)$$

The quantities listed above are those needed for point estimation and confidence intervals involving $\underline{\theta}$ and σ^2.

5.1 VOR Bearing Angle Estimator

According to the reparameterization of the original nonlinear model, the relevant phase angle for VOR applications is θ_1 . The estimator of θ_1 can be obtained by algebraic expansion of the matrix equation in (5.0.2), and its corresponding variance is the first element of the square matrix in (5.0.3). The estimator of VOR bearing angle and its estimated variance are

$$\hat{\theta}_1 = Q_1 - C_k \sum_{j=1}^{K} j\, J_j^2(16)[Q_{332+j} - Q_{332-j}] \qquad (5.1.1)$$

$$\hat{\mathrm{Var}}[\hat{\theta}_1] = (1/N)(1+C_K)\hat{\sigma}^2$$

where

$$C_K = [2 \sum_{j=1}^{K} j^2 J_j^2(16)]^{-1}$$

Note that the variance of the bearing angle estimator depends on the selected number of phase values through C_K. Because the Bessel function $J_j(16)$ approaches zero as j increases, it is clear that C_K, and hence $Var[\hat{\theta}_1]$, approaches a lower bound as K increases. In practice, computational speed and/or memory constraints may require that only a few phase observations be used to estimate θ_1 . Values of C_j listed in Table 2 suggest that K=10 is sufficient to minimize C_K .

Table 2. Bessel Functions and Weights

j	$J_j^2(16)$	$j^2 J_j^2(16)$	C_j
0	0.0305896861	0.0000000000	–
1	0.0081716494	0.0081716494	61.2080
2	0.0346699637	0.1386798547	3.4032
3	0.0019226029	0.0173034257	3.0448
4	0.0410635904	0.6570174461	.5872
5	0.0033031768	0.0825794204	.5360
6	0.0277958044	1.0006489577	.2800
7	0.0333112958	1.6322534965	.1264
8	0.0000492964	0.0031549718	.1264
9	0.0359235032	2.9098037600	.0752
10	0.0425207884	4.2520788410	.0240
11	0.0046542621	0.5631657107	.0240
12	0.0126338128	1.8192690450	.0240
13	0.0560848988	9.4783478893	.0240
14	0.0742215568	14.5474251288	.0240
15	0.0575717228	12.9536376362	.0240
16	0.0314896359	8.0613467818	.0240
17	0.0132170213	3.8197191653	.0240
18	0.0044686657	1.4478476965	.0240
19	0.0012561973	0.4534872292	.0240
20	0.0003002854	0.1201141759	.0240
21	0.0000620785	0.0273766197	.0240
22	0.0000112467	0.0054433896	.0240
23	0.0000018048	0.0009547344	.0240
24	0.0000002588	0.0001490789	.0240

However, the apparent gain from using using only a few phase variates is balanced by a corresponding loss in precision for estimating σ^2 . According to the specifications for equipment designed to provide the sample values from a VOR signal, it is likely that K=10 will provide adequate precision for estimating θ_1 .

The procedure described above can be used to achieve acceptable precision bounds using only a few lines in the phase spectrum of the observed VOR signal. An alternative method, which may require fewer observations, is to obtain an estimate of θ_1 from a few phase variables not adjacent to Q_{332} . To illustrate this approach, values of $j^2J_j^2(16)$ that appear in C_K are listed in Table 2. The maximum value of this quantity occurs if j=14, corresponding to phase observations at k=318 and k=346. Clearly, to minimize the variance of an estimator of θ_1 based on nonadjacent Q_k's, one should add observations in order of decreasing values on $j^2J_j^2(16)$. Thus, phase variate pairs would be included in the order j=14,15,13,16, etc., corresponding to k=(318,346), (317,347),(319,345),(316,348), etc. It is easy to show that the multiplier analogous to C_K is already near the lower bound of Table 2 after only four phase pairs are included to estimate θ_1. Hence, if estimation of σ^2 is not severely affected, a significant saving in computational requirements is achieved using phase variables nonadjacent to Q_{332}. Assuming that N is moderately large, the gain is especially desirable if Discrete Fourier Transforms are used to obtain the Q_k.

5.2 Discussion

It can be argued that a transformation of the nonlinear model to linear form is unnecessary because suitable nonlinear least squares methods can be applied directly to the sample data. These methods are iterative and require initial estimates of the unknown phase angles. However, for VOR applications, software for estimating unknown angles will be implemented on desktop computers, which will also serve as controllers in VOR calibration systems. In this case, the phase spectrum transformation and subsequent estimation of phase angles using the linear models approach is computationally efficient and is to be preferred if there are no serious deficiencies in the technique.

From a mathematical standpoint, estimation of $\underline{\theta}$ using the phase spectrum transformation depends on two related assumptions. First, it was implicitly assumed that formulas for the mean and variance of phase random variables approximate the true mean and variance to the extent that departures from the correct values are negligible. A second assumption, which requires further study, concerns the adjustment of computed values of the Arctan function. Recall that $Q_k=Q_k^*+\gamma_k 2\pi$, where Q_k^* is an observable random variable in the interval $(-\pi,\pi]$. In the derivation of the estimators of $\underline{\theta}$ and σ^2, it was assumed that the integers $\{\gamma_k\}$ can be determined from the data. If the γ_k's are not known, then $\underline{\hat{\theta}}$ and $\hat{\sigma}^2$ in (5.0.2) are not estimators because they are not observable.

A computer simulation of the VOR signal with independent Gaussian errors was used to determine if the assumptions described above severely limit applicability of the phase spectrum transformation. Results of this investigation indicate that linearity of the mean and the ability to adjust the Arctan function depend, primarily, on variability in measurement errors. The technique was applied consistently for values of σ^2 less than .001. For somewhat greater values of σ^2, straightforward determination of the γ's is usually successful, and indications are that the method can be refined, perhaps by developing a search technique for the γ's. Analyses were conducted with N=1024 time samples.

It should be emphasized that values of σ^2 used in computer simulations are believed to be representative of expected variability of measurement errors for a VOR audiofrequency generator currently being constructed. Computations based on simulated data were used to affirm the mathematical results of previous sections and are not reported here.

To conclude this section we remark that hardware specifications for VOR generators and the method for sampling the continuous time signal will together determine the accuracy and reliability of specified values for frequency, amplitude, offset, and modulation index. Because properties of measurement errors, such as stability and independence, can be affected by hardware and software specifications, examination of estimated residuals for the time samples can be useful to validate assumptions about sampling errors. Estimated residuals for the VOR model are given by

$$\hat{\varepsilon}_j = Y_j - y_j(\underline{\hat{\theta}}) \quad , \quad j=1,\ldots,N \quad .$$

Plots of residuals and/or tests for serial correlation can be expected to reveal inconsistent or unusual properties of a particular VOR measurement system. Detection of a problem may require redesign of the system or a modification of the estimation method developed in this report.

6. ROBUSTNESS OF ASSUMPTIONS

In this section, two generalizations of the VOR model are examined to understand the consequences if values of some parameters assumed to be known are in error. To facilitate the discussion, we state a modification of the VOR model which is sufficient for the generalizations considered in this section:

$$Y(t) = v(t;\underline{\theta}) + s(t;\underline{\theta}) + e(t)$$

where

$$v(t;\underline{\theta}) = \alpha_1 \cos[2\pi 30t + \theta_1 + \theta_2]$$

and

$$s(t;\underline{\theta}) = \alpha_2 \cos[2\pi 9960t + 16\sin[2\pi 30t + \theta_2]] \quad .$$

Recall that for VOR applications we assumed that $\alpha_1=\alpha_2=2^{1/2}$.

6.1 Signal Offset

Let Y(t) denote a signal with $\alpha_1=\alpha_2=2^{1/2}$. Suppose that instead of Y(t) we observe $Y(t)+\mu$ where $\mu\neq 0$. The Fourier transform of the observed process is:

$$60 \int_{-1/60}^{1/60} [Y(t) + \mu]\exp[i2\pi 30kt]\, dt$$

$$= \begin{cases} S_0 + 30\mu & , \quad k=0 \\ S_k & , \quad k\neq 0 \end{cases} \quad ,$$

where S_k is the transform of Y(t). Therefore, signal offset does not affect the previous results because S_0 was not used to estimate $\underline{\theta}$.

6.2 Signal Amplitude

In subsection 3.1 we obtained the Fourier coefficients of a signal more general than the VOR requires. The derivation of essential results that followed in no way depended on the particular values of the amplitude parameters α_1 and α_2. If $y(t;\underline{\theta})=v(t;\underline{\theta})+s(t;\underline{\theta})$ denotes a VOR signal with α_1 and α_2 unspecified, and $s_k=a_k+ib_k$ denotes the corresponding transform of $y(t;\underline{\theta})$, then it is easily shown that

$$a_k \cong \begin{cases} \alpha_1 \cos[\theta_1 + \theta_2] & , k=1 \\ \alpha_2 J_{k-332}(16)\cos[\theta_3+(k-332)\theta_2], & k>2 \end{cases}$$

and

$$b_k \cong \begin{cases} -\alpha_1 \sin[\theta_1 + \theta_2] & , k=1 \\ -\alpha_2 J_{k-332}(16)\sin[\theta_3+(k-332)\theta_2], & k>2 \end{cases}$$

Clearly, phase computations using $\tan(q_k)=-b_k/a_k$, which are fundamental to the estimation method, are invariant with respect to particular values of α_1 and α_2, even if $\alpha_1 \neq \alpha_2$.

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APPENDIX

The appendix presents the derivation of Fourier coefficients in equation (3.1.1), and obtains the distribution of phase variables when errors in the VOR model are Gaussian.

A.1 Derivation of Fourier Coefficients

In this section we determine the Fourier coefficients for the function

$$y(t;\underline{\theta}) = \mu + v(t;\underline{\theta}) + s(t;\underline{\theta}) \tag{A.1.1}$$

where

$$v(t;\underline{\theta}) = \alpha_1 \cos[2\pi f_1 t + \theta_1 + \theta_2]$$

and

$$s(t;\underline{\theta}) = \alpha_2 \cos[2\pi m f_1 + \theta_3 + \beta \sin[2\pi f_1 + \theta_2]]$$

The following results and trigonometric identities are needed in the derivation:

$$\int_{-1/2f}^{1/2f} \cos(2\pi fjt)\cos(2\pi fkt)dt = \begin{cases} 1/2f, & j=k\neq 0 \\ 0, & j\neq k \end{cases} \tag{A.1.2}$$

$$\int_{-1/2f}^{1/2f} \cos(2\pi fjt)\sin(2\pi fkt)dt = 0, \quad \text{all } j,k \tag{A.1.3}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{A.1.4}$$

$$\cos\alpha\cos\beta = (1/2)[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \tag{A.1.5}$$

$$\int_0^{\pi} \cos(nt - z\sin t)dt = \pi J_n(z)\,. \tag{A.1.6}$$

The Fourier coefficients of $y(t;\underline{\theta})$ are obtained from the complex integral

$$a_k + ib_k = 2f_1 \int_{-1/2f_1}^{1/2f_1} y(t;\underline{\theta})\exp[i2\pi f_1 kt]dt$$

where a_k and b_k can be represented by

$$\left.\begin{aligned} a_k &= a_k(\mu) + a_k(v) + a_k(s) \\ b_k &= b_k(\mu) + b_k(v) + b_k(s) \end{aligned}\right\} k=0,\pm1,\pm2,\cdots.$$

The components of a_k and b_k represent Fourier coefficients of respective terms in (A.1.1). Since μ is a constant, it is clear that

$$a_k(\mu) = \begin{cases} 2\mu , & k=0 \\ 0 , & k\neq 0 \end{cases}$$

and

$$b_k(\mu) = 0 , \quad \text{all } k$$

Following is a derivation of $a_k(v)$ and $a_k(s)$. The corresponding Fourier sine transforms, $b_k(v)$ and $b_k(s)$, are easily deduced from these results.

For non-negative integer values of k, we have

$$a_k(v) = 2f_1 \int_{-1/2f_1}^{1/2f_1} \alpha_1 \cos[2\pi f_1 t + \theta_1 + \theta_2]\cos(2\pi f_1 kt)dt$$

$$= 2f_1\alpha_1 \cos(\theta_1 + \theta_2) \int_{-1/2f_1}^{1/2f_1} \cos(2\pi f_1 t)\cos(2\pi f_1 kt)dt$$

$$- 2f_1\alpha_1 \sin(\theta_1 + \theta_2) \int_{-1/2f_1}^{1/2f_1} \sin(2\pi f_1 t)\cos(2\pi f_1 kt)dt$$

$$= \begin{cases} \alpha_1 \cos(\theta_1 + \theta_2) , & k=1 \\ 0 , & k\neq 1 \end{cases}$$

where we have used (A.1.2) to (A.1.4). Similarly,

$$b_k(v) = \begin{cases} -\alpha_1 \sin(\theta_1 + \theta_2) , & k=1 \\ 0 , & k\neq 1 \end{cases}$$

The Fourier coefficients $a_k(s)$ are given by

$$a_k(s) = 2f_1 \int_{-1/2f_1}^{1/2f_1} \alpha_2 \cos[2\pi m f_1 t + \theta_3 + \beta \sin[2\pi f_1 t + \theta_2]]\cos(2\pi f_1 kt)dt.$$

Substituting $u=2\pi f_1 t+\theta_2$, so $dt=(2\pi f_1)^{-1}$, we get

$$a_k(s) = \pi^{-1}\alpha_2 \int_{-\pi}^{\pi} \cos[mu - m\theta_2 + \theta_3 + \beta \sin u]\cos(ku - k\theta_2)\, du,$$

where the limits $-\pi+\theta_2 \leq u \leq \pi+\theta_2$ can be replaced by $-\pi \leq u \leq \pi$ because the integrand has period 2π. Furthermore, using the identity (A.1.5), we obtain

$$a_k(s) = (2\pi)^{-1}\alpha_2 \int_{-\pi}^{\pi} \cos\left[(m-k)u - (m-k)\theta_2 + \theta_3 + \beta \sin u\right]du$$

$$+ \int_{-\pi}^{\pi} \cos\left[(m+k)u - (m+k)\theta_2 + \theta_3 + \beta \sin u\right]du,$$

and using (A.1.4) we get

$$a_k(s) = (2\pi)^{-1}\alpha_2 \cos\left[\theta_3 - (m-k)\theta_2\right] \int_{-\pi}^{\pi} \cos\left[(m-k)u + \beta \sin u\right]du$$

$$- \sin\left[\theta_3 - (m-k)\theta_2\right] \int_{-\pi}^{\pi} \sin\left[(m-k)u + \beta \sin u\right]du$$

$$+ \cos\left[\theta_3 - (m+k)\theta_2\right] \int_{-\pi}^{\pi} \cos\left[(m+k)u + \beta \sin u\right]du$$

$$- \sin\left[\theta_3 - (m+k)\theta_2\right] \int_{-\pi}^{\pi} \sin\left[(m+k)u + \beta \sin u\right]du.$$

The second and fourth integrals are zero because the integrands are odd functions with period 2π. Since the first and third integrands are even functions, it follows from (A.1.6) that

$$a_k(s) = \alpha_2\{J_{m-k}(\beta)\cos\left[\theta_3-(m-k)\theta_2\right] + J_{m+k}(\beta)\cos\left[\theta_3-(m+k)\theta_2\right]\}\ .$$

Similarly,

$$b_k(s) = -\alpha_2\{J_{m-k}(\beta)\sin\left[\theta_3-(m-k)\theta_2\right] - J_{m+k}(\beta)\sin\left[\theta_3-(m+k)\theta_2\right]\}\ .$$

We have proved that the Fourier coefficients of $y(t;\underline{\theta})$ are

$$a_k = \begin{cases} \mu & , k=0 \\ \alpha_1 \cos[\theta_1 + \theta_2] + \alpha_2 a_k(s) & , k=1 \\ \alpha_2 a_k(s) & , k \geq 2 \end{cases}$$

and

$$b_k = \begin{cases} 0 & , k=0 \\ -\alpha_1 \sin[\theta_1 + \theta_2] - \alpha_2 b_k(s) & , k=1 \\ -\alpha_2 b_k(s) & , k \geq 2 \end{cases}$$

where $a_k(s)$ and $b_k(s)$ are defined above.

A.2 Distribution of Phase Random Variables

A justification for using approximate formulas for the mean and variance of phase random variables is the complexity of the exact distribution of the Q_k*'s even when errors are assumed to be Gaussian. For completeness, the distribution of $\text{Arctan}[-B_k/A_k]$ is derived in this section. The subscript is dropped in the derivation.

If errors are Gaussian, then A and B are independent Gaussian random variables and

$$E[A]=a;\ E[B]=b;\ a^2+b^2=r^2;$$

$$\text{Var}[A]=\text{Var}[B]=(2/N)\sigma^2.$$

Particular values of a and b are given by (4.1.1). The joint distribution of A and B is

$$f_{A,B}(u,v)=(N/4\pi\sigma^2)\exp\{-(N/4\sigma^2)[(u-a)^2+(v-b)^2]\},\ -\infty<u,v<\infty.$$

Let X=A and Y=Arctan[-B/A]. Then because u=x and v=-x tan y, the Jacobian of the transformation is $J=|x|\sec^2 y$, and the joint distribution of X and Y is given by

$$f(x,y)=(N/4\sigma^2)|x|\sec^2 y\ \exp\{-(N/4\sigma^2)[(x-a)^2+(x\tan y+b)^2]\},\ -\infty<x<\infty, -\pi/2<y<\pi/2.$$

Expanding the exponent and completing the square, we obtain

$$\begin{aligned} f(x,y) &= (N\sec^2 y/4\pi\sigma^2)^{1/2}\exp\{(N/4\sigma^2)[(a-b\tan y)^2\cos^2 y-r^2]\} \\ &\quad \cdot|x|(N\sec^2 y/4\pi\sigma^2)^{1/2}\exp\{(-N/4\sigma^2)\sec^2 y[x-(a-b\tan y)\cos^2 y]^2\} \\ &= K(y)\cdot|x|\,\phi[x:(a-b\tan y)\cos^2 y,(2/N)\sigma^2\cos^2 y] \end{aligned}$$

where $\phi[z:\xi,\tau^2]$ denotes the probability density function of a Gaussian distribution with mean ξ and variance τ^2. Integration of $f(x,y)$ over x to obtain the distribution of Y shows that $f(y)=K(y)E[|X|]$, where X is Gaussian with mean $(a-b \tan y)\cos^2 y$ and variance $(2/N)\sigma^2\cos^2 y$.

If Z is a Gaussian random variable with mean ξ and variance τ^2, then

$$E[|Z|] = -\int_{-\infty}^{0} z\phi[z:\xi,\tau^2]dz + \int_{0}^{\infty} z\phi[z:\xi,\tau^2]dz$$

$$= -\xi + 2\int_{0}^{\infty} z\phi[z:\xi,\tau^2]dz.$$

Integration by parts gives

$$E[|Z|] = \xi\cdot(1-2\Phi[-\xi/\tau:\xi,\tau^2]) + (2\tau^2/\pi)^{1/2}\exp\{-\xi^2/2\tau^2\}.$$

where $\Phi(w:\xi,\tau^2)=\int_{-\infty}^{w}\phi[z:\xi,\tau^2]dz$. It follows that the distribution of $Y=\text{Arctan}[-B/A]$ is given by $f(y)=K(y)E[|Z|]$ with $\xi=(a-b \tan y)\cos^2 y$ and $\tau^2=(2/N)\sigma^2\cos^2 y$. Substition of these values above gives the following distribution for a phase random variable:

$$f(y) = \exp(-Nr^2/4\sigma^2)\{\pi^{-1}+(N\cos^2 y/4\pi\sigma^2)^{1/2}(a-b \tan y)$$
$$\cdot\left(1-\Phi[-(N\cos^2 y/2\sigma^2)^{1/2}(a-b \tan y):\xi,\tau^2]\right)$$
$$\cdot\exp\left[(N\cos^2 y/4\sigma^2)(a-b \tan y)^2\right]\},\quad -\pi/2<y<\pi/2,$$

where ξ and τ^2 are defined above.