In-Phase/Quadrature Covariance-Matrix Representation of the Uncertainty of Vectors and Complex Numbers

Dylan F. Williams, C. M. Wang, and Uwe Arz

***Abstract*—We describe the in-phase/quadrature covariance-matrix representation of the uncertainty in complex vectors, and transformations between this representation and the magnitude/phase and real/imaginary uncertainty representations.**

***Index Terms*—Complex numbers, in-phase, magnitude, phase, quadrature, translation, uncertainty, vectors.**

I. Introduction

We describe the magnitude/phase, real/imaginary, and in-phase/quadrature representation for describing uncertainties in complex numbers and vectors, as well as transformations between them.

Uncertainties in microwave engineering are most commonly expressed in a polar or magnitude/phase representation. However, the phase uncertainty in this representation has a singularity at small magnitudes that complicates its use.

Recently Ridler and Salter [1;2] suggested replacing the magnitude/phase uncertainty representation with a covariance-matrix-based real/imaginary representation. This representation avoids the problems of the magnitude/phase representation at small magnitudes, is complete, and captures all correlations. While uncertainties can be represented rigorously in terms of uncertainty regions in the complex plane, it is difficult to plot these regions as a function of frequency and relate them to the more commonly used magnitude/phase uncertainty representation.

Here we describe an in-phase/quadrature covariance-matrix representation based upon a straightforward coordinate rotation of the real/imaginary covariance matrix into an in-phase/quadrature coordinate system, as described in [3;4]. The in-phase/quadrature covariance-matrix uncertainty representation is an alternate way of expressing the same information captured in the real/imaginary representation. Yet the in-phase/quadrature covariance matrix representation maintains much of the convenience and intuitive nature of the magnitude/phase uncertainty representation in common use in the microwave community.

Manuscript received March 9, 2004. This work is a publication of the National Institute of Standards and Technology, and agency of the U.S. government, and is not subject to copyright.

D. F. Williams and C. M. Wang are with the National Institute of Standards and Technology, Boulder, CO 80305 USA (phone: 303-497-3138; e-mail: dylan@ boulder.nist.gov).

U. Arz is with the Physikalisch-Technische Bundesanstalt (PTB) in Braunschweig, Germany.

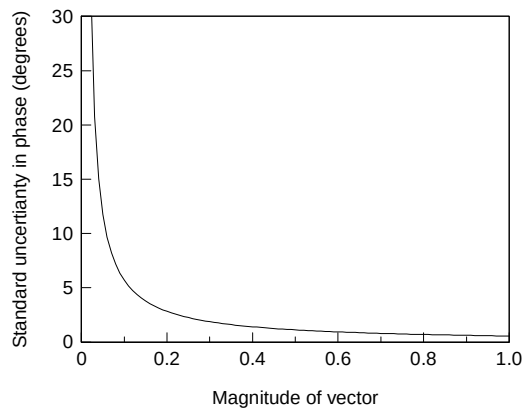

Fig. 1. The standard uncertainty of the phase of a vector subject to uncorrelated additive noise in its real and imaginary parts of Gaussian distribution with standard deviation of 0.01.

II. Magnitude/Phase Uncertainty Representation

Frequency-domain electrical quantities are usually represented by complex numbers in terms of a magnitude and a phase. In addition, many error mechanisms in electrical instrumentation lead to uncertainties that are independent of the phase of the vector, and are thus more easily represented in a magnitude/phase uncertainty representation than in real/imaginary representation. For example, consider uncertainty in the phase of a vector caused by additional delay due to cable bending. This uncertainty is independent of the phase of the vector describing the signal, and can be described by a single number (the variance of the phase of the vector) in the magnitude/phase uncertainty representation. As a consequence, the uncertainties in frequency-domain electrical quantities are usually represented in terms of the uncertainty in the magnitude and the phase of a vector.

On the other hand, representing uncertainties in the magnitude and phase of a vector when the size of the errors approach the magnitude of the base vector to which the

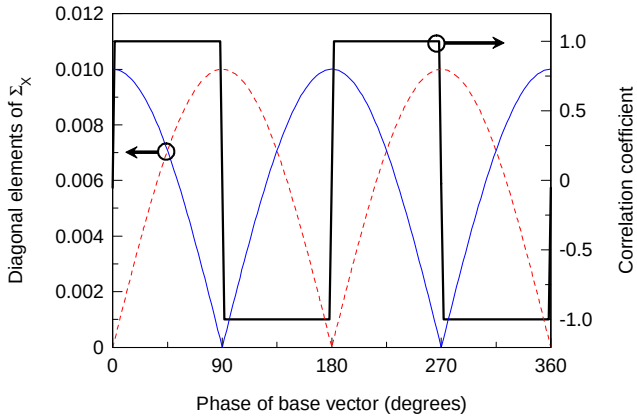

Fig. 2. The elements of the covariance matrix $\sum_X$ of a vector x subject to a random delay error, as a function of this vector's phase. The thin dashed line corresponds to the standard deviation of the real part of x, and the thin solid line to the standard deviation of the imaginary part of x. The thick solid line corresponds to the correlation coefficient of the two diagonal elements of $\sum_X$.

uncertainties correspond can be problematic, as illustrated in Fig. 1. Figure 1 plots the uncertainty in the phase of a vector in the presence of additive noise as a function of the magnitude of the base vector. The figure shows that the uncertainty in phase rises sharply as the magnitude of the vector decreases. This sharp rise in uncertainty is accompanied by a change in the distribution of the uncertainty in phase, and becomes especially difficult to interpret when the magnitude of the phase uncertainty approaches 180 degrees. In addition, because the magnitude of a vector is always positive, the distribution of uncertainty in the magnitude of the vector changes as well. For example, if the uncertainties in the real and imaginary parts of the underlying vector have Gaussian distributions, the uncertainty in the magnitude will be described by a Rayleigh distribution.

III. Real/Imaginary Covariance-Matrix Representation

The variance of a single scalar measurand quantifies the uncertainty in its measured value. Likewise, a covariance matrix can be used to express the uncertainty of a vector quantity [1;2;5;6]. We can write a complex number $x = x_1 + i\,x_2$ as a vector $X = [x_1, x_2]^T$, where i is the square root of -1, the superscript T represents the transpose, x_1 is the real part of x, and x_2 is the imaginary part of x. Thus we can treat complex numbers, such as reflection coefficients or signal amplitudes, as vectors, and express their uncertainties in the form of covariance matrices as well.

The covariance matrix $\sum_X$ of $X = [x_1, x_2]^T$ is defined by [1;2;5;6]

$$\Sigma_X = \begin{bmatrix} E\left((x_1 - E(x_1))^2\right) & E\left((x_1 - E(x_1))(x_2 - E(x_2))\right) \\ E\left((x_2 - E(x_2))(x_1 - E(x_1))\right) & E\left((x_2 - E(x_2))^2\right) \end{bmatrix}, \quad (1)$$

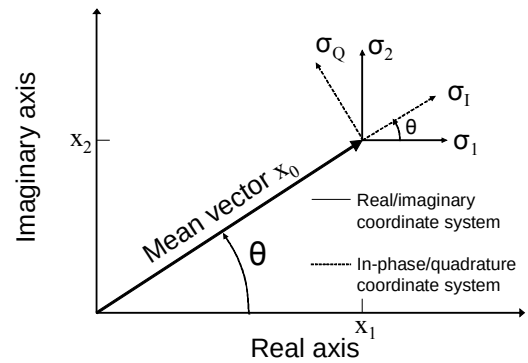

Fig. 3. The base vector x, the conventional real/imaginary coordinate system used to define the uncertainties of x, and the in-phase/quadrature coordinate system.

where $E(y)$ is the "expected value" of y, and is defined by

$$E(y) = \int_{-\infty}^{\infty} y\, p(y)\, \mathrm{d}y, \quad (2)$$

where $p(y)$ is the probability density function of y [5;6]. Here we call $\sum_X$ the real/imaginary covariance matrix of X because its elements correspond to the real and imaginary parts of $x = x_1 + i\,x_2$.

The two diagonal elements $E\left((x_k - E(x_k))^2\right)$ of $\sum_X$ are the variances of x_1 and x_2. The square root of these variances are the standard deviations σ_1 of x_1 and σ_2 of x_2. The sample estimate of σ_k is the standard uncertainty s_k.

The covariance matrix $\sum_X$ is symmetric, and the two off-diagonal elements of $\sum_X$ are equal to the covariance $\mathrm{cov}(x_1, x_2) = E\left((x_1 - E(x_1))(x_2 - E(x_2))\right)$ of x_1 and x_2. The covariance of x_1 and x_2 is equal to $\sigma_1\sigma_2\rho$, where ρ is the correlation coefficient of x_1 and x_2, and satisfies $-1 \le \rho \le 1$. The sample estimate of $\sigma_1\sigma_2\rho$ is $s_1 s_2 r$.

Figure 2 plots the elements of the covariance matrix describing the uncertainty of a vector subject to a random delay error, as might be introduced by cable bending. This uncertainty is described by a single number, the variance of the phase of the vector. However, the figure shows that, while the real/imaginary covariance matrix $\sum_X$ offers a complete description of the uncertainty of X, the elements of $\sum_X$ depend greatly on the phase of X. Thus, while this representation is rigorous and complete, it lacks the intuitive nature of the magnitude/phase representation. Furthermore, while it is certainly possible to plot uncertainty regions for complex vectors, as suggested in [1;2], and is difficult to do this as a function of frequency.

IV. In-Phase/Quadrature Uncertainty Representation

The matrix transform

$$\Theta(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3}$$

rotates a vector by θ radians. The transform Θ is unitary, and $\Theta(\theta)\,\Theta(\theta)^{\mathsf{T}} = \mathrm{I}$. We also have $\Theta(\theta)^{\mathsf{T}} = \Theta(\theta)^{-1} = \Theta(-\theta)$.

Applying Θ to X is equivalent to multiplying x by $e^{i\theta}$. Thus the vector $\Theta(-\theta_0)X$ is the vector x after it has been rotated so that its mean lies on the real axis. However, other than this rotation of coordinate system, the uncertainties and uncertainty regions associated with x are left unchanged by this transformation.

We define the in-phase/quadrature covariance matrix $\sum_X^{IQ}$ of x as the covariance matrix of $\Theta(-\theta_0)X$. In general, the covariance matrix $\sum_Y$ of $Y=AX$, where A is a linear matrix transformation, is given by $\sum_Y = A \sum_X A^{\mathsf{T}}$ [5;6]. Thus

$$\Sigma_X^{IQ} = \Theta(-\theta_0)\,\Sigma_X\,\Theta(-\theta_0)^{\mathsf{T}} = \Theta(-\theta_0)\,\Sigma_X\,\Theta(\theta_0), \tag{4}$$

where θ_0 is the angle of x_0 when $|x_0|$ is nonzero, and $\theta_0 = 0$ when $|x_0|$ is equal to zero. Equation (4) can be inverted to obtain

$$\Sigma_X = \Theta(\theta_0)\,\Sigma_X^{IQ}\,\Theta(-\theta_0). \tag{5}$$

The in-phase/quadrature covariance matrix $\sum_X^{IQ}$ is also the covariance matrix of x in the in-phase/quadrature coordinate system shown in dashed lines in Fig. 3. This is the coordinate system that has been aligned to x, as illustrated in the figure.

Earlier we alluded to the problems of singularities in magnitude/phase uncertainty representations, and the non-intuitive nature of the real/imaginary representation. The in-phase/quadrature uncertainty representation discussed here maintains some of the best features of each: it does not suffer from singularities at low frequencies and the diagonal elements of $\sum_X^{IQ}$ are easily and intuitively related to the magnitude/phase uncertainty representations. Of course, either the real/imaginary or the in-phase/quadrature covariance matrices can be used to determine and plot the uncertainty regions for the data, when that is appropriate. Furthermore, $\sum_X$ and $\sum_X^{IQ}$ can be easily related with (5).

V. Transformation Between In-phase/Quadrature and Magnitude/Phase Uncertainty Representations

The in-phase/quadrature covariance-matrix offers a convenient way of transforming between the real/imaginary covariance-matrix and the commonly used magnitude/phase uncertainty representation when the magnitude of the vector is large compared with its uncertainties. This is because the first diagonal element σ_I^2 of $\sum_X^{IQ}$ corresponds to the variance of x in the direction of (in phase with) x_0, and thus is related to the uncertainty σ_m in the magnitude $|x_0|$ of x_0. Likewise, the second diagonal element σ_Q^2 of $\sum_X^{IQ}$ corresponds to the variance of x in the direction perpendicular to (in quadrature with) x_0, and is related to the uncertainty σ_θ in the phase θ_0 of x_0 in radians. Thus, when the magnitude of x_0 is large compared to σ_I and σ_Q, we can write $\sum_X^{IQ}$ as

$$\Sigma_X^{IQ} \approx \begin{bmatrix} \sigma_m^2 & \sigma_m\,|x_0|\,\sigma_\theta\,\rho_{m\theta} \\ \sigma_m\,|x_0|\,\sigma_\theta\,\rho_{m\theta} & |x_0|^2\,\sigma_\theta^2 \end{bmatrix}, \tag{6}$$

where $\rho_{m\theta}$ is the correlation coefficient of $|x|$ and θ.

We can use (5) and (6) to find $\sum_X$ from σ_m, σ_θ, and $\rho_{m\theta}$. We can also determine σ_m, σ_θ, and $\rho_{m\theta}$ from $\sum_X$ by using (4) and then inverting (6).

VI. Extension to a Collection of Vectors

We can extend the rules captured in (3), (4), and (5) to a collection of (possibly) correlated complex numbers as follows. Let $\mathbf{X} = [X_1, X_2, \ldots, X_N]^{\mathsf{T}}$ be a vector containing the $2N$ real and imaginary parts of N complex numbers. If θ_k is the phase of the sub-vector X_{0k} of $\mathbf{X}_0$, where $\mathbf{X}_0$ is the mean of $\mathbf{X}$, then

$$\Sigma_{\mathbf{X}}^{IQ} = \mathbf{\Theta}^{-1}\,\Sigma_{\mathbf{X}}\left(\mathbf{\Theta}^{-1}\right)^{\mathsf{T}} = \mathbf{\Theta}^{-1}\,\Sigma_{\mathbf{X}}\,\mathbf{\Theta}, \tag{7}$$

where

$$\mathbf{\Theta} = \begin{bmatrix} \Theta(\theta_1) & 0 & \cdots \\ 0 & \Theta(\theta_2) & \\ \vdots & & \ddots \end{bmatrix}. \tag{8}$$

Likewise,

$$\Sigma_{\mathbf{X}} = \mathbf{\Theta}\;\Sigma_{\mathbf{X}}^{IQ}\,\mathbf{\Theta}^{-1}. \tag{9}$$

VII. Conclusion

We have described transformations between the real/imaginary, in-phase/quadrature, and magnitude/phase uncertainty representations. The in-phase/quadrature representation maintains the rigor of the real/imaginary covariance matrix uncertainty representation, preserves much of the simplicity of the magnitude and phase uncertainty representations when errors do not depend on the phase of the vector, and offers a straightforward way of transforming between the two.

References

[1] N. M. Ridler and M. J. Salter, "Evaluating and Expressing Uncertainty in Complex S-Parameter Measurements," *Automatic RF Techniques Group Conference Digest*, vol. 56, pp. 63-75, Dec.2000.

[2] N. M. Ridler and M. J. Salter, "Propagating S-parameter uncertainties to other measurement quantities," *Automatic RF Techniques Group Conference Digest*, vol. 58, pp. 71-89, Dec.2001.

[3] D. F. Williams, C. M. Wang, and U. Arz, "An optimal multiline TRL calibration algorithm," *IEEE MTT-S Int. Microwave Symp. Dig.*, vol. 3, pp. 1819-1822, June2003.

[4] D. F. Williams, C. M. Wang, and U. Arz, "An optimal vector-network-analyzer calibration algorithm," *IEEE Trans. Microwave Theory Tech.*, vol. 51, no. 12, pp. 2391-2401, Dec.2003.

[5] W. Mendenhall and T. Sincich, *Statistics for Engineering and the Sciences*, 3 ed. San Francisco: Dellen Publishing Company, 1988.

[6] S. Ross, *A first course in probability*. New York: Macmillan Publishing Co., Inc., 1976.