Self-Study Manual on Optical Radiation Measurements: Part I—Concepts, Chapter 6

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Chapter 6. Distribution of Optical Radiation with Respect to Polarization, John B. Shumaker

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PREFACE

This is the third in a series of Technical Notes (910-) entitled "Self-Study Manual on Optical Radiation Measurements." It contains the sixth chapter of that Manual. Inadvertently, this chapter has been completed before chapters 4 and 5, which are nearly ready for publication in NBS TN 910-2. Chapter 4, "More on the Distribution of Optical Radiation with Respect to Position and Direction," deals with all of the radiometric quantities of Table A1-3 in Appendix A of NBS TN 910-1 that were not covered in detail in the first three chapters. Chapter 5, "The Measurement Equation," introduces the central topic that ties together our entire treatment of optical radiation measurements. The measurement-equation concept is introduced by deriving the appropriate measurement equations for three different illustrative problems. All of these chapters are still in Part I--Concepts, but our plans call for us next to turn to one or two chapters in Part III--Applications. We feel that we now have established enough of the fundamentals to be able to usefully deal with some specific applications, the topic in which we realize many readers will be most interested.

For background information on the whole project and on the plans for the "Self-Study Manual" (SSM), we reproduce here (immediately following) the Preface to the first Technical Note 910-1, of March 1976. Although the exact details of the outline and plans for the individual chapters are still developing, our principal aims and overall plans are still as set forth in that Preface.

Here, in chapter 6 of Part I, we are primarily concerned with establishing the concepts needed to deal with polarization in relation to radiometry (the measurement of optical radiation). There already exist excellent treatments of polarization that are concerned primarily with polarization phenomena, and we have not tried to compete with them. This is not a treatise on polarization phenomena, as such, but only on as much about them as is needed to deal adequately with the measurement of power propagated in the form of optical radiation -- with radiometry. To our knowledge, this is a topic that has not previously been treated in the literature.

We are grateful to many for valuable comments and criticisms received, both informally as well as in formal reviews, concerning all phases of the SSM. In connection with this polarization chapter, we are particularly indebted to Drs. Jean M. and Harold E. Bennett of the Michelson Laboratory, Naval Weapons Center, China Lake, California, and Dr. Elio Passaglia of NBS for helpful discussions and critical reviews. Mrs. Betty Castle has again done an outstanding job of typing a difficult text.

Fred E. Nicodemus, Editor

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PREFACE to NBS TN 910-1

This is the initial publication of a new series of Technical Notes (910) entitled "Self-Study Manual on Optical Radiation Measurements." It contains the first three chapters of this Manual. Additional chapters will be published, similarly, as they are completed. The Manual is being written by the Optical Radiation Section of NBS. In addition to writing some of the chapters, themselves, Fred E. Nicodemus is the Editor of the Manual and Henry J. Kostkowski, Chief of the Section, heads the overall project.

In recent years, the economic and social impact of radiometric measurements (including photometric measurements) has increased significantly. Such measurements are required in the manufacture of cameras, color TV's, copying machines, and solid-state lamps (LED's). Ultraviolet radiation is being used extensively for the polymerization of industrial coatings, and regulatory agencies are concerned with its effects on the eyes and skin of workers. On the other hand, phototherapy is usually the preferred method for the treatment of jaundice in the newborn. Considerable attention is being given to the widespread utilization of solar energy. These are just a few examples of present day applications of optical radiation. Most of these applications would benefit from simple measurements of one to a few per cent uncertainty and, in some cases, such accuracies are almost essential. But this is rarely possible. Measurements by different instruments or techniques commonly disagree by 10% to 50%, and resolving these discrepancies is time-consuming and costly.

There are two major reasons for the large discrepancies that occur. One is that optical radiation is one of the most difficult physical quantities to measure accurately. Radiant power varies with the radiation parameters of position, direction, wavelength, time, and polarization. The responsivity of most radiometers also varies with these same radiation parameters and with a number of environmental and instrumental parameters, as well. Thus, the accurate measurement of optical radiation is a difficult multi-dimensional problem. The second reason is that, in addition to this inherent difficulty, there are few measurement experts available. Most of the people wanting to make optical radiation measurements have not been trained to do so. Few schools have had programs in this area and tutorial and reference material that can be used for self-study is only partially available, is scattered throughout the literature, and is generally inadequate. Our purpose in preparing this Self-Study Manual is to make that information readily accessible in one place and in systematic, understandable form.

The idea of producing such a manual at NBS was developed by one of us (HJK) in the latter part of 1973. Detailed planning got under way in the summer of 1974 when a full-time editor (FEN) was appointed. The two of us worked together for about one year developing an approach and format while writing and rewriting several drafts of the first few chapters. These are particularly important because they will serve as a model for the rest of the Manual. During this period, a draft text for the first four chapters was distributed, along with a questionnaire, for comment and criticism to some 200 individuals representing virtually every technical area interested in the Manual. About 50 replies were received, varying widely in the reactions and suggestions expressed. Detailed discussions were also held with key individuals, including most of the Section staff, particularly those that will be writing some of the later chapters. In spite of the very wide range of opinions encountered, all of this feedback has provided valuable guidance for the final decisions about objectives, content, style, level of presentation, etc.

In particular, we have been able to arrive at a clear solution to difficult questions about the level of presentation. Both of us started out with the firm conviction that, with enough time and effort, we should be able to present the subject so that readers with the equivalent of just elementary college mathematics and science could easily follow it. That conviction was based on our experience of success in explaining the subtleties of radiometric measurements to technicians at that level. What we failed to consider, however, was that, in making such explanations to individuals we always were able to relate what we said to the particular background and immediate problem of the individual. That's just not possible in a text intended for broad use by workers in astronomy, mechanical heat-transfer engineering, illumination engineering, photometry, meteorology, photo-biology and photo-chemistry, optical pyrometry, remote sensing, military infrared applications, etc. To deal directly and explicitly with each individual's problems in a cook-book approach would require an impossibly large and unwieldy text. So we must fall back on general principles which immediately and unavoidably require more knowledge and familiarity with science and mathematics, at the level of a bachelor's degree in some branch of science or engineering, or the equivalent in other training and experience.

In its present form, the Manual is a definitive tutorial treatment of the subject that is complete enough for self instruction. This is what is meant by the phrase "self-study" in the title. The Manual does not contain explicitly programmed learning steps as that phrase sometimes denotes. In addition, through detailed, yet concise, chapter summaries, the Manual is designed to serve also as a convenient and authoritative reference source. Those already familiar with a topic should turn immediately to the summary at the end of the appropriate chapter. They can determine from that summary what, if any, of the body of the chapter they want to read for more details.

The basic approach and focal point of the treatment in this Manual is the measurement equation. We believe that every measurement problem should be addressed with an equation relating the quantity desired to the data obtained through a detailed characterization of the instruments used and the radiation field observed, in terms of all of the relevant parameters. The latter always include the radiation parameters, as well as environmental and instrumental parameters, as previously pointed out. The objective of the Manual is to develop the basic concepts and characteristics required so that the reader will be able to use this measurement-equation approach. It is our belief that this is the only way that uncertainties in the measurement of optical radiation can generally be limited to one, or at most a few, per cent.

Currently, the Manual deals only with the classical radiometry of incoherent radiation. The basic quantitative relations for the propagation of energy by coherent radiation (e.g., laser beams) are just being worked out [19-22].[1] Without that basic theory, a completely satisfactory general treatment of the measurement of coherent (including partially coherent) optical radiation is not possible. Accordingly, in spite of the urgent need for improved measurements of laser radiation, we won't attempt to deal with it now. Possibly this situation will be changed before the current effort has been completed and a supplement on laser measurements can be added.

As stated above, we first hoped to prepare this Manual on a more elementary level but found that it was impossible to avoid making use of both differential and integral calculus of more than one variable. However, to help those that might be a bit "rusty" with such mathematics, we go back to first principles each time a mathematical concept or procedure beyond those of simple algebra or trigonometry is introduced. This should also throw additional light on the physical and geometrical relationships involved. Where it seems inappropriate to do this in the text, we cover such mathematical considerations in appendices. It is also assumed that the reader has had an introductory college course in physics, or the equivalent.

The Manual is being organized into three Parts, as follows:

Part I. Concepts

Step by step build up of the <u>measurement equation</u> in terms of the radiation parameters, the properties and characteristics of sources, optical paths, and receivers, and the environmental and instrumental parameters. Useful quantities are defined and discussed and their relevance to various applications in many different fields (photometry, heat-transfer engineering, astronomy, photo-biology, etc.) is indicated. However, discussions of actual devices and measurement situations in this Part are mainly for purposes of illustrating concepts and basic principles.

Part II. Instrumentation

Descriptions, properties, and other pertinent data concerning typical instruments, devices, and components involved in common measurement situations. Included is material dealing with sources, detectors, filters, atmospheric paths, choppers (and other types of optical modulators), prisms, gratings, polarizers, radiometers, photometers, spectroradiometers, spectrophotometers, etc.

Part III. Applications

Measurement techniques for achieving a desired level of, or improving, the accuracy of a measurement. Included will be a very wide variety of examples of environmental and instrumental parameters with discussion of their effects and how to deal with them. This is where we deal with real measurements in the real world. The examples will also be drawn from the widest possible variety of areas of application in illumination engineering, radiative heat transfer, military infrared devices, remote sensing, meteorology, astronomy, photo-chemistry and photo-biology, etc.

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[1]Figures in brackets indicate literature references listed at the end of this Technical Note.

Individual chapter headings have been assigned only to the first five chapters:

Chapter 1. Introduction

Chapter 2. Distribution of Optical Radiation with respect to Position and Direction -- Radiance

Chapter 3. Spectral Distribution of Optical Radiation

Chapter 4. Optical Radiation Measurements -- a Measurement Equation

Chapter 5. More on the Distribution of Optical Radiation with respect to Position and Direction

Other subjects definitely planned for Part I are thermal radiation, photometry, distribution with respect to time, polarization, diffraction, and detector concepts. It is not our intention, however, to try to complete all of Part I before going on to Parts II and III. In fact, because we realize that a great many readers are probably most interested in the material on applications to appear in Part III, we will try to complete and publish some chapters in Parts II and III just as soon as adequate preparation has been made in the earlier chapters of Part I. However, because our approach to radiometry differs so much from the traditional treatment, we feel that unnecessary confusion and misunderstanding can be avoided if at least the first nine chapters of Part I are published first and so are available to readers of later chapters.

Finally, we invite the reader to submit comments, criticisms, and suggestions for improving future chapters in this Manual. In particular, we welcome illustrative examples and problems from as widely different areas of application as possible.

As previously stated, we are indebted to a great many individuals for invaluable "feedback" that has helped us to put this text together more effectively. Notable are the inputs and encouragement from the Council on Optical Radiation Measurements (CORM), especially the CORM Coordinators, Richard J. Becherer, John Eby, Franc Grum, Alton R. Karoli, Edward S. Steeb, and Robert B. Watson, and the Editor of *Electro-Optical Systems Design*, Robert D. Compton. In addition, for editorial assistance, we are grateful to Donald A. McSparron, Joseph C. Richmond, and John B. Shumaker, and particularly to Albert T. Hattenburg.

We are especially grateful to Mrs. Betty Castle for the skillful and conscientious effort that produced the excellent typing of this difficult text. We also want to thank Henry J. Zoranski for his capable help with the figures.

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SELF-STUDY MANUAL on OPTICAL RADIATION MEASUREMENTS

Part I. Concepts

Chapter 6. Distribution of Optical Radiation with Respect to Polarization

by John B. Shumaker

This chapter develops and illustrates the concepts necessary to include polarization rigorously in classical radiometry. The treatment is based upon the Stokes polarization vector of spectral radiance and Mueller transmittance matrices. The Mueller matrices of many common polarizing optical components are discussed. Considerable attention is paid to the measurement of the Stokes spectral-radiance-vector components and to the measurement of the Mueller-transmittance-matrix elements. The concepts and techniques are illustrated by discussions of such subjects as radiometer calibrations, three-polarizer attenuators, depolarizers, and the characterization of Polaroid-type polarizers. Also included are an appendix on matrix multiplication and an appendix showing one way by which, in principle, any Mueller transmittance matrix may be measured.

Key Words: Mueller matrix; polarization; polarizer; Stokes parameters.

<u>In this CHAPTER</u>. We discuss polarization, another of the radiation parameters introduced in Chapter 1 [1]. The other radiation parameters -- position, direction, wavelength, and time -- are continuous variables and we have used functional notations like $L(x,y,\theta,\phi,\lambda)$ to show the dependence of radiometric quantities upon these parameters. Polarization, however, enters somewhat differently. We include polarization by introducing four polarization components of spectral radiance, $L_{\lambda,0}(x,y,\theta,\phi,\lambda)$, $L_{\lambda,1}(x,y,\theta,\phi,\lambda)$, $L_{\lambda,2}(x,y,\theta,\phi,\lambda)$, and $L_{\lambda,3}(x,y,\theta,\phi,\lambda)$. The first of these, $L_{\lambda,0}$, is identical with the spectral radiance we have been using in earlier chapters. The other three, $L_{\lambda,1}$, $L_{\lambda,2}$, and $L_{\lambda,3}$, provide the polarization information. The expressions that result from this four-fold extension of the quantities appearing in the measurement equation are adequate to handle any kind of polarizing optical element and any polarization state within the approximations of classical (geometrical optics) radiometry of incoherent radiation.

We will not in this chapter discuss in detail specific polarizing or depolarizing instruments. These will be described in Part II and are also covered in several of the references [2,3,4,5]. We concentrate here on developing and illustrating the use of the concepts and equations necessary to include polarization rigorously in classical radiometry. We begin the chapter with a review of the relationship between the oscillating electric field and the polarization state of radiation. We then introduce the four components of spectral radiance and study how they are transformed by ideal polarizers and retarders. Next we discuss the response of a radiometer to polarized light and how it may be calibrated. After this we examine real, non-ideal, optical components and finally conclude the chapter with a brief look at some experimental precautions in the use of polarizing optics.

The object of this chapter is to extend the equations of radiometry which have been developed in preceding chapters so that they can include polarization rigorously. We make little attempt to provide an intuitive understanding of the polarization phenomenon. Our approach is directed toward the radiometrist who may regard polarization as an unfortunate complication which, like the common cold, has been inflicted upon him by the whimsical gods. We therefore have tried to present a systematic measurement approach to the polarization problem which does not require the reader to become an expert on polarization optics. As a consequence, we illustrate our equations and the kinds of measurement information involved with conceptual measurements which employ simple, easily available instruments. Although some very precise relative polarization measurements can be made using high quality optical components and specialized instruments such as ellipsometers, such precision is usually not required in radiometry. These special devices and techniques are described in some of the references [2,5,6] and will also be considered in Parts II and III. This chapter also does not discuss the many remarkable uses which man and nature have made of polarized light. These are briefly discussed in reference [2] which includes extensive further literature references.

THE STOKES COMPONENTS of POLARIZATION

Polarization can be thought of in terms of the orientation of an electromagnetic field vector oscillating about the line of sight between an observer and a light source. All of the consequences of polarization can be derived from this starting point and the names "linear" polarization and "circular" polarization originate in this description. However, within the approximations of geometrical optics a more direct phenomenological approach is possible. This simply requires the reinterpretation of spectral radiance as being fully characterized by four independent polarization components, called Stokes components. All four components are measurable, as we will see, and form a phenomenological basis for the quantitative inclusion of polarization in radiometry and photometry. There are other ways of describing polarization [2,4,5] but for our purposes the use of Stokes components is the simplest because for light with a negligible degree of polarization -- frequently the most important situation in radiometry -- the treatment reduces directly to the equations we have been considering in the preceding chapters. The Stokes components provide no particular insight into why or how polarizing optical components work but we will not be interested in that aspect of polarization in this chapter. We are interested in how the polarization state of a beam affects the response of a radiometer and in how to describe the flux of a polarized beam so that the beam is uniquely defined from a radiometric point of view -- so that it could be used to calibrate a radiometer, for example. The four Stokes components of spectral radiance at every point and direction in a radiation field provide that kind of description. Since most people prefer to think in the familiar terms of electric field vectors and linear and circular polarization, we will indicate how the Stokes components are related to these common concepts, but we emphasize that this is not essential to radiometry.

We begin with the familiar microscopic description of polarization. We consider a single steadily radiating oscillator at a distant point on the negative Z-axis of a rectangular coordinate system and imagine that we can continuously record the instantaneous electric field[1] present at the origin due to this oscillator. Except at distances from the light source of a few wavelengths or less, the Z component of the electric field -- the component along the direction of propagation -- will be completely negligible and the field will lie in the X-Y plane. The X and Y field components then will be of the form

$$\begin{aligned} E_x &= V_x \cdot \cos(2\pi \cdot \nu \cdot t + \delta_x) \\ E_y &= V_y \cdot \cos(2\pi \cdot \nu \cdot t + \delta_y) \end{aligned} \qquad [V \cdot m^{-1}] \qquad (6.1)$$

where V_x and V_y are the amplitudes $[V \cdot m^{-1}]$, ν is the frequency [Hz], δ_x and δ_y are the phases [rad] of the electromagnetic wave train, and t is the time [s]. In figure 6.1 are shown some examples to illustrate how the electric-field vector changes with time for some special combinations of amplitudes and phases. In each case the sketch on the left shows the pattern traced out in the X-Y plane by the tip of the electric-field vector and the graphs on the right show the time behavior of the X and Y components. In diagrams (a) we see light which is described as <u>linearly polarized</u>[2] in a vertical direction; the horizontal component, E_x, of the electric field is always zero.[3] Diagrams (b) show a more general example of linear polarization. E_y and E_x are exactly in phase with one another, $\delta_y = \delta_x$, but their amplitudes V_y and V_x differ. Diagrams (c) show <u>right circularly polarized light</u>; the electric vector appears to trace a circle in the clockwise

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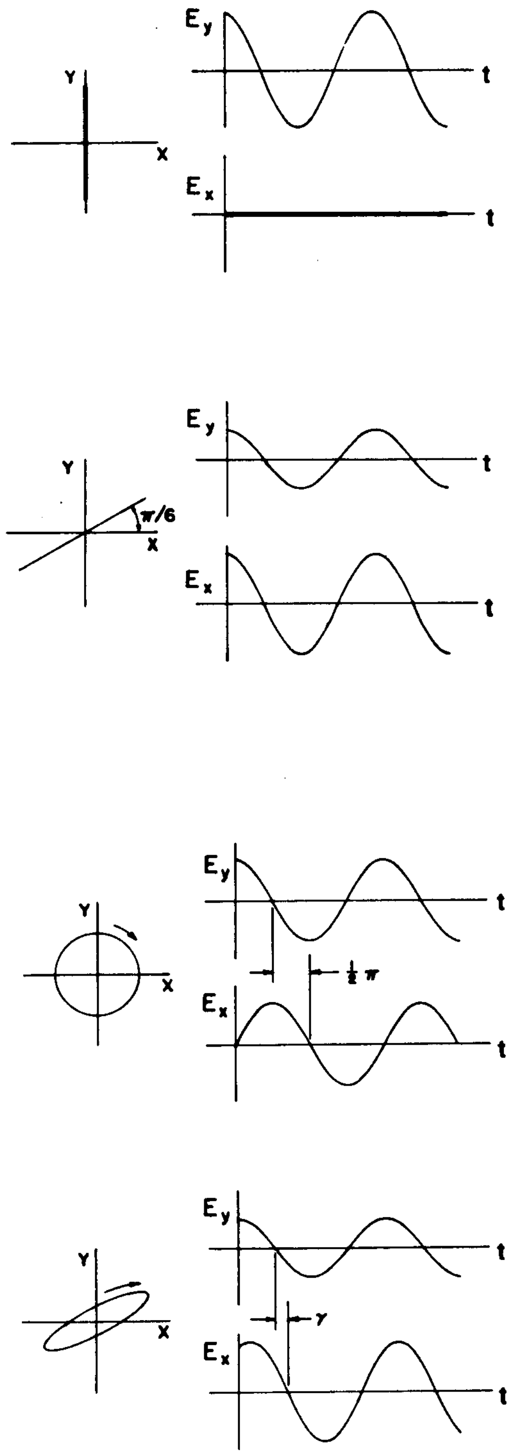
[1]The electric and magnetic field vectors are perpendicular to each other and to the direction of propagation of the radiation. Our development could equally well have been based upon the magnetic field, as it is in much of the older literature.

[2]"Linear" and "plane" polarization are synonymous terms. "Linear" seems more consistent with the terms "circular" and "elliptical" and so is used exclusively here.

[3]The terms "horizontal" and "vertical" linear polarization and "right" and "left" circular polarization are somewhat arbitrary and are used differently by different authors. Light which contains no vertical component of the electric field may be described as horizontally polarized; however, if the magnetic field vector is emphasized instead of the electric vector then such light would be called vertically polarized. Right and left handedness are likewise ambiguous descriptions of circular polarization. Fortunately, as long as one is consistent within his own calculations it rarely matters what conventions one adopts for these definitions. To insure consistency one must usually take all his formulas and equations from the same source. Our conventions used in this chapter agree with those of Shurcliff [2] where, for instance, light reflected from horizontal surfaces such as from a desk top or the ocean is defined as possessing enhanced horizontal polarization. Polarizing sunglasses, consequently, are vertical polarizers as customarily worn.

Figure 6.1

Electric-field-vector behavior for monochromatic polarized radiation



(a) Linearly polarized in the Y-Z plane

(b) Linearly polarized at +30°

(c) Right circularly polarized

(d) Right elliptically polarized

direction (as we look back toward the source). E_y and E_x have equal amplitudes but are exactly a quarter cycle out of phase with one another: $\delta_y = \delta_x + \pi/2$. If $\delta_y = \delta_x - \pi/2$ the electric vector will rotate in a counterclockwise direction and the polarization is called <u>left circular</u>. The final example shows <u>elliptically polarized</u> light which is the general case of eq. (6.1). In this case no special relationships exist between V_y and V_x and between δ_y and δ_x. E_y and E_x have unequal amplitudes and are out of phase by an angle $\gamma = \delta_y - \delta_x$.[1] If γ is positive the polarization is called right elliptical and if γ is negative it is called left elliptical. Elliptical polarization represents the most general expression of a perfectly monochromatic ray which is possible. Linear and circular polarizations are special cases of elliptical polarization.

As an aid to visualizing polarization we might think of a classical source mechanism in terms of the oscillations of charged particles. At radio wavelengths a vertically polarized field can be produced by a vertical transmitting antenna with a cluster of electrons sloshing up and down at the radio frequency. At optical wavelengths we can similarly imagine a vertical vibration of electrons at the source as one way of generating vertically polarized light. Circular polarization then can be thought of as being generated by charges moving in circles at the source. If the charges are moving in circular orbits as seen by an observer in one direction then observers in other directions will see the charges moving in ellipses or even vibrating in straight lines thus observing elliptical or linear polarization. Such directional dependence of the polarization form is actually observed in the Zeeman effect, for example, where some spectral lines emitted by a gas in a magnetic field show circular polarization if viewed along the direction of the magnetic field and change continuously through elliptical to linear polarization as the viewing direction changes toward the perpendicular to the magnetic field.

Three parameters are required to describe the general polarization ellipse. They can be chosen in several ways. For example the magnitude of the parameter $\gamma = \delta_y - \delta_x$ specifies the ellipticity of the polarization and its sign specifies its handedness -- left or right. The parameter V_y/V_x specifies the orientation of the ellipse in the X-Y plane. And the parameter $V_x^2 + V_y^2$ specifies the size of the ellipse. A set of parameters which has proven to be somewhat more useful, however, is the set of Stokes parameters[2]:

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[1]γ is further required to be adjusted by adding or subtracting integral multiples of 2π if necessary to give $|\gamma| \leq \pi$.

[2]In eq. (6.2) that follows, the proportionality constant κ is given by $\kappa = n/(2\mu \cdot c)$ where n is the refractive index, μ the magnetic permeability and c is the vacuum velocity of light. If V_x and V_y are expressed in volts/meter and $E_{n,i}$ in watts/m^2 then $\kappa/n = 1.327 \times 10^{-3}$ [$W \cdot V^{-2}$] for non-magnetic materials. Equations (6.2) contain less information than eqs. (6.1). The absolute phase information has been lost. Ultimately this will result in the exclusion of interference effects from our treatment and in the additivity of radiometric quantities such as spectral radiance. These, of course, are just the limitations (and advantages) of geometrical optics.

$$
\begin{aligned}
E_{n,0} &= \kappa(V_x{}^2 + V_y{}^2) \quad [\mathrm{W\cdot m^{-2}}] \\
E_{n,1} &= \kappa(V_x{}^2 - V_y{}^2) \\
E_{n,2} &= \kappa(2V_x \cdot V_y \cdot \cos\gamma) \\
E_{n,3} &= \kappa(2V_x \cdot V_y \cdot \sin\gamma).
\end{aligned} \tag{6.2}
$$

There are four of these parameters, the first of which is redundant in the present context[1] but, as we shall see later, forms the basis for including unpolarized light. The second Stokes parameter $E_{n,1}$ is a measure of the excess of horizontal linear polarization over vertical linear polarization. It is called the <u>horizontal preference parameter</u>. The third parameter $E_{n,2}$ is a measure of the excess of linear polarization at 45° (from the horizontal) to that at 135°. It is called the <u>+45° preference parameter</u>. The last Stokes parameter is a measure of the excess of right-circular component to left-circular component. It is called the <u>right-circular preference parameter</u>. These parameters all have the unit-dimensions of irradiance and constitute the four Stokes polarization components of normal irradiance from a single steady oscillator.

A real, non-monochromatic, non-point radiation source will also generate a varying electric field vector at any point. But the motion of this electric field vector will generally not be easy to describe. For one thing the motion need not be confined to a plane. For another, it may not be periodic. In the limit as the solid angle and wavelength interval approach zero the motion of the electric field vector will become planar and elliptical and so the Stokes parameters are appropriate for describing the polarization state of the radiometric quantity spectral radiance. Accordingly, the average values of the Stokes spectral radiance components that a real radiometer would measure are given by

$$
\begin{aligned}
L_{\lambda,0} &= \frac{1}{\Delta A \cdot \Delta t} \iint_{\Delta A} \int_{\Delta t} \kappa \frac{V_x{}^2 + V_y{}^2}{\Delta\omega \cdot \Delta\lambda} \, dx \cdot dy \cdot dt \\
L_{\lambda,1} &= \frac{1}{\Delta A \cdot \Delta t} \iint_{\Delta A} \int_{\Delta t} \kappa \frac{V_x{}^2 - V_y{}^2}{\Delta\omega \cdot \Delta\lambda} \, dx \cdot dy \cdot dt \\
L_{\lambda,2} &= \frac{1}{\Delta A \cdot \Delta t} \iint_{\Delta A} \int_{\Delta t} \kappa \frac{2V_x \cdot V_y \cdot \cos\gamma}{\Delta\omega \cdot \Delta\lambda} \, dx \cdot dy \cdot dt \\
L_{\lambda,3} &= \frac{1}{\Delta A \cdot \Delta t} \iint_{\Delta A} \int_{\Delta t} \kappa \frac{2V_x \cdot V_y \cdot \sin\gamma}{\Delta\omega \cdot \Delta\lambda} \, dx \cdot dy \cdot dt
\end{aligned} \tag{6.3}
$$

where Δt is the time required for a measurement, ΔA is the area of the detector surface perpendicular to the ray direction, $\Delta\omega$ is the solid angle from which rays can illuminate

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[1]Because $E_{n,0}^2 = E_{n,1}^2 + E_{n,2}^2 + E_{n,3}^2$.

the detector and $\Delta\lambda$ is the wavelength band reaching the detector. V_x, V_y and γ are the field amplitudes and phase difference at the detector surface.

In the mathematical limit as ΔA, Δt, $\Delta\omega$ and $\Delta\lambda$ approach zero

$$L_{\lambda,0}^2 = L_{\lambda,1}^2 + L_{\lambda,2}^2 + L_{\lambda,3}^2. \tag{6.4}$$

This is the condition for complete polarization. With sufficient radiometer resolution in these four parameters, A, t, ω and λ, all measurement results would show complete polarization. However, as was discussed in Chapter 2, [1] radiometric measurements cannot be made if any of these resolution intervals goes to zero because zero power would then be received.[1] The practical limits on the smallness of ΔA, Δt, $\Delta\omega$ and $\Delta\lambda$ result in values of Stokes spectral-radiance components which obey

$$L_{\lambda,0}^2 \geq L_{\lambda,1}^2 + L_{\lambda,2}^2 + L_{\lambda,3}^2.$$

Thus, we speak of a <u>polarized component</u> of spectral radiance

$$L_{\lambda,p} = \sqrt{L_{\lambda,1}^2 + L_{\lambda,2}^2 + L_{\lambda,3}^2} \tag{6.5}$$

and an <u>unpolarized component</u>

$$L_{\lambda,u} = L_{\lambda,0} - L_{\lambda,p}$$

and define the <u>degree of polarization</u> as

$$P = L_{\lambda,p}/L_{\lambda,0}. \tag{6.6}$$

Although at first glance it would seem that a definition of unpolarized light which obviously depends upon instrument resolution would be highly unsatisfactory, it turns out that this is rarely a problem. Usually unpolarized light arises because of inadequate

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[1] By considering the coherence properties of radiation one can deduce that the state of polarization in the detector plane remains essentially constant over areas of the order of $\delta A \sim \lambda^2 \cdot (\delta\omega)^{-1}$. Similarly, the state of polarization is essentially constant in time over a time interval of the order of $\delta t \sim (\delta\nu)^{-1} = \lambda^2 \cdot (c \cdot \delta\lambda)^{-1}$. Therefore, the resolution intervals need be no smaller than these values. A perfect radiometer with these resolution intervals would require a source with spectral radiance

$$L_{\lambda,0} \sim 10^4\ h \cdot \nu / (\delta\omega \cdot \delta A \cdot \delta\lambda \cdot \delta t) = 10^4 \cdot \frac{h \cdot c}{\lambda} \cdot \left(\lambda^2 \cdot \frac{\lambda^2}{c}\right)^{-1} \sim 2 \cdot 10^{19} \quad [\mathrm{W \cdot m^{-3} \cdot sr^{-1}}]$$

to achieve a 1% measurement accuracy at $\lambda = 500$ [nm]. This is a factor of 10^6 times the solar spectral radiance. For any less intense source the radiometer resolution intervals must be larger if useful accuracy is desired.

resolution in the time dimension. For typical spectrometers and for steady incandescent sources (so-called natural light), measurement times less than 10^{-11} seconds would be required to observe complete polarization. That is, V_x, V_y, and γ will not change appreciably over periods of 10^{-11} seconds and non-vanishing values of $L_{\lambda,1}$, $L_{\lambda,2}$ or $L_{\lambda,3}$ can be expected. If Δt is very much longer, however, the fluctuations in V_x, V_y and γ may cause cancellations in computing the mean values of these three Stokes components. $L_{\lambda,0}$, being the sum of positive quantities V_x^2 and V_y^2, will not be reduced by averaging. Thus, $L_{\lambda,0}$ will appear to include a contribution which has disappeared from the $L_{\lambda,1}$, $L_{\lambda,2}$ and $L_{\lambda,3}$ components and is consequently called unpolarized. Since the time resolution required to resolve the unpolarized component is so far beyond the capabilities of customary radiometry there is, for all practical purposes, no ambiguity in the definition of unpolarized light. Occasionally a (usually periodic) rapid variation in polarization over t or over one of the other parameters A, ω or λ is deliberately introduced to simulate the "natural" unpolarized light we have just described. These variations are usually sufficiently coarse that they could conceivably be resolved by some radiometers. For this reason such light is frequently called depolarized instead of unpolarized.

The first Stokes component, $L_{\lambda,0}$, is the spectral radiance that we have been considering in earlier chapters. It is a measure of the total radiant flux associated with a ray and is therefore usually the quantity of greatest interest. It is the value of spectral radiance which a polarization-indifferent radiometer would measure. $L_{\lambda,0}$ can never be negative. The other three Stokes components $L_{\lambda,1}$, $L_{\lambda,2}$ and $L_{\lambda,3}$ may be either negative or positive. If the radiation contains more vertical linear polarization than horizontal linear polarization then the horizontal preference component $L_{\lambda,1}$ will be negative. If the radiation is completely vertically polarized then $L_{\lambda,1} = -L_{\lambda,0}$. If the radiation contains equal horizontally and vertically linearly polarized contributions as in unpolarized light or in 45° polarized light then $L_{\lambda,1} = 0$. If the horizontal content is in excess then $L_{\lambda,1}$ is positive; its maximum value is $L_{\lambda,0}$ when the radiation is completely horizontally linearly polarized. The +45°-preference Stokes component of spectral radiance, $L_{\lambda,2}$, describes the +45° and +135° linearly polarized content of the radiation in an exactly analogous manner. And finally, the right-circular preference component, $L_{\lambda,3}$, similarly takes on values from $-L_{\lambda,0}$ for completely left-circularly polarized light through zero for light with no net circular polarization to $+L_{\lambda,0}$ for completely right-circularly polarized light. Radiation linearly polarized at other angles (other than 0° and 45° and the complementary angles 90° and 135°) will have non-zero values for both $L_{\lambda,1}$ and $L_{\lambda,2}$. Likewise if either or both of $L_{\lambda,1}$ and $L_{\lambda,2}$ are non-zero and in addition $L_{\lambda,3}$ is non-zero then the radiation contains attributes of both linear and circular polarization and is, of course, elliptically polarized.

Although for spectral radiance the four Stokes components provide a complete and, for all practical purposes, unique description of the polarization state of radiation, the same cannot be said for the integrals of spectral radiance such as radiance, spectral intensity, spectral irradiance, etc. For such cases, as we have indicated, the trajectory of the tip of the electric vector can no longer be described, in general, by three parameters. The only Stokes component of these integrated radiometric quantities which is always unambiguous is the first one -- the integrals of $L_{\lambda,0}$. As in earlier chapters where polarization was neglected, an integral of spectral radiance does not uniquely characterize a complete radiation field; one must, in general, trace the spectral radiance by determining the propagance, ray by ray, from source to receiver and then, at the receiver, perform the desired integration of spectral radiance. Fortunately, at the ultimate detector the usual quantity of interest is the total power, and to calculate this requires only an appropriate integral of $L_{\lambda,0}$. Therefore, we shall discuss polarization only in terms of rays and spectral radiance.

STOKES VECTORS and MUELLER MATRICES

The complete description of polarization in classical radiometry requires the specification of four parameters such as the Stokes components of spectral radiance associated with each ray. These Stokes components are measurable with, for example, a linear polarizer and a quarter-wave plate. We will return to this later after we have generalized the concept of propagance[1] or transmittance to include polarization.

In Chapter 3, eq. (3.14) [1] we related[2] the spectral radiance L' of a ray emerging from an optical medium to the incident spectral radiance L by means of a transmittance or propagance T:

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[1]For simplicity we'll usually speak only of transmittance and won't distinguish between propagance and transmittance. However, our results apply equally well when reflectance is involved provided some care is exercised in defining the coordinate system used for the incident and reflected rays. The simplest coordinate system to use is one with the origin at the point of reflection of the ray. The incident ray is referred to such a coordinate system with its Z-axis along the ray in the direction of propagation, its Y-axis in the plane of incidence -- that is, the plane defined by the incident and reflected rays -- and its X-axis perpendicular to the plane of incidence and, therefore, lying in the reflecting surface or, more generally, tangent to it. The reflected ray is referred to a coordinate system with its Z-axis in the direction of propagation of the reflected ray, with its X-axis coincident with the X-axis of the incident ray and with its Y-axis in the plane of incidence but with its sense chosen to preserve the handedness of the coordinate system. If the incident-ray coordinate system is right-handed then that of the reflected ray is also.

[2]In the interests of typographical simplicity several of the symbols used in earlier chapters will be modified in this chapter. In particular, since we will be dealing only with spectral radiance we will have no need to distinguish between L and L_λ, so we will usually omit the subscript λ on this quantity. The spectral-radiance input to a device, formerly denoted $L_{\lambda i}$, will usually be written L and the output, $L_{\lambda p}$, will usually be written L'. The transmittance, τ, or propagance, τ^*, will usually be written T.

$$L'(x,y,\theta,\phi,\lambda) = T(x,y,\theta,\phi,\lambda)\cdot L(x,y,\theta,\phi,\lambda).$$

L and L' we now see consist of four components each so we must expect four equations of this type -- one for each component. Moreover, since the propagance may differ for different forms of polarization and since every component of L may in general contribute to each component of L' the propagance T must now contain sixteen elements:

$$\begin{aligned}
L_0' &= T_{00}\cdot L_0 + T_{01}\cdot L_1 + T_{02}\cdot L_2 + T_{03}\cdot L_3 \\
L_1' &= T_{10}\cdot L_0 + T_{11}\cdot L_1 + T_{12}\cdot L_2 + T_{13}\cdot L_3 \\
L_2' &= T_{20}\cdot L_0 + T_{21}\cdot L_1 + T_{22}\cdot L_2 + T_{23}\cdot L_3 \\
L_3' &= T_{30}\cdot L_0 + T_{31}\cdot L_1 + T_{32}\cdot L_2 + T_{33}\cdot L_3.
\end{aligned} \tag{6.7}$$

This is the generalization of eq. (3.14) which is required to describe the changes in the Stokes spectral-radiance components of a ray upon passing through an optical medium. Just as the spectral radiance in a beam may vary from ray to ray and from wavelength to wavelength, so may the propagance elements T_{00}, T_{01}, . . . vary from ray to ray, etc. Thus the values of these elements depend upon the ray we are examining and we should write them as $T_{00}(x,y,\theta,\phi,\lambda)$, etc., where the x,y,θ, and ϕ are the same reference-surface quantities which identify the ray of spectral radiance $L_{\lambda,0}(x,y,\theta,\phi,\lambda)$. However, throughout most of this chapter we deal with the behavior of a single ray so we will usually not explicitly display this dependence.

Because the four Stokes polarization components are independent [within the restriction $L_0 \geq (L_1^2 + L_2^2 + L_3^2)^{\frac{1}{2}}$] it is common to describe the set of four as a vector, the Stokes spectral-radiance vector

$$\mathsf{L} = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} \tag{6.8}$$

By expressing spectral radiance this way, as a column vector, eqs. (6.7) can be compactly written as

$$\mathsf{L}' = \mathsf{T}\cdot\mathsf{L}, \tag{6.9}$$

where T is the 4×4 matrix

$$\mathsf{T} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{6.10}$$

and eq. (6.9) means exactly the same thing as eqs. (6.7) -- no more and no less. We shall generally use this vector and matrix notation[1] throughout the rest of this chapter, not because of any useful four-dimensional mental image that might be envisioned, but simply for economy of notation.

There is one precaution about matrix equations which must be observed. That is that the product $\mathsf{T} \cdot \mathsf{L}$ is not always the same as $\mathsf{L} \cdot \mathsf{T}$ and the product of two matrices $\mathsf{T}_1 \cdot \mathsf{T}_2$ is not generally the same as $\mathsf{T}_2 \cdot \mathsf{T}_1$. The rule for writing the equations correctly in accordance with the conventional matrix multiplication formulas is to write the incident radiation vector and the transmittance matrices in the order, from right to left, in which the ray encounters the optical elements they represent. So, if a ray with spectral-radiance vector L passes through one optical device with transmittance matrix T_1 and then through a second with transmittance matrix T_2, the emerging spectral radiance will be

$$\mathsf{L}'(x,y,\theta,\phi,\lambda) = \mathsf{T}_2(x,y,\theta,\phi,\lambda) \cdot \mathsf{T}_1(x,y,\theta,\phi,\lambda) \cdot \mathsf{L}(x,y,\theta,\phi,\lambda). \tag{6.11}$$

The effect of a succession of optical devices can be combined into a single effective transmittance matrix by simply multiplying the individual matrices together in the proper order. Thus, eq. (6.11) can be written

$$\mathsf{L}' = \mathsf{T} \cdot \mathsf{L} , \tag{6.12}$$

where the product matrix $\mathsf{T} = \mathsf{T}_2 \cdot \mathsf{T}_1$ can be obtained by following the rules for matrix multiplication. Of course the same result would be obtained by writing out eqs. (6.7) for the radiance components passing device 1 and then again using eqs. (6.7) inserting these expressions for the radiance input to device 2 to obtain expressions for the final emerging radiance components.

The matrices which describe the transmittance or propagance through optical elements which influence polarization are known in the literature as <u>Mueller matrices</u>. In very simple cases the proper form of the Mueller matrix is obvious by inspection. For optical elements for which equations can be written relating the amplitudes and phases of the

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[1]Readers who are not familiar with matrix manipulations, or who tend to forget which subscript means what, will find everything needed to convert these matrix equations to workable formulas in Appendix 6.

emerging radiation field to the amplitudes and phases of the entering field, the matrix can be derived by a straightforward procedure [7,8]. This includes all perfect polarizing optical elements such as ideal polarizers, quarter-wave plates, mirrors, etc. For a real, non-ideal device, however, the Mueller matrix can only be obtained ultimately by experiment. Mueller matrices for ideal optical components have been tabulated by many authors [2,7].

THE SIMPLE SPECTRAL FILTER

Let us now study some examples of the use of Mueller matrices and Stokes components. Perhaps the simplest useful optical element is just a glass absorbing spectral filter with transmittance $\tau(\lambda)$. The Mueller transmittance matrix for this should have the effect of simply multiplying each component of L by τ with no mixing by or interaction with other components. That is, we want

$$\begin{aligned} L_0' &= \tau(\lambda)\cdot L_0 \\ L_1' &= \tau(\lambda)\cdot L_1 \\ L_2' &= \tau(\lambda)\cdot L_2 \\ L_3' &= \tau(\lambda)\cdot L_3. \end{aligned} \tag{6.13}$$

Comparing these equations with eq. (6.7) we see that $T_{00} = \tau(\lambda)$, $T_{11} = \tau(\lambda)$, $T_{22} = \tau(\lambda)$, $T_{33} = \tau(\lambda)$ and all other $T_{ij} = 0$. Thus, for a simple filter we have

$$\mathsf{T} = \begin{bmatrix} \tau(\lambda) & 0 & 0 & 0 \\ 0 & \tau(\lambda) & 0 & 0 \\ 0 & 0 & \tau(\lambda) & 0 \\ 0 & 0 & 0 & \tau(\lambda) \end{bmatrix} = \tau(\lambda)\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.14}$$

THE IDEAL POLARIZER

A more interesting optical element is the ideal linear polarizer. An ideal polarizer possesses a polarization axis (perpendicular to the ray direction) with the property that the transmittance to light linearly polarized in this direction is 1 and the transmittance to light linearly polarized at right angles to this direction is 0. Aside from surface-reflection effects, crystal-prism polarizers such as the Nicol-prism polarizer come close to this ideal. If the angle between the polarization axis of the polarizer and the horizontal reference axis or x-axis of the optical system is ϕ the Mueller transmittance matrix of an ideal linear polarizer is

$$\mathsf{P}(\phi) = \tfrac{1}{2}\begin{bmatrix} 1 & \cos 2\phi & \sin 2\phi & 0 \\ \cos 2\phi & \cos^2 2\phi & \sin 2\phi\cdot\cos 2\phi & 0 \\ \sin 2\phi & \sin 2\phi\cdot\cos 2\phi & \sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.15}$$

Let us use eq. (6.9) to see the effect of such a polarizer on unpolarized light. Since unpolarized light is characterized by $L_1 = L_2 = L_3 = 0$, we have

$$\mathsf{L}' = \mathsf{P}(\phi)\cdot\mathsf{L} = \tfrac{1}{2}\begin{bmatrix} 1 & \cos 2\phi & \sin 2\phi & 0 \\ \cos 2\phi & \cos^2 2\phi & \sin 2\phi\cdot\cos 2\phi & 0 \\ \sin 2\phi & \sin 2\phi\cdot\cos 2\phi & \sin^2 2\phi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} L_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \tfrac{1}{2}\, L_0\cdot\begin{bmatrix} 1 \\ \cos 2\phi \\ \sin 2\phi \\ 0 \end{bmatrix}. \qquad (6.16)$$

The vector equation (6.16) is simply a shorthand for the set of four equations:

$$\begin{aligned} L_0' &= \tfrac{1}{2}L_0 \\ L_1' &= \tfrac{1}{2}L_0\cdot\cos 2\phi \\ L_2' &= \tfrac{1}{2}L_0\cdot\sin 2\phi \\ L_3' &= 0. \end{aligned}$$

By using the trigonometric identity $\cos^2 2\phi + \sin^2 2\phi = 1$ we see that $L_0' = \left(L_1'^2 + L_2'^2 + L_3'^2\right)^{\frac{1}{2}}$. This shows [eq. (6.4)] that the output radiance is completely polarized. Since $L_3' = 0$ there is no circular component so the polarization is strictly linear. If we turn the polarizer so that its axis is parallel to the x axis we would expect horizontally polarized light. Putting $\phi = 0°$ in eq. (6.16) we obtain

$$\mathsf{L}' = \mathsf{P}(0°)\cdot\mathsf{L} = \tfrac{1}{2}L_0\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, light with Stokes components like

$$\begin{bmatrix} L \\ L \\ 0 \\ 0 \end{bmatrix}$$

is pure horizontally polarized light. If the polarizer is turned to $\phi = 45°$ we expect light polarized in that orientation. From eq. (6.16) we get

$$\mathsf{L}' = \tfrac{1}{2}L_0\cdot\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, light whose Stokes components are a multiple of

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is linearly polarized at 45°. Similarly inserting $\phi = 90°$ and $135°$ in eq. (6.16) shows that vertically polarized light is a multiple of

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

and light polarized at 135° is characterized by Stokes components proportional to

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Notice that the spectral radiance L'_0 is always positive, as its name suggests it must be, while the other components may have either sign.

Suppose instead of starting with unpolarized light we had put arbitrarily polarized light through the ideal linear polarizer. We have

$$\mathsf{L}' = \mathsf{P}(\phi)\cdot\mathsf{L} = \tfrac{1}{2}\begin{bmatrix} 1 & C & S & 0 \\ C & C^2 & S\cdot C & 0 \\ S & S\cdot C & S^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} = \tfrac{1}{2}(L_0 + L_1\cdot C + L_2\cdot S)\cdot\begin{bmatrix} 1 \\ C \\ S \\ 0 \end{bmatrix} \tag{6.17}$$

where we have used the abbreviations $\cos 2\phi = C$ and $\sin 2\phi = S$. As with initially unpolarized radiation, the transmitted radiation is completely linearly polarized at the orientation ϕ of the polarizer axis. Equation (6.17) takes on an especially simple form if the initial ray is completely linearly polarized. In this case, as we have seen, L has the form

$$\mathsf{L} = L_0 \cdot \begin{bmatrix} 1 \\ \cos 2\psi \\ \sin 2\psi \\ 0 \end{bmatrix}$$

where ψ is the polarization direction. Substituting from this expression $L_1 = L_0\cdot\cos 2\psi$ and $L_2 = L_0\cdot\sin 2\psi$ in eq. (6.17) we find

$$\mathsf{L}' = \tfrac{1}{2}L_0\cdot[1 + \cos 2(\phi-\psi)]\cdot \begin{bmatrix} 1 \\ C \\ S \\ 0 \end{bmatrix} = L_0\cdot\cos^2(\phi-\psi)\cdot \begin{bmatrix} 1 \\ \cos 2\phi \\ \sin 2\phi \\ 0 \end{bmatrix} \tag{6.18}$$

When $\phi = \psi$ the polarizer has no effect on the ray but when $\phi = \psi \pm 90°$ the ray is completely extinguished. Equation (6.18) is known as Malus' Law and expresses the familiar behavior of a pair of ideal polarizers.

Before going on to consider other polarizing components we should remark that the discussion above can be turned around and used to derive the Mueller transmittance matrix of an ideal polarizer [eq. (6.15)]. The argument runs as follows: We define an ideal linear polarizer as an object (1) which cannot be used to distinguish circularly polarized light from unpolarized light, (2) whose output is always completely linearly polarized in a direction parallel to its polarization axis, and (3) which attenuates completely linearly polarized light in accordance with Malus' Law [eq. (6.18)]. If we now examine eqs. (6.7) we see that, because the first property says that the L_k' components on the left do not depend upon the value of L_3, the last column of the matrix must consist entirely of zeros. The second property says that the output of the polarizer is proportional to

$$\begin{bmatrix} 1 \\ \cos 2\phi \\ \sin 2\phi \\ 0 \end{bmatrix}$$

where ϕ is the orientation of the polarizer axis. Combining this with the third property we see that if the input ray has a radiance vector

$$\mathsf{L} = L_0\cdot \begin{bmatrix} 1 \\ \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{bmatrix}$$

the output ray will be described by

$$\mathsf{L}' = L_0\cdot\cos^2(\phi-\alpha)\cdot \begin{bmatrix} 1 \\ \cos 2\phi \\ \sin 2\phi \\ 0 \end{bmatrix}.$$

Therefore the Mueller matrix $\mathsf{P}(\phi)$ for an ideal linear polarizer must satisfy

$$\mathsf{L}' = \tfrac{1}{2}L_0 \cdot \begin{bmatrix} 1 + \cos2\phi\cdot\cos2\alpha + \sin2\phi\cdot\sin2\alpha \\ \cos2\phi + \cos^2 2\phi\cdot\cos2\alpha + \sin2\phi\cdot\cos2\phi\cdot\sin2\alpha \\ \sin2\phi + \sin2\phi\cdot\cos2\phi\cdot\cos2\alpha + \sin^2 2\phi\cdot\sin2\alpha \\ 0 \end{bmatrix} = \mathsf{P}(\phi)\cdot L_0 \cdot \begin{bmatrix} 1 \\ \cos2\alpha \\ \sin2\alpha \\ 0 \end{bmatrix}$$

where we have substituted $\cos^2(\phi-\alpha) = \tfrac{1}{2}(1 + \cos2\phi\cdot\cos2\alpha + \sin2\phi\cdot\sin2\alpha)$. Comparing this equation with eqs. (6.7) and remembering that the last column of $\mathsf{P}(\phi)$ must be zero we immediately obtain eq. (6.15). This derivation illustrates the possibility of deducing the form of a Mueller matrix from the observed experimental behavior of the object it is meant to describe. We will later use this approach again with a Mueller matrix form which more accurately describes real (non-ideal) polarizers.

THE IDEAL LINEAR RETARDER

A linear retarder is an optical device frequently made of a thin plate of an anisotropic crystal such as calcite or quartz with its faces cut parallel to the crystal optic axis. Such plates have the (birefringence) property that light linearly polarized in one direction, called the fast axis of the retarder, travels with a higher velocity than light linearly polarized in a direction at right angles to this, called the slow axis. A mixture of both polarization forms entering a retarder then will emerge with the components phase shifted with respect to one another and consequently with its polarization form changed. The amount of phase shift is called the retardance, δ. The retardance is related to the thickness, a, of the plate and the indices of refraction n_f and n_s for the fast and slow polarizations, respectively, according to

$$\delta = \frac{2\pi a}{\lambda}(n_f - n_s) \quad [\text{rad}]. \tag{6.19}$$

The Mueller transmittance matrix for an ideal retarder is:

$$\mathsf{R}(\delta,\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C^2 + S^2\cdot\cos\delta & S\cdot C\cdot(1-\cos\delta) & -S\cdot\sin\delta \\ 0 & S\cdot C\cdot(1-\cos\delta) & S^2 + C^2\cdot\cos\delta & C\cdot\sin\delta \\ 0 & S\cdot\sin\delta & -C\cdot\sin\delta & \cos\delta \end{bmatrix} \tag{6.20}$$

where $C = \cos2\phi$, $S = \sin2\phi$ and ϕ is the angle between the fast axis of the retarder and the horizontal.

Retarders in which the retardance is one-quarter cycle $(\delta = \pi/2)$ or one-half cycle $(\delta = \pi)$ are called quarter- and half-wave plates. Quarter-wave plates are useful for converting linearly polarized light to circularly polarized light, and vice-versa; half-wave plates, for rotating the direction of polarization of linearly polarized light. If we set $\delta = \pi/2$ in eq. (6.20) we obtain the Mueller matrix for an ideal quarter-wave plate:

$$Q(\phi) = R(\pi/2,\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C^2 & S\cdot C & -S \\ 0 & S\cdot C & S^2 & C \\ 0 & S & -C & 0 \end{bmatrix}. \tag{6.21}$$

DETERMINING STOKES COMPONENTS with IDEAL INSTRUMENTS

With an ideal linear polarizer and an ideal quarter-wave plate we are now in a position to measure, in principle, the four Stokes components of an arbitrary light ray. We pass the ray through the quarter-wave plate and then through the polarizer and measure the output with a polarization-indifferent detector. The Mueller transmittance matrix for the combination of quarter-wave plate at angle $\phi = \alpha$ and polarizer at angle $\phi = \beta$ is

$$A = P(\beta)\cdot Q(\alpha)$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 & \cos 2\beta & \sin 2\beta & 0 \\ \cos 2\beta & \cos^2 2\beta & \sin 2\beta\cdot\cos 2\beta & 0 \\ \sin 2\beta & \sin 2\beta\cdot\cos 2\beta & \sin^2 2\beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\alpha & \sin 2\alpha\cdot\cos 2\alpha & -\sin 2\alpha \\ 0 & \sin 2\alpha\cdot\cos 2\alpha & \sin^2 2\alpha & \cos 2\alpha \\ 0 & \sin 2\alpha & -\cos 2\alpha & 0 \end{bmatrix} \quad (6.22)^1$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 & \cos 2\alpha\cdot\cos 2(\beta-\alpha) & \sin 2\alpha\cdot\cos 2(\beta-\alpha) & \sin 2(\beta-\alpha) \\ \cos 2\beta & \cos 2\beta\cdot\cos 2\alpha\cdot\cos 2(\beta-\alpha) & \cos 2\beta\cdot\sin 2\alpha\cdot\cos 2(\beta-\alpha) & \cos 2\beta\cdot\sin 2(\beta-\alpha) \\ \sin 2\beta & \sin 2\beta\cdot\cos 2\alpha\cdot\cos 2(\beta-\alpha) & \sin 2\beta\cdot\sin 2\alpha\cdot\cos 2(\beta-\alpha) & \sin 2\beta\cdot\sin 2(\beta-\alpha) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If a ray with spectral-radiance vector

$$L = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

passes through this combination, the spectral radiance vector L' of the output ray will be given by

$$L' = A\cdot L = \tfrac{1}{2}\left[L_0 + L_1\cdot\cos 2\alpha\cdot\cos 2(\beta-\alpha) + L_2\cdot\sin 2\alpha\cdot\cos 2(\beta-\alpha) + L_3\cdot\sin 2(\beta-\alpha)\right]\cdot\begin{bmatrix} 1 \\ \cos 2\beta \\ \sin 2\beta \\ 0 \end{bmatrix}. \tag{6.23}$$

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[1]Obviously, unless one can find simplifying trigonometric identities a long train of polarizing optics will quickly lead to an almost unmanageable problem. The operator form of Mueller matrix manipulation described by Priebe [9] permits many simplifications to be performed before the matrix elements become too unwieldy. His technique is especially useful for weakly (linearly) polarizing components, small retardances, and small angles of rotation.

Since we are only interested in the response of a polarization-indifferent detector to this ray we actually need only the L_0' component of L' which is

$$L_0' = \tfrac{1}{2}\left[L_0 + L_1\cdot\cos2\alpha\cdot\cos2(\beta-\alpha) + L_2\cdot\sin2\alpha\cdot\cos2(\beta-\alpha) + L_3\cdot\sin2(\beta-\alpha)\right]. \tag{6.24}$$

Had we been interested in passing the output into or through any further polarization-sensitive components we would, in general, have needed all four components of L'. From eq. (6.24) we see that if we make four measurements of L_0' at four combinations of α and β, we can solve the four resulting equations for the Stokes components L_0, L_1, L_2, and L_3 of the original ray. A particularly simple solution is obtained when α and β are set at 0°, 45°, and 90° as follows:

$$\begin{aligned}
L_0'\ (\alpha = 0,\ \beta = 0) &= \tfrac{1}{2}(L_0 + L_1)\\
L_0'\ (\alpha = 0,\ \beta = 90°) &= \tfrac{1}{2}(L_0 - L_1)\\
L_0'\ (\alpha = 45°,\ \beta = 45°) &= \tfrac{1}{2}(L_0 + L_2)\\
L_0'\ (\alpha = 0,\ \beta = 45°) &= \tfrac{1}{2}(L_0 + L_3).
\end{aligned} \tag{6.25}$$

From these four measured values of L_0' the polarization components of the original ray can be obtained at once.

In general the result of passing a ray through an ideal retarder followed by an ideal linear polarizer is a ray whose spectral radiance is a sum of the Stokes components of the original ray each multiplied by sines and cosines of the orientation angles and the retardance.[1] Such a relationship readily lends itself to Fourier analysis, and many automatic techniques for measuring some or all of the Stokes components are based on this idea [10,11].

It is possible to deduce relative values of L_1, L_2, and L_3 by measuring the pair of angular settings α and β of the quarter-wave plate and polarizer which simultaneously minimize L_0'. This is a common technique in ellipsometry, for example, because such a determination requires only monotonicity of detector response -- not linearity or, indeed, not even significant stability. The minimum value of L_0' which can be achieved by adjusting α and β is

$$L_0' = \tfrac{1}{2}\left[L_0 - (L_1^2 + L_2^2 + L_3^2)^{\frac{1}{2}}\right].$$

This occurs when

$$\tan2\alpha = L_2/L_1$$

and

$$\tan2(\beta-\alpha) = L_3/(L_1^2 + L_2^2)^{\frac{1}{2}}, \tag{6.26}$$

[1]The complete expression is given in eq. (6.41).

as can be obtained by differentiation of eq. (6.24). From a measurement of the angles α and $\beta-\alpha$ at this minimum[1] then we can calculate the relative values of the polarization components from eq. (6.26):

$$\begin{aligned} L_1/L_p &= \cos 2\alpha \cdot \cos 2(\beta-\alpha) \\ L_2/L_p &= \sin 2\alpha \cdot \cos 2(\beta-\alpha) \\ L_3/L_p &= \sin 2(\beta-\alpha), \end{aligned} \tag{6.27}$$

where $L_p = (L_1^2 + L_2^2 + L_3^2)^{\frac{1}{2}}$ is the polarized component of L_0. Unfortunately the determination of L_0 and L_p cannot be carried out by measurements of angles only but requires flux measurements with the ubiquitous problems of temporal, positional, directional, and wavelength responsivity and of detector linearity which characterize and bedevil radiometric measurements.

An alternative way to measure relative values of L_1, L_2, and L_3 is by using a variable retarder -- usually called a <u>compensator</u>. This is a retarder composed essentially of two wedge-shaped retarders arranged so that sliding them across one another changes the total thickness of the composite block and hence the retardance of the device. By simultaneously adjusting the retardance and angular orientation of a perfect compensator, arbitrarily polarized light can be converted to any other arbitrary polarization state which possesses the same degree of polarization. Thus, we may pass radiation, whose Stokes components are to be determined, through a compensator and then through a fixed perfect polarizer and adjust the compensator so that the polarizer is able to extinguish the polarized component of the light. From the retardance and orientation settings of the compensator when this adjustment has been achieved the Stokes component values can be computed. Taking for simplicity a polarizer with its polarization axis horizontal we have from eqs. (6.15) and (6.20)

$$P(0°)\cdot R(\delta,\phi)\cdot L = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C^2 + S^2\cdot\cos\delta & S\cdot C\cdot(1-\cos\delta) & -S\cdot\sin\delta \\ 0 & S\cdot C\cdot(1-\cos\delta) & S^2 + C^2\cdot\cos\delta & C\cdot\sin\delta \\ 0 & S\cdot\sin\delta & -C\cdot\sin\delta & \cos\delta \end{bmatrix} \cdot \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}.$$

The output radiance is

$$L_0' = \tfrac{1}{2}\left[L_0 + L_1\cdot(C^2 + S^2\cdot\cos\delta) + L_2\cdot S\cdot C\cdot(1-\cos\delta) - L_3\cdot S\cdot\sin\delta\right].$$

[1]In the literature of ellipsometry the angles α and $\chi = \beta-\alpha$ defined by eq. (6.26) are known respectively as the azimuth and ellipticity of the polarized component. See reference [6] for a review of ellipsometry.

By differentiation we find that L_0' has a minimum value of

$$L_0' = \tfrac{1}{2}\left[L_0 - (L_1^2 + L_2^2 + L_3^2)^{\frac{1}{2}}\right] = \tfrac{1}{2}(L_0 - L_p)$$

at values of the orientation angle ϕ and retardance δ which satisfy:

$$\begin{aligned} L_1/L_p &= -(\cos^2 2\phi + \sin^2 2\phi \cdot \cos\delta) \\ L_2/L_p &= -\sin 2\phi \cdot \cos 2\phi \cdot (1 - \cos\delta) \\ L_3/L_p &= \sin 2\phi \cdot \sin\delta. \end{aligned} \tag{6.28}$$

These ellipsometric techniques can be extremely accurate and are used extensively in studying surfaces and surface films by the analysis of the relative polarization form of reflected light [5,6]. Note that these techniques require an accurate means of sensing a minimum flux level and that L_p and the more important component L_0 remain undetermined.

THE IDEAL CIRCULAR POLARIZER

In eqs. (6.25) we saw how a quarter-wave plate followed by an ideal linear polarizer with its polarization axis at 45° to the fast axis of the retarder could be used to gather information about the Stokes component L_3, which is the circular polarization component. In this case the Mueller transmittance matrix [eq. (6.22)] reduces to:

$$\mathcal{A}_{cir} = \mathcal{P}(\beta)\cdot\mathcal{Q}(\alpha = \beta \mp 45°) = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ C & 0 & 0 & \pm C \\ S & 0 & 0 & \pm S \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.29}$$

where $C = \cos 2\beta$, $S = \sin 2\beta$, β describes the orientation of the linear polarizer and the + or - signs are chosen according to the sign of $\beta - \alpha = \pm 45°$. Arbitrarily polarized light passing through this combination will emerge with Stokes vector

$$\mathcal{L}' = \mathcal{A}_{cir}\cdot\begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} = \tfrac{1}{2}(L_0 \pm L_3)\cdot\begin{bmatrix} 1 \\ \cos 2\beta \\ \sin 2\beta \\ 0 \end{bmatrix}. \tag{6.30}$$

Suppose, for example, that the incoming light is completely circularly polarized so that $L_0 = \pm L_3$ and that its handedness is opposite to that of $\mathcal{A}_{cir}$. We will then observe $\mathcal{L}' = 0$, i.e. complete extinction. Notice that although a measurement of the spectral radiance of the emergent ray yields information primarily about the circular polarization component L_3 the emergent ray itself is linearly polarized. In order to generate circularly polarized light we must reverse the retarder-polarizer combination: the light must first pass through the linear polarizer and then through the quarter-wave plate. Taken in this order the combination has the Mueller matrix

$$\mathsf{P}_{cir} = \mathsf{Q}(\beta \pm 45°)\cdot\mathsf{P}(\beta) = \tfrac{1}{2}\begin{bmatrix} 1 & C & S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & \pm C & \pm S & 0 \end{bmatrix}, \tag{6.31}$$

where $C = \cos 2\beta$, $S = \sin 2\beta$ and β is the orientation of the linear polarizer. The quarter-wave plate is at the angle $\alpha = \beta \pm 45°$. Now if arbitrarily polarized light passes through this combination the result is

$$\mathsf{L}' = \mathsf{P}_{cir}\cdot\begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} = \tfrac{1}{2}(L_0 + L_1\cos 2\beta + L_2\sin 2\beta)\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{bmatrix} \tag{6.32}$$

which represents pure circularly polarized light. In addition to illustrating the mathematical quirk that the product of two matrices may depend upon the order in which they are taken, the difference between A_{cir} and P_{cir} [(or eqs. (6.29) and (6.31)] is important experimentally since commercial circular polarizers are usually made by cementing a quarter-wave plate to a linear polarizer with their axes 45° apart. Such a device will act as a circular polarizer only if light enters from the linear polarizer side and will act as an analyzer of circularly polarized light only if the light enters from the quarter-wave plate side.

An ideal circular polarizer without these idiosyncracies can be made by adding a second quarter-wave plate to make a sandwich with the linear polarizer in the middle. Each element must be turned 45° from the previous one to give

$$\mathsf{P}_{cir} = \mathsf{Q}(\psi \pm 90°)\cdot\mathsf{P}(\psi \pm 45°)\cdot\mathsf{Q}(\psi) \tag{6.33}$$

where $\mathsf{Q}(\phi)$ is the Mueller matrix [eq. (6.21)] for a quarter-wave plate and $\mathsf{P}(\phi)$ is that for a linear polarizer [eq. (6.15)]. If we multiply out the matrices we find

$$\mathsf{P}_{cir} = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix}. \tag{6.34}$$

Plus signs in eq. (6.33) lead to plus signs in eq. (6.34) and a right-circular polarizer. Minus signs throughout give a left-circular polarizer. This circular polarizer now has properties analogous to those we associate with the ideal linear polarizer: it can be used as polarizer or analyzer from either side; it will pass circularly polarized light of the proper handedness but will extinguish any component of the opposite handedness; and it cannot distinguish between linearly polarized light and unpolarized light.

RADIOMETER CALIBRATION

Equation (6.32) is a special case of

$$L' = Q(\alpha)\cdot P(\beta)\cdot L$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 & \cos 2\beta & \sin 2\beta & 0 \\ \cos 2\alpha\cdot\cos 2(\alpha-\beta) & \cos 2\alpha\cdot\cos 2\beta\cdot\cos 2(\alpha-\beta) & \cos 2\alpha\cdot\sin 2\beta\cdot\cos 2(\alpha-\beta) & 0 \\ \sin 2\alpha\cdot\cos 2(\alpha-\beta) & \sin 2\alpha\cdot\cos 2\beta\cdot\cos 2(\alpha-\beta) & \sin 2\alpha\cdot\sin 2\beta\cdot\cos 2(\alpha-\beta) & 0 \\ \sin 2(\alpha-\beta) & \cos 2\beta\ \sin 2(\alpha-\beta) & \sin 2\beta\cdot\sin 2(\alpha-\beta) & 0 \end{bmatrix}\cdot L \qquad (6.35)$$

$$= \tfrac{1}{2}(L_0 + L_1\cdot\cos 2\beta + L_2\cdot\sin 2\beta)\cdot\begin{bmatrix} 1 \\ \cos 2\alpha\cdot\cos 2(\alpha-\beta) \\ \sin 2\alpha\cdot\cos 2(\alpha-\beta) \\ \sin 2(\alpha-\beta) \end{bmatrix},$$

which can be obtained by multiplying the matrices of eqs. (6.21) and (6.15). This describes the behavior of a polarizer at orientation angle β followed by a quarter-wave plate at orientation angle α. By the proper orientation of the quarter-wave plate and polarizer any arbitrary state of polarization of the emerging beam can be achieved. If $\beta = \alpha$, for example, the output is linearly polarized at the orientation α, and if $\beta = \alpha \pm 45°$ the output is right (+) or left (-) circularly polarized. Such a beam can be used to measure Mueller transmittance matrix elements and to calibrate a radiometer.

The complete Mueller transmittance matrix for an optical component must be known if the polarization state of a ray after passing through this component is wanted. However, in many cases all that is needed is a prediction of the response of a complete instrument to the input radiation. This is a simpler problem because the instrument response depends only upon the spectral-radiance component L_0' which triggers the final electronic, thermal, or chemical response.[1] Here, only the top row of the Mueller matrix of the instrument is required. That is, only the first of eqs. (6.7) is needed. We can write

$$dS = (R_{00}\cdot L_0 + R_{01}\cdot L_1 + R_{02}\cdot L_2 + R_{03}\cdot L_3)\cdot d\Theta\cdot d\lambda$$

where dS is the element of radiometer signal output associated with the ray whose Stokes spectral-radiance components are L_0, L_1, L_2, and L_3 and whose elements of throughput

[1]Some detectors may respond to properties of the electromagnetic radiation other than power -- angular momentum, for example. It appears that such detectors can still be treated in a purely formal way as we describe here, although the resulting responsivities, R_{0i}, may not be proportional to the Mueller matrix elements of any physically realizable optical component.

and wavelength are $d\Theta = d\omega \cdot \cos\theta \cdot dA$ and $d\lambda$, respectively. The coefficients R_{00}, R_{01}, etc., are the first-row Mueller transmittance-matrix elements of the optical part of the radiometer, including the optical polarization characteristics of the detector- or transducer-element responsivity, multiplied by the polarization-indifferent responsivity of that element in combination with any signal-processing and display components of the instrument.[1] The total response of the instrument is the integral of this expression over all rays which can reach the photo-sensitive surface.

$$S = \int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} (R_{00}\cdot L_0 + R_{01}\cdot L_1 + R_{02}\cdot L_2 + R_{03}\cdot L_3)\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda \qquad (6.36)$$

where $\Delta\lambda$, ΔA, and $\Delta\omega$ are the acceptance intervals of the radiometer. If these acceptance intervals are small enough that the Stokes radiance vector can be considered uniform over all of them then[2]

$$S = \mathcal{R}_{00}\cdot L_0 + \mathcal{R}_{01}\cdot L_1 + \mathcal{R}_{02}\cdot L_2 + \mathcal{R}_{03}\cdot L_3 \qquad (6.36a)$$

where $\mathcal{R}_{0i} = \int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} R_{0i}\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda$. These four responsivity factors $\mathcal{R}_{0i}$ can be determined by observing the response of the instrument to four sufficiently different, known polarization states of a test beam with fixed geometry and spectral distribution. We thus obtain four equations of the form of eq. (6.36a) to be solved for the four responsivity factors $\mathcal{R}_{0i}$. The L_i are the Stokes components of the input-beam radiance produced, for example, as described above, following eq. (6.35), and measured, if necessary, as described in connection with eq. (6.24) using a polarization-indifferent detector and a second quarter-wave plate and linear polarizer.

If the acceptance intervals of the instrument are too large to assume uniform radiance, then it may be that the coefficients R_{0i} in eq. (6.36) can be considered constant, independent of wavelength and of ray position and direction. In this case

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[1]Just as the electrical engineer replaces an actual generator by an equivalent circuit consisting of either an ideal voltage generator and series impedance or an ideal current generator and shunt impedance, it is convenient, analytically, to replace the actual detector or transducer element by an equivalent combination of (1) a pure transmittance (whose Mueller transmittance matrix can be combined with that of the other optical elements by matrix multiplication) followed by (2) a polarization-indifferent detector element that, in combination with the following signal-processing and display components, determines the magnitude of the overall responsivity.

[2]As in earlier chapters we assume linearity throughout this discussion. That is, R_{0i} may depend upon ray position, direction and wavelength but not upon Stokes spectral-radiance components.

$$S = R_{00} \cdot \Phi_0 + R_{01} \cdot \Phi_1 + R_{02} \cdot \Phi_2 + R_{03} \cdot \Phi_3 \qquad (6.36b)$$

where $\Phi_i = \int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} L_i \cdot d\omega \cdot \cos\theta \cdot dA \cdot d\lambda$. Again the four responsivity coefficients can be determined from observations of the output response S to four beams with known, different, Stokes spectral-radiance-vector distributions. A common way of achieving approximately isotropic (directionally uniform) responsivity, at least over the acceptance solid angle $\Delta\omega$, is, as we have seen in Chapter 5 [12], to introduce a diffuser as the first component of the radiometer. In this case, since such a diffuser is also a depolarizer, we have the further simplification: $R_{01} = R_{02} = R_{03} = 0$. The quantity Φ_0 is the total radiant flux in the wavelength pass band of the radiometer.

If neither the R_{0i} nor the L_i are uniform, the dependence of the R_{0i} upon ray direction, position and wavelength can still be determined by probing the instrument with well-defined, narrow pencils of rays with uniform Stokes spectral-radiance vector. The general usefulness of such a radiometer, however, is questionable since its response is a sum of weighted integrals of the spectral radiance distribution falling upon it. For comparison of sources with identical relative spectral radiance distribution over direction, position and wavelength, of course, such a radiometer might be satisfactory and its calibration in this case again calls for only four measurements.

THE ROTATION MATRICES

If we look back at eqs. (6.15) and (6.21) we notice a similarity in the way the angle ϕ enters into the matrices for a polarizer and for a quarter-wave plate. This is not accidental. It arises in the following way: suppose we somehow were given numerical Mueller propagance-matrix-element values for an optical component at a particular orientation, say $\phi = 0$. How can we generalize this to other orientations, $\phi \neq 0$? We do it in a kind of backward way by first finding the Stokes components of an entering ray in a coordinate frame which has been rotated through the angle ϕ so as to match the orientation for which we know the Mueller matrix of the (rotated) optical component. Then we can multiply this Stokes vector by the Mueller matrix and finally rotate the coordinate frame back to the original "horizontal" orientation.

Consider an arbitrarily polarized ray described by

$$\mathsf{L} = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} .$$

If we rotate our coordinate frame, only the linear polarization components, L_1 and L_2, will be affected because these are the only components whose definitions require specification of a reference, horizontal direction. So, thinking of L_1 and L_2 for the moment as orthogonal components of a two-component vector (see fig. 6.2) we can easily find the new components of this vector in a rotated frame of reference. In real space we know that if we rotate the "horizontal" direction by 45° we will have rotated the direction of L_1 into that of $\pm L_2$. In our two-dimensional vector space, however, the angle between L_1 and L_2 is 90°; therefore a rotation through the angle ϕ in real space corresponds to a rotation through the angle 2ϕ in our (L_1,L_2) vector space. We see from figure 6.2 that such a rotation of our coordinate axes will leave us with new components

$$L_1' = L_1 \cdot \cos 2\phi + L_2 \cdot \sin 2\phi$$

and

$$L_2' = -L_1 \cdot \sin 2\phi + L_2 \cdot \cos 2\phi.$$

This result, combined with

$$L_0' = L_0$$

and

$$L_3' = L_3$$

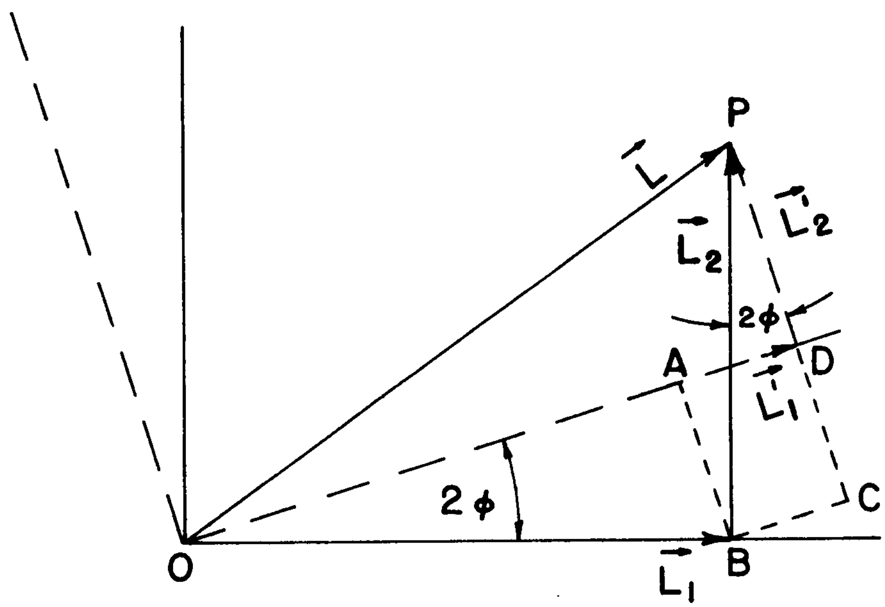
is very suggestive of eq. (6.7): that is, by defining a matrix $\mathsf{M}(\phi)$ we can write

$$\mathsf{L}' = \mathsf{M}(\phi)\cdot\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi & 0 \\ 0 & -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathsf{L} \tag{6.37}$$

to describe this rotation. The procedure for handling an optical component turned at an angle ϕ is now fairly clear. Rotating the coordinates of L through the angle ϕ so that "horizontal" agrees with the optical-component orientation, applying its Mueller propagance matrix, and rotating again through $-\phi$ to get back to the real horizontal is equivalent to the following

$$\mathsf{L}'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-2\phi) & \sin(-2\phi) & 0 \\ 0 & -\sin(-2\phi) & \cos(-2\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathsf{T} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi & 0 \\ 0 & -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathsf{L} \tag{6.38}$$

in which T is the Mueller matrix. The matrix $\mathsf{M}(\phi)$ defined in eq. (6.37) is known as a <u>rotation matrix</u>. If the multiplication of the three 4×4 matrices in eq. (6.38) is carried out we obtain the equivalent Mueller matrix evaluated at the angle ϕ. That is,



$$L_1' = OD = OA + AD = OA + BC = L_1 \cdot \cos 2\phi + L_2 \cdot \sin 2\phi$$

$$L_2' = DP = CP - DC = CP - AB = L_2 \cdot \cos 2\phi - L_1 \cdot \sin 2\phi$$

Figure 6.2. The components of $\vec{L}$ in the L_1, L_2 and L_1', L_2' coordinate systems.

$$\mathcal{T}(\phi) = \mathcal{M}(-\phi)\cdot\mathcal{T}(0)\cdot\mathcal{M}(\phi)$$

$$= \begin{bmatrix} T_{00} & T_{01}\cdot C - T_{02}\cdot S & T_{01}\cdot S + T_{02}\cdot C & T_{03} \\ T_{10}C - T_{20}S & T_{11} - A & T_{12} + B & T_{13}\cdot C - T_{23}\cdot S \\ T_{10}\cdot S + T_{20}\cdot C & T_{21} + B & T_{22} + A & T_{13}\cdot S + T_{23}\cdot C \\ T_{30} & T_{31}\cdot C - T_{32}\cdot S & T_{31}\cdot S + T_{32}\cdot C & T_{33} \end{bmatrix} \qquad (6.39)$$

where $A = (T_{11} - T_{22})\cdot S^2 + (T_{12} + T_{21})\cdot S\cdot C$

$B = (T_{11} - T_{22})\cdot S\cdot C - (T_{12} + T_{21})\cdot S^2$

$S = \sin 2\phi$

$C = \cos 2\phi$

and T_{00}, T_{01}, T_{02}, etc., are taken to be the matrix elements [eq. (6.10)] at $\phi = 0$. Thus, in their dependence upon the orientation angle ϕ, all Mueller matrices must resemble the matrix product of eq. (6.39). This multiplication has already been performed for most of the matrices we discuss in this chapter and accounts for the angular orientation dependence built into them. Many Mueller matrices in the literature, however, are written only for a special angle, usually $\phi = 0$, and require the transformation of eq. (6.39) before they can be used at other orientations.

In addition to being useful in providing the dependence of the transmittance of optical components upon orientation, the rotation matrix $\mathcal{M}(\phi)$ of eq. (6.37) is the Mueller transmittance matrix for an ideal circular retarder. Circular retarders are made of materials which exhibit the property called optical activity or circular birefringence. Crystal quartz and solutions of dextrose are examples of such materials. Linearly polarized light entering a circular retarder will emerge with its direction of polarization rotated by the angle ϕ appearing in $\mathcal{M}(\phi)$. This angle is the circular retardance and is related to the wavelength λ and the thickness a of the material by

$$\phi = \frac{2\pi a}{\lambda}(n_r - n_\ell) \quad [\text{rad}]$$

[compare with eq. (6.19)] where n_r is the refractive index for right-handed circularly polarized light at wavelength λ and n_ℓ is for left-handed. For solutions of an optically active material $n_r - n_\ell$ will be proportional to the concentration of solute so that measurements of the angular rotation of the direction of polarization for a fixed path length provide a convenient means of measuring solute concentrations. This technique has been developed to a high degree of accuracy in instruments called saccharimeters or polarimeters.

A different kind of rotation problem arises if we have the Mueller matrix elements of an optical component or train of components and ask what happens if the radiation passes in the opposite direction through the system. How does the Mueller matrix change? The answer is that for an original matrix of elements T_{ij} given by eq. (6.10) the matrix for the reversed optical system is

$$T^{(r)} = \begin{bmatrix} T_{00} & T_{10} & -T_{20} & T_{30} \\ T_{01} & T_{11} & -T_{21} & T_{31} \\ -T_{02} & -T_{12} & T_{22} & -T_{32} \\ T_{03} & T_{13} & -T_{23} & T_{33} \end{bmatrix}; \tag{6.40}$$

the rows and columns have been interchanged[1] and the signs of the matrix elements are changed once for every subscript 2 which appears. The interchange of rows and columns is understandable because the individual columns operate on the individual components of the Stokes vector of the input radiation while the rows generate the components of the Stokes vector of the output radiation. So if we interchange the roles of input and output it isn't surprising that the roles of the rows and columns must be interchanged too. The change of sign associated with the subscript 2 is caused by the transformation of the +45° polarization component into its complementary +135° component when an object is reversed.[2] Imagine a plate containing milled slots in horizontal, vertical, and 45° directions and also a hole tapped for a right-handed screw. Now if the plate is viewed from the back the horizontal and vertical slots and the tapped hole will appear unchanged but the 45° slot will look like a 135° slot. Thus, the +45° and +135° directions change roles upon reversal, and this requires the sign changes shown in eq. (6.40).

REAL OPTICAL COMPONENTS

So far in this chapter we have seen that, if we have a source of unpolarized light, a polarization-indifferent detector, a perfect linear polarizer, and a perfect quarter-wave plate, we can determine the state of polarization of any arbitrary ray and by varying the polarization of such a ray we can use it to characterize or calibrate the response of an instrument to polarized light. Unfortunately real polarizers and quarter-wave plates exhibit Mueller transmittance matrices which differ somewhat from those given in

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[1]If the rows and columns of a matrix are interchanged the result is known as the transpose of the original matrix.

[2]It is not obvious that the relationship between the matrix for the combination of a polarizer followed by a quarter-wave plate in eq. (6.35) and its reverse in eq. (6.22) satisfies eq. (6.40) This is because the angles used in eq. (6.35) are the supplements of those in eq. (6.22); when an optical component at orientation ϕ is reversed it will find itself at orientation (180° - ϕ). If the matrices in eq. (6.35) and (6.22) are both expressed in terms of the same angles it will be found that eq. (6.40) is obeyed.

eqs. (6.15) and (6.21). High quality crystal polarizers and retarders will probably approach the matrices of eqs. (6.15) and (6.20) closely except for the effects of surface reflections. So, for many radiometric applications, where polarization complications are no more than small corrections to measurements often made with complete neglect of polarization, such optical elements can be used and the Mueller matrices of eqs. (6.15) and (6.20) will be adequate to account for their behavior (to within 10% or so).

Regardless of whether a polarizer and retarder are perfect, if their Mueller matrices are known the Stokes components of a ray can still generally be measured essentially as described in connection with eq. (6.24). Four or more combinations of angular settings of the polarizer and retarder are needed as before. The difference is that the Mueller matrix of the combination of imperfect elements will differ from eq. (6.22) so that eq. (6.24) will not be valid. However, the analogue to eq. (6.24) will still be linear in the components L_0, L_1, L_2, and L_3 and the four measurements at suitable angular settings will permit these components to be evaluated. In the simplest case the retarder will not be exactly a quarter-wave plate due, for example, to its use at a wavelength other than the one for which it was designed. For this case eq. (6.24) becomes [using eq. (6.20) in (6.22) instead of (6.21)]:

$$\begin{aligned} L_0' = \tfrac{1}{2} \Big\{ L_0 &+ L_1\cdot[\cos 2\alpha\cdot\cos 2(\beta-\alpha) - \sin 2\alpha\cdot\sin 2(\beta-\alpha)\cdot\cos\delta] \\ &+ L_2\cdot[\sin 2\alpha\cdot\cos 2(\beta-\alpha) + \cos 2\alpha\cdot\sin 2(\beta-\alpha)\cdot\cos\delta] \\ &+ L_3\cdot[\sin 2(\beta-\alpha)\cdot\sin\delta] \Big\} . \end{aligned} \tag{6.41}$$

Obviously if δ is known, from eq. (6.19) for example, this equation is no more difficult to work with than eq. (6.24).

The real problem arises when one doesn't know the Mueller matrices of his polarizers or retarders and has no sources of polarized light with known Stokes components which can be used to calibrate them. One can, with considerable confidence, produce unpolarized light and a polarization-indifferent detector (using diffusers, such as integrating spheres [13], for instance). With such a source and detector, one can determine only element T_{00} of the Mueller matrix of an optical component or combination of components. However, as we have seen in eq. (6.41), for example, when polarizing components are combined the transmittance to unpolarized radiation of the combination depends upon additional matrix elements of the individual components. Thus, we might expect that measurements of the transmittances of all combinations of several polarizing components as a function of their orientation angles can yield considerable information about the matrix elements of the individual components. Using this idea we show in Appendix 7 how almost any Mueller matrix can be measured using nothing more than an unpolarized light source and a polarization-indifferent detector. In the following paragraphs we show how the same technique can be applied to measure a simple matrix which approximately describes the common Polaroid type of polarizer.

Our procedure simultaneously measures three linear-polarizing filters.[1] We assume that each is mounted in such a way that it can be reproducibly inserted and removed from the unpolarized measurement radiation beam and that it can be rotated about the beam axis, with its angular orientation read from a scale fixed to the filter mount. Real polarizers, especially sheet polarizers, can be expected to exhibit a small amount of retardance. Likewise, real retarders may be slightly linearly polarizing. Therefore, for the Mueller transmittance matrix of such real components we assume a form which is general enough to include both functions:

$$\mathsf{F}(\phi) = \begin{bmatrix} s & d\cdot C & d\cdot S & 0 \\ d\cdot C & s\cdot C^2 + p\cdot S^2 & (s-p)\cdot S\cdot C & -q\cdot S \\ d\cdot S & (s-p)\cdot S\cdot C & s\cdot S^2 + p\cdot C^2 & q\cdot C \\ 0 & q\cdot S & -q\cdot C & p \end{bmatrix}. \tag{6.42}$$

In this expression ϕ is the orientation of the filter as read from the scale attached to the filter mount. $C = \cos 2(\phi-\phi^o)$ and $S = \sin 2(\phi-\phi^o)$, where ϕ^o is the orientation of the polarization axis of the polarizing filter. The angle ϕ^o and the parameters s, d, p, and q are to be determined by experiment. The form of $\mathsf{F}(\phi)$, of course, includes the ϕ-dependence of eq. (6.39).

The parameters s, d, p, and q represent a model for dichroic polarizers of the Polaroid type with allowance for some retarder behavior.[2] In this model

$$s = \tfrac{1}{2}(k_1 + k_2) \qquad d = \tfrac{1}{2}(k_1 - k_2)$$
$$p = \sqrt{k_1\cdot k_2}\cdot\cos\delta \qquad q = \sqrt{k_1\cdot k_2}\cdot\sin\delta, \tag{6.43}$$

where the retardance of the filter is δ and the quantities k_1 and k_2 are known as the <u>principle transmittances</u> of the filter.[3] The maximum transmittance to linearly polarized light is k_1 and the minimum transmittance (when the direction of polarization is rotated 90°) is k_2. The relationships (6.43) imply that

$$q^2 = s^2 - d^2 - p^2. \tag{6.44}$$

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[1]We use the word "filter" here and in the rest of this chapter in a general sense to refer to any optical element which may modify the distribution of radiation with respect to any parameter, including polarization.

[2]Equation (6.42) can also be used to describe the polarizing effects of reflecting surfaces [4,14]. If the coordinate system described in the footnote on page 9 is used then $\phi = \phi^o$.

[3]Nominal values of k_1 and k_2 have been published [2] for the common Polaroid polarizers.

Thus we really have only three parameters to measure -- say s, d, and p -- in order to define the Mueller matrix F completely. Note that the matrix includes, as special cases, both perfect polarizers ($k_1 = 1$, $k_2 = 0$) and perfect retarders ($k_1 = 1$, $k_2 = 1$). It also provides for the most important imperfections encountered in real linear polarizers and retarders.

We introduce the notation $\bar{F}(\phi)$ to indicate the result of a measurement with our polarization-indifferent detector of the transmittance to unpolarized radiation of filter F when it is turned to an orientation angle ϕ on its mount. Likewise $\overline{GF}(\phi_g,\phi_f)$ is the measured transmittance of the pair of filters F and G at angular settings of ϕ_f and ϕ_g, respectively. Our notation also indicates the direction of radiation travel -- from right to left through the symbols; i.e. through filter F first, then through filter G to the detector.

We first measure the transmittances $\bar{F}(\phi_f)$, $\bar{G}(\phi_g)$, and $\bar{H}(\phi_h)$ of the three individual filters. From eq. (6.42) we see that these are given by

$$\bar{F}(\phi_f) = s_f, \quad \bar{G}(\phi_g) = s_g, \quad \text{and} \quad \bar{H}(\phi_h) = s_h$$

where the subscripts on s identify the filter to which it pertains. This result should be independent of ϕ_f, ϕ_g, and ϕ_h, so, in principle, a measurement at any orientation should suffice. However, a real filter will probably have small inhomogeneities which may move into and out of the beam as the filter rotates. In this case the best that can be done is to obtain an average value of s by sampling many orientations ϕ. These averages of $\bar{F}(\phi_f)$, etc., we will simply write as $\bar{F}$, $\bar{G}$, and $\bar{H}$. Thus,

$$s_f = \bar{F}, \quad s_g = \bar{G}, \quad \text{and} \quad s_h = \bar{H} \; . \tag{6.45}$$

We now measure the transmittance $\overline{FH}(\phi_f,\phi_h)$ of a pair of the filters. From eq. (6.42) we have

$$\mathsf{F}(\phi_f)\cdot\mathsf{H}(\phi_h) = \begin{bmatrix} [s_f\cdot s_h + d_f\cdot d_h\cdot C_{fh}] & \cdot & \cdot & \cdot \\ [d_f\cdot s_h\cdot C_f + s_f\cdot d_h\cdot C_f\cdot C_{fh} + p_f\cdot d_h\cdot S_f\cdot S_{fh}] & \cdot & \cdot & \cdot \\ [d_f\cdot s_h\cdot S_f + s_f\cdot d_h\cdot S_f\cdot C_{fh} - p_f\cdot d_h\cdot C_f\cdot S_{fh}] & \cdot & \cdot & \cdot \\ [q_f\cdot d_h\cdot S_{fh}] & \cdot & \cdot & \cdot \end{bmatrix} \tag{6.46}$$

where we have shown only the first column of the product transmittance matrix because it turns out that this is all that enters into the final expressions. We have also used the shorthand notation

$$C_{fh} = \cos 2[(\phi_f - \phi_f^{\,o}) - (\phi_h - \phi_h^{\,o})]$$

and

$$S_{fh} = \sin 2[(\phi_f - \phi_f^{\,o}) - (\phi_h - \phi_h^{\,o})].$$

The transmittance $\overline{FH}(\phi_f,\phi_h)$ is just the (0,0) element of this matrix or:

$$\overline{FH}(\phi_f,\phi_h) = s_f \cdot s_h + d_f \cdot d_h \cdot C_{fh} \quad .$$

By expanding C_{fh} we can rewrite this as

$$\overline{FH}(\phi_f,\phi_h) = \overline{FH}_1 + \overline{FH}_2 \cdot \cos 2(\phi_f - \phi_h) + \overline{FH}_3 \cdot \sin 2(\phi_f - \phi_h) \tag{6.47}$$

where $\overline{FH}_1 = s_f \cdot s_h$

$$\overline{FH}_2 = d_f \cdot d_h \cdot \cos 2(\phi_f^{\,o} - \phi_h^{\,o})$$

and $\overline{FH}_3 = d_f \cdot d_h \cdot \sin 2(\phi_f^{\,o} - \phi_h^{\,o})$.

If we measure $\overline{FH}(\phi_f,\phi_h)$ at two or more sets of angles ϕ_f and ϕ_h we will be able to solve the resulting set of equations (6.47) -- either by the usual methods for simultaneous equations or, better, by least squares fitting -- for the coefficients $\overline{FH}_2$ and $\overline{FH}_3$. $\overline{FH}_1$, of course, is already known from eq. (6.45).[1] In this way the orientation-independent coefficients $\overline{FH}_2$ and $\overline{FH}_3$ are measured. From these we obtain

$$\tan 2(\phi_f^{\,o} - \phi_h^{\,o}) = \overline{FH}_3/\overline{FH}_2 \tag{6.48}$$

and

$$d_f \cdot d_h = \sqrt{\overline{FH}_2^{\,2} + \overline{FH}_3^{\,2}} \tag{6.49}$$

By repeating this process for the other two possible ways of pairing the three filters we obtain the other polarization-axis angle differences and the other $d \cdot d$ products like eq. (6.49). The three equations of the form (6.49) can be further solved for the individual values of d:

$$\begin{aligned}
d_f &= \left[\frac{\left(\overline{GH}_2^{\,2} + \overline{GH}_3^{\,2}\right)\cdot\left(\overline{FH}_2^{\,2} + \overline{FH}_3^{\,2}\right)}{\overline{HG}_2^{\,2} + \overline{HG}_3^{\,2}}\right]^{1/4} \\
d_g &= \left[\frac{\left(\overline{HG}_2^{\,2} + \overline{HG}_3^{\,2}\right)\cdot\left(\overline{GF}_2^{\,2} + \overline{GF}_3^{\,2}\right)}{\overline{FH}_2^{\,2} + \overline{FH}_3^{\,2}}\right]^{1/4} \\
d_h &= \left[\frac{\left(\overline{FH}_2^{\,2} + \overline{FH}_3^{\,2}\right)\cdot\left(\overline{HG}_2^{\,2} + \overline{HG}_3^{\,2}\right)}{\overline{GF}_2^{\,2} + \overline{GF}_3^{\,2}}\right]^{1/4} .
\end{aligned} \tag{6.50}$$

[1] If $\overline{FH}_1$ is determined from eqs. (6.47) and does not agree with $\bar{F} \cdot \bar{H}$ [eq. (6.45)] then the model of eq. (6.42) may be inadequate. Most likely some of the (0,3) or (3,0) elements don't vanish as assumed in eq. (6.42).

So far our measurements have given us the orientation of the three filter polarization axis directions ϕ^o with respect to one another but have not revealed the absolute orientation of any one of them. This is obviously because there is no unique transverse reference direction associated with our unpolarized measurement beam. The absolute polarization axis orientation can be determined, if desired, by passing a beam known to be partially horizontally linearly polarized through one of the filters and rotating the filter for maximum transmission. At this orientation $\phi = \phi^o$ which then serves to determine the ϕ^o values of the other two filters from eqs. (6.48). A suitable source of partially horizontally polarized light is light reflected from a glass surface oriented so that the plane of incidence (the plane defined by the incident and reflected rays) is vertical.

The final measurements are of the three-filter transmittances $\overline{HGF}(\phi_h,\phi_g,\phi_f)$. From eqs. (6.46) and (6.42) we find, after some simplification, that the (0,0) element of the three-filter product matrix is:

$$\begin{aligned}\overline{HGF}(\phi_h,\phi_g,\phi_f) &= s_h \cdot s_g \cdot s_f + s_h \cdot d_g \cdot d_f \cdot C_{gf} + d_h \cdot d_g \cdot s_f \cdot C_{hg} \\ &\quad + d_h \cdot s_g \cdot d_f \cdot C_{gf} \cdot C_{hg} - d_h \cdot p_g \cdot d_f \cdot S_{gf} \cdot S_{hg} .\end{aligned} \tag{6.51}$$

As before $C_{gf} = \cos 2\left[(\phi_g - \phi_g^o) - (\phi_f - \phi_f^o)\right] = \cos 2\left[(\phi_g - \phi_f) - (\phi_g^o - \phi_f^o)\right]$, etc. Everything in this equation is now known [from eqs. (6.45), (6.48), and (6.49)] except p_g, so p_g can be calculated directly:

$$\begin{aligned}p_g &= \frac{\bar{H}\cdot\bar{G}\cdot\bar{F} - \overline{HGF}(\phi_h,\phi_g,\phi_f) + \bar{H}\cdot(\overline{GF}_2^{\,2} + \overline{GF}_3^{\,2})^{\frac{1}{2}}\cdot C_{gf} + \bar{F}\cdot(\overline{HG}_2^{\,2} + \overline{HG}_3^{\,2})^{\frac{1}{2}}\cdot C_{hg}}{(\overline{FH}_2^{\,2} + \overline{FH}_3^{\,2})^{\frac{1}{2}}\cdot S_{gf}\cdot S_{hg}} \\ &\quad + \bar{G}\cdot\frac{C_{gf}\cdot C_{hg}}{S_{gf}\cdot S_{hg}} .\end{aligned} \tag{6.52}$$

Again this result should be averaged over many settings of ϕ_f, ϕ_g, and ϕ_h in order to minimize errors due to filter inhomogeneities. Similar three-filter measurements with filters F and H in the middle will permit p_f and p_h to be determined. Finally the values of q can be calculated[1] from eq. (6.44) and with that the three matrices corresponding to eq. (6.42) are completely evaluated.

[1]The algebraic sign of q is not determined by the procedure we have outlined here. In many applications this will be of no consequence. The ambiguity is, in part, due to the arbitrariness in the definition of circular handedness. The sign of q can be determined if a beam containing circularly polarized light of specified handedness is passed through the filter and the output is analyzed with a linear polarizer. See also the discussion of these sign ambiguities in Appendix 7.

The procedure we have outlined applies only to filters which are at least partial linear polarizers. This is because the right-hand side of eq. (6.48) can become indeterminate (0/0) for a pure retarder or a simple attenuating spectral filter. Any such component can, however, be measured by putting it in the middle of a three-filter train between two polarizing filters which have already been characterized. Equation (6.51) is still valid and the symbol G now refers to the unknown filter. The quantity s_g can be obtained by the single filter transmittance measurement as before. This leaves d_g, p_g, and ϕ_g^o (appearing in C_{gf}, etc.) to be determined. The angle ϕ_g^o can be established by setting $\phi_f = \phi_h + \phi_f^o - \phi_h^o$ (i.e., the polarization axes of F and H are made parallel to each other) and rotating the unknown filter G until a maximum transmittance is obtained. With the polarization axes of F and H parallel to each other the transmittance is

$$\overline{HGF}(\phi_h,\phi_g,\phi_h + \phi_f^o - \phi_h^o) = s_h \cdot s_g \cdot s_f + d_h \cdot p_g \cdot d_f + (s_h \cdot d_f + d_h \cdot s_f) \cdot d_g \cdot C_{hg} + d_h \cdot d_f \cdot (s_g - p_g) \cdot C_{hg}^2 . \tag{6.53}$$

This expression has a maximum at $\phi_g = \phi_h + \phi_g^o - \phi_h^o$.[1] This serves to determine the angle ϕ_g^o, which once again should be averaged for many settings of ϕ_h. Once ϕ_g^o is known the other parameters of filter G can easily be obtained by making many measurements and using linear-least-squares fitting in eq. (6.51) or by making measurements at two suitable angles, ϕ_g, and solving the resulting two equations given by eq. (6.53) for the unknowns d_g and p_g. q_g then follows as before from eq. (6.44). If the transmittance, eq. (6.53), exhibits no significant dependence on $\phi_h - \phi_g$ after averaging over ϕ_h then $d_g = 0$, and $p_g = s_g$. Such a filter is just an attenuating spectral filter.

Needless to say, with real measurements on real filters some uncertainties in determining s, d, p, and q can be expected. For example, if 1% measurements are being made of a good linear polarizer the uncertainties in s, d, p, and q will all be similar and of the order of s/100. But since $p \sim q \sim 0$ the <u>relative</u> uncertainty in these two parameters may appear alarmingly large. Fortunately, the relative contribution to any final instrument response from a single matrix element T_{ij} of an optical component cannot be larger[2] than the fraction $|T_{ij}/T_{00}|$. Thus, in the example of the 1% measurements of a linear polarizer the uncertainties in p and q can never contribute more than about a 1% uncertainty to any measurement made using this polarizer.

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[1]If d_g is small but not zero there may be a lesser maximum 90° removed from this angle.

[2]This is true if the ultimate instrument detector responds to a property of the electromagnetic radiation such as power, linear momentum, or photon number, but need not be true if it responds to a property characteristic of a particular polarization state such as angular momentum.

Once the Mueller matrix elements have been determined for two real linear polarizers and two approximate quarter-wave plates the polarizer and wave-plate pairs can be used to generate and analyze a wide range of polarization states of light. And this in turn makes possible the characterization of more complicated Mueller matrices -- in particular those of circular polarizers, which are not included in our general imperfect filter expression F in eq. (6.42) because we have set $F_{03} = F_{30} = 0$. We thus see how we can start with real, inexpensive, polarizing filters and build up a complete polarization measurement capability.

POLARIZER-ATTENUATORS

A three-polarizer train is sometimes used as an absolute or predictable variable optical attenuator [15,16]. The first and last polarizers of this train are aligned with their polarization axes parallel to each other and the middle element is rotated to achieve the desired degree of attenuation. Equation (6.53) describes just this arrangement. The transmittance of the train relative to its maximum transmittance is therefore:

$$\tau = \frac{\overline{HGF}(\phi_h^o,\phi_g,\phi_f^o)}{\overline{HGF}(\phi_h^o,\phi_g^o,\phi_f^o)} = [s_h\cdot s_g\cdot s_f + d_h\cdot d_f\cdot p_g + (s_h\cdot d_f + d_h\cdot s_f)\cdot d_g\cdot C_{hg}$$
$$+ d_h\cdot d_f\cdot(s_g - p_g)\cdot C_{hg}^2]\,/[s_h\cdot s_g\cdot s_f + s_h\cdot d_g\cdot d_f + d_h\cdot s_g\cdot d_f \qquad (6.54)$$
$$+ d_h\cdot d_g\cdot s_f].$$

This expression for the relative transmittance, of course, is valid only for unpolarized light and a polarization-indifferent detector. If the polarizers are all perfect linear polarizers we have $s_f = s_g = s_h = 1$, $d_f = d_g = d_h = 1$, and $p_f = p_g = p_h = 0$. Then

$$\tau = \frac{1 + 2C_g + C_g^2}{4} = \cos^4(\phi_g - \phi_g^o)\ .$$

If the outer polarizers are perfect and the middle component is a perfect half-wave plate instead of a polarizer the only change is that $d_g = 0$ and $p_g = -1$. This results in

$$\tau = \frac{2C_g^2}{2} = \cos^2 2(\phi_g - \phi_g^o)\ .$$

In both these cases the light emerging from the attenuator is always completely linearly polarized in the direction determined by the final polarizer. The relative transmittance depends only on the easily measured angle ϕ_g.

Unfortunately, with real optical elements the values of s, d, and p must be known if the transmittance is to be calculated accurately as a function of ϕ_g from eq. (6.54). Moreover, the state of polarization of radiation emerging from the attenuator will depend weakly upon the angle ϕ_g and upon the state of polarization of the entering radiation. Thus, the Mueller matrix of the rest of the optical system must be known and additional matrix elements of the attenuator must be evaluated if the effective relative transmittance is to be calculated exactly.

DEPOLARIZERS

We briefly mentioned depolarized radiation in the early part of this chapter and would now like to say a few words about the means of achieving this. The Mueller transmittance matrix for a depolarizer is obviously proportional to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.55}$$

Unfortunately no single optical element exhibits this property. However, if we consider a complete measurement situation with a source radiance vector $\mathbb{L}$, an optical component with Mueller matrix $\mathbb{D}$, and a radiometer with responsivity matrix $\mathcal{R}$, with the elements (responsivity coefficients) R_{00}, R_{01}, etc., we have for the radiometer output signal:

$$S = \int_{\Delta t}\int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} \mathcal{R}\cdot\mathbb{D}\cdot\mathbb{L}\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda\cdot dt \tag{6.56}$$

where $\Delta\omega$, ΔA, $\Delta\lambda$, and Δt are the resolution intervals of the radiometer in these parameters. (As in our discussion of the calibration of a radiometer, we actually use only the upper row of the responsivity matrix $\mathcal{R}$ to obtain the output signal S.) If the responsivity of the radiometer and the Stokes spectral radiance vectors of the source are sufficiently uniform over the radiometer resolution intervals, they can be removed from this integral to give

$$\begin{aligned} S &\approx \mathcal{R}\cdot\left[\frac{\int_{\Delta t}\int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega}\mathbb{D}\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda\cdot dt}{\Delta t\cdot\Delta\lambda\cdot\Delta A\cdot\Delta\Theta}\right]\cdot\mathbb{L}\cdot\Delta t\cdot\Delta\lambda\cdot\Delta A\cdot\Delta\Theta \\ &= \mathcal{R}\cdot <\mathbb{D}>\cdot\mathbb{L}\cdot\Delta t\cdot\Delta\lambda\cdot\Delta A\cdot\Delta\Theta \end{aligned} \tag{6.57}$$

where

$$<\mathbb{D}> = \begin{bmatrix} <D_{00}> & <D_{01}> & <D_{02}> & <D_{03}> \\ <D_{10}> & <D_{11}> & <D_{12}> & <D_{13}> \\ <D_{20}> & <D_{21}> & <D_{22}> & <D_{23}> \\ <D_{30}> & <D_{31}> & <D_{32}> & <D_{33}> \end{bmatrix}$$

and $\langle D_{ij}\rangle = \frac{1}{\Delta t\cdot\Delta\lambda\cdot\Delta A\cdot\Delta\Theta}\int_{\Delta t}\int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} D_{ij}\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda\cdot dt$. Now we see that if all $\langle D_{ij}\rangle$ except $\langle D_{00}\rangle$ can be made to vanish, the Mueller matrix $\langle \mathbb{D}\rangle$ will become

$$\langle \mathbb{D}\rangle = \begin{bmatrix} \langle D_{00}\rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.58}$$

which is the form for a depolarizer. Most depolarizers consist of two retarders [eq. (6.20)] placed one behind the other. The (0,0) element of the resulting product matrix is 1 and all the other elements are zero or are trigonometric functions of the ϕ's and δ's of the two retarders. By carefully choosing the ϕ values (or the δ values) and averaging over a range of δ's (or ϕ's) these trigonometric matrix elements can be made negligibly small. The δ's (or ϕ's), over which the device averages, are arranged to vary in time, wavelength, position or direction. Depolarizers which depolarize by averaging over wavelength ($\sim$100 nm) [17] or over time ($\sim$1 sec) [17] or over the cross-sectional area of the beam ($\sim$1 cm^2) [3] have been used and will be described more completely in Part II. Depolarizers are frequently known as <u>pseudo-depolarizers</u> or as <u>polarization scramblers</u>.

A SIMPLIFIED CHARACTERIZATION of POLARIZATION

Many radiometric measurements involving polarized light are carried out using just two measurements -- one with a linear polarizer set in a horizontal orientation in front of the detector and the second with the polarizer turned 90°. Let us see what the limitations of such measurements are and under what conditions they will be sufficient to characterize a light ray. In order not to complicate matters unduly, let us assume that the polarizer is perfect so that its Mueller transmittance matrix in the horizontal orientation is:

$$\mathbb{P}(0°) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and in the vertical orientation:

$$\mathbb{P}(90°) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We will also assume that the detector is polarization indifferent or has been compensated or calibrated for horizontally and vertically polarized radiation. When an arbitrarily polarized ray with Stokes vector

$$\mathsf{L} = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

passes through the linear polarizer in the two orientations, we have

$$\mathsf{L}'(0°) = \mathsf{P}(0°) \cdot \mathsf{L} = \begin{bmatrix} L_0 + L_1 \\ L_0 + L_1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathsf{L}'(90°) = \mathsf{P}(90°) \cdot \mathsf{L} = \begin{bmatrix} L_0 - L_1 \\ -L_0 + L_1 \\ 0 \\ 0 \end{bmatrix}.$$

The detector will record signals proportional to $L_0'(0°) = L_0 + L_1$ and $L_0'(90°) = L_0 - L_1$, respectively. Thus the original components L_0 and L_1 can be calculated from $\frac{1}{2}[L_0'(0°) + L_0'(90°)]$ and $\frac{1}{2}[L_0'(0°) - L_0'(90°)]$. If we can assume that $L_2 = L_3 = 0$ then the light beam has been completely characterized. Usually L_0, the spectral radiance, and the degree of polarization [eq. (6.6)]

$$P = \frac{L_1}{L_0} = \frac{L_0'(0°) - L_0'(90°)}{L_0'(0°) + L_0'(90°)}$$

are the quantities reported. Thus we see that the two-measurement description of polarized light is limited to those situations in which (as we could have guessed) there is no 45° (or 135°) linearly polarized component and no circularly polarized component. The elimination of the +45°-preference component can always be arranged by suitable choice of the reference "horizontal" direction. Note, however, that unless one has some *a priori* knowledge of this direction this entails the equivalent of at least one additional measurement to find this direction and there is thus no real saving in attempting to eliminate this component. In many situations, such as scattering studies of isotropic samples irradiated, first, by radiation polarized in the plane of incidence and then by radiation polarized perpendicular to the plane of incidence, a reference direction parallel to or perpendicular to the plane of incidence can be taken and the +45°-preference component may then be safely neglected (but only if the sample is isotropic). The neglect of the circularly polarized component is probably usually justified in most radiometric measurements where uncertainties of a percent or so are tolerable. For the highest accuracies circular polarization must be included, however, because it can creep in whenever partially polarized light at non-normal incidence is reflected from mirrors or transmitted through films. Thus, any but the simplest instruments will be likely to exhibit some circular-polarization selectivity.

When circular polarization can be neglected most of the discussions of this chapter can be greatly simplified. For example, no measurements involving retarders such as quarter-wave plates need be considered; sources, optical components and radiometers can be completely characterized using only linear polarizers. Also the procedure outlined in eqs. (6.42) to (6.53) for measuring the Mueller matrix of a polarizer can be simplified by omitting the three-filter measurements because we can now set $q = 0$ and $p^2 = s^2 - d^2$ for each filter.

EXPERIMENTAL CONCERNS

We have tried in this chapter to present a complete, exact treatment of polarization and an indication of how this treatment can be applied in the laboratory. We have not mentioned the experimental problems which make high accuracy polarization measurements difficult -- these properly belong in later chapters where we deal with specific radiometric instruments. However, some brief mention of some of these problems seems appropriate here. The proper experimental inclusion of polarization in measuring instrumentation, as we have seen, requires the insertion into the optical path of one, two, or three polarizing filters of one kind or another. These must be rotatable and it must be possible to measure the angles of rotation, or at least to set each filter reproducibly at several preselected angular positions. Unless the filters are absolutely uniform there will be a dependence of filter transmittance on angular orientation which may be unrelated to the polarization property being investigated. For anything more than qualitative observations it is therefore important that polarization measurements be carried out and suitably averaged over many orientations of the polarizing components. Another experimental difficulty common to all measurements using multiple filters in tandem but which is exacerbated with polarizing filters is that of filter-filter interreflections. Such interreflections have the effect of increasing the apparent transmittance of the filter train and thereby introducing errors. In the case of polarizing filters this problem can be eliminated by the usual technique of tilting the filters, but the axis of rotation of the filters must remain parallel to the beam axis. Thus, the filter must wobble as it is rotated. If a tilted polarizing filter were rotated about an axis perpendicular to its face, then entering radiation would be partially linearly polarized at a fixed orientation by the tilted first surface, resulting in observations of partial polarization even of unpolarized light. If the tilt rotates with the filter, however, then this polarizing effect becomes simply an integral property of the filter and will be accounted for in the measured Mueller transmittance matrix elements. Of course, the filter must always be used at the same tilt orientation. Another complication which may require experimental attention is that the lateral displacement of the beam as it passes through the tilted filter now rotates with the filter.

SUMMARY of CHAPTER 6.

Polarization can be included rigorously in classical radiometry (geometrical optics) by treating spectral radiance as a four-component mathematical column vector called a Stokes vector

$$\mathsf{L}_\lambda = \begin{bmatrix} L_{\lambda,0} \\ L_{\lambda,1} \\ L_{\lambda,2} \\ L_{\lambda,3} \end{bmatrix}. \tag{6.8a}$$

The four components of L_λ are known as Stokes components. The Stokes vector and its components, in general, depend upon ray position and direction and upon wavelength, i.e., upon x,y,θ,ϕ, and λ. The first Stokes component $L_{\lambda,0}$ is the spectral radiance which a polarization-indifferent detector would measure. It is the measure of the spectral radiant flux associated with the ray and is identical to the spectral radiance discussed in earlier chapters. Each of the other three components of L_λ represents the excess of one form of polarization over its complementary form. $L_{\lambda,1}$ describes the excess of horizontal over vertical linear polarization, $L_{\lambda,2}$ describes the excess of +45° over +135° linear polarization and $L_{\lambda,3}$ describes the excess of right over left circular polarization. The spectral radiance $L_{\lambda,0}$, is the sum of two parts, a polarized part, $L_{\lambda,p}$, given by

$$L_{\lambda,p} = \sqrt{L_{\lambda,1}^2 + L_{\lambda,2}^2 + L_{\lambda,3}^2} \tag{6.5}$$

and an unpolarized part

$$L_{\lambda,u} = L_{\lambda,0} - L_{\lambda,p}.$$

The fraction

$$P = L_{\lambda,p}/L_{\lambda,0} \tag{6.6}$$

is known as the degree of polarization.

In this description of polarized radiation the transmittance or propagance of an optical path is represented by a 4 × 4 matrix known as a Mueller matrix. For example (compare eq. (3.14) [1])

$$\mathsf{L}_\lambda' = \mathsf{T}\cdot\mathsf{L}_\lambda \tag{6.9a}$$

relates the spectral radiance L_λ at one point on a ray and the propagance T along the path between that point and a second point to the spectral radiance L_λ' at the second point. The Mueller propagance matrix T in this equation is given by

$$\mathsf{T} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{6.10}$$

so that eq. (6.9a) is equivalent to

$$\begin{aligned} L_0' &= T_{00}\cdot L_0 + T_{01}\cdot L_1 + T_{02}\cdot L_2 + T_{03}\cdot L_3 \\ L_1' &= T_{10}\cdot L_0 + T_{11}\cdot L_1 + T_{12}\cdot L_2 + T_{13}\cdot L_3 \\ L_2' &= T_{20}\cdot L_0 + T_{21}\cdot L_1 + T_{22}\cdot L_2 + T_{23}\cdot L_3 \\ L_3' &= T_{30}\cdot L_0 + T_{31}\cdot L_1 + T_{32}\cdot L_2 + T_{33}\cdot L_3 \end{aligned} \tag{6.7}$$

where we have dropped the subscript λ for simplicity.

When polarization is included, the spectral radiance of a ray of radiation transmitted by a train of optical components depends upon the order in which the propagating flux in the ray encounters the components. This is reflected in matrix equations by the order of writing the Mueller matrices and the Stokes vectors. The entering Stokes vector is written on the right and the Mueller matrices are written from right to left in the order in which the optical components they represent are encountered. The resulting matrix equations can then be evaluated by the usual matrix multiplication rules (Appendix 6).

The Mueller transmittance matrix of an <u>ideal linear polarizer</u> is given by

$$\mathsf{P}(\phi) = \tfrac{1}{2}\begin{bmatrix} 1 & C & S & 0 \\ C & C^2 & S\cdot C & 0 \\ S & S\cdot C & S^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.15a}$$

where $C = \cos 2\phi$, $S = \sin 2\phi$, and ϕ is the angle between the polarization axis of the polarizer and the horizontal polarization reference axis of the ray. When a ray of arbitrary spectral radiance is passed through such a polarizer the polarization state of the transmitted ray can be predicted by performing the matrix multiplication indicated in eq. (6.9a):

$$\mathsf{L}' = \mathsf{P}(\phi)\cdot\mathsf{L} = \mathsf{P}(\phi)\cdot\begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} = \tfrac{1}{2}(L_0 + L_1\cdot C + L_2\cdot S)\cdot\begin{bmatrix} 1 \\ C \\ S \\ 0 \end{bmatrix} . \tag{6.17}$$

The transmitted ray has a spectral radiance of $\frac{1}{2}(L_0 + L_1 \cdot \cos 2\phi + L_2 \cdot \sin 2\phi)$. It has no net circularly polarized component $(L_3' = 0)$ but is completely linearly polarized at the angle ϕ, [$P = 1$, from eq. (6.6)].

The Mueller transmittance matrix of an <u>ideal linear retarder</u> is

$$R(\delta,\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C^2 + S^2 \cdot \cos\delta & S \cdot C \cdot (1 - \cos\delta) & -S \cdot \sin\delta \\ 0 & S \cdot C \cdot (1 - \cos\delta) & S^2 + C^2 \cdot \cos\delta & C \cdot \sin\delta \\ 0 & S \cdot \sin\delta & -C \cdot \sin\delta & \cos\delta \end{bmatrix} \tag{6.20}$$

where $C = \cos 2\phi$, $S = \sin 2\phi$, and ϕ is the angle between the fast axis of the retarder and the horizontal reference axis of the ray. δ is the <u>retardance</u> of the device and is proportional to the thickness of the retarder plate. If $\delta = \pi/2$ [rad] the retarder is called a <u>quarter-wave plate</u>, which has a Mueller transmittance matrix

$$Q(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C^2 & S \cdot C & -S \\ 0 & S \cdot C & S^2 & C \\ 0 & S & -C & 0 \end{bmatrix} . \tag{6.21}$$

The combination of an ideal quarter-wave plate and an ideal linear polarizer can be used either to determine the state of polarization of an arbitrarily polarized ray or to produce any given state of polarization. If the radiation passes first through a rotatable quarter wave plate at angle α and then through a fixed polarizer at angle β the transmitted ray is given by

$$L' = P(\beta) \cdot Q(\alpha) \cdot \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$= \tfrac{1}{2}[L_0 + L_1 \cdot \cos 2\alpha \cdot \cos 2(\beta-\alpha) + L_2 \cdot \sin 2\alpha \cdot \cos 2(\beta-\alpha) + L_3 \cdot \sin 2(\beta-\alpha)] \cdot \begin{bmatrix} 1 \\ \cos 2\beta \\ \sin 2\beta \\ 0 \end{bmatrix} . \tag{6.23}$$

If the spectral radiance of the transmitted ray is measured at four suitable orientations α of the quarter-wave plate the four simultaneous linear equations of the form of eq. (6.23) can be solved for the Stokes components L_0, L_1, L_2, and L_3 of the original ray. Since the transmitted beam is always completely linearly polarized at the fixed orientation β, the measurement can, in principle, be carried out by a detector of any arbitrary polarization selectivity. For absolute measurements, however, a calibration of the detector will be required using (usually) a differently polarized source. In this case it will be simpler if the detector is polarization-indifferent so that the spectral-radiance,

L_0', is the quantity measured.

If the radiation passes first through the polarizer and then through the quarter-wave plate, the transmitted ray will be described by

$$\mathsf{L}' = \mathsf{Q}(\alpha)\cdot\mathsf{P}(\beta)\cdot\begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} = \tfrac{1}{2}(L_0 + L_1\cdot\cos 2\beta + L_2\cdot\sin 2\beta)\cdot\begin{bmatrix} 1 \\ \cos 2\alpha\cdot\cos 2(\alpha-\beta) \\ \sin 2\alpha\cdot\cos 2(\alpha-\beta) \\ \sin 2(\alpha-\beta) \end{bmatrix}. \qquad (6.35)$$

By suitable choice of the orientation angles of the quarter-wave plate and polarizer any pure polarization state from circular $(|\alpha-\beta| = 45°)$ to linear $(\alpha-\beta = 0°)$ can be realized.

The final electrical, thermal, or chemical response, S, of a radiometer depends upon all four Stokes components of the entering radiance:

$$S = \int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} (R_{00}\cdot L_0 + R_{01}\cdot L_1 + R_{02}\cdot L_2 + R_{03}\cdot L_3)\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda. \qquad (6.36)$$

The quantities $\Delta\lambda$, ΔA, and $\Delta\omega$ are the acceptance intervals of the radiometer. If these acceptance intervals are sufficiently small that the Stokes spectral-radiance components can be assumed constant and removed from the integrals in eq. (6.36) then we can write

$$S = \mathcal{R}_{00}\cdot L_0 + \mathcal{R}_{01}\cdot L_1 + \mathcal{R}_{02}\cdot L_2 + \mathcal{R}_{03}\cdot L_3. \qquad (6.36a)$$

The values of the responsivity factors $\mathcal{R}_{0i}$ can be determined experimentally from observations of the radiometer output for a beam in four different, known states of polarization such as can be obtained from the polarizer and quarter-wave plate combination described by eq. (6.35). If the first optical component of the radiometer is a perfect diffuser, eq. (6.36) reduces to

$$S = R_{00}\cdot\Phi_0 \qquad (6.36c)$$

where $\Phi_0 = \int_{\Delta\lambda}\int_{\Delta A}\int_{\Delta\omega} L_0\cdot d\omega\cdot\cos\theta\cdot dA\cdot d\lambda$ is the total radiant flux falling on the diffuser within the wavelength band pass of the radiometer. If neither of these simplifications is appropriate the dependence of the coefficients R_{0i} upon ray position, direction and wavelength can, in principle, be determined by probing the radiometer with rays of known polarization state. With these functions known, then, a radiometer measurement of an unknown beam provides one relationship among integrals of the four $L_i(x,y,\theta,\phi,\lambda)$ functions. If the use of the radiometer is restricted to comparisons of beams with identical relative spectral-radiance distributions then such detailed knowledge of its responsivity is unnecessary and four measurements of different known polarization states will characterize it for this purpose.

The ideal Mueller transmittance matrices of eqs. (6.15) and (6.20) are only approximations to the transmittance matrices of real optical components. Many real linear polarizers and retarders can, however, be adequately represented by Mueller matrices of the form

$$F(\phi) = \begin{bmatrix} s & d\cdot C & d\cdot S & 0 \\ d\cdot C & s\cdot C^2 + p\cdot S^2 & (s-p)\cdot S\cdot C & -q\cdot S \\ d\cdot S & (s-p)\cdot S\cdot C & s\cdot S^2 + p\cdot C^2 & q\cdot C \\ 0 & q\cdot S & -q\cdot C & p \end{bmatrix}, \tag{6.42}$$

where

$$C = \cos 2(\phi-\phi^o), \quad S = \sin 2(\phi-\phi^o)$$

and

$$q^2 = s^2 - d^2 - p^2. \tag{6.44}$$

The values of s, d, and p as well as the filter polarization axis direction ϕ^o can be determined experimentally by appropriate transmittance measurements. If no sources of known variable polarization state are available it is possible to measure these parameters with sufficient accuracy for most radiometric applications by making transmittance measurements in unpolarized light on three or more such objects as a function of their orientation angles. Once the Mueller transmittance matrices of a polarizer and an approximate quarter-wave plate have been established they can be used for analyzing or generating a wide range of arbitrary polarization states by generalizing eqs. (6.23) and (6.35).

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Appendix 6. Matrix Multiplication

by John B. Shumaker

In this appendix we review the rules for matrix multiplication. We will confine our attention to the 4-component Stokes polarization vectors and the 4 × 4 Mueller matrices encountered in the study of polarization in Chapter 6. For a more general treatment the reader should consult mathematics or mathematical physics texts [7,18].

A Mueller matrix is a two-dimensional array of 16 numbers, the symbol for any one of which may be labeled by two subscripts. The first subscript denotes the row and the second the column of the element in question. We can write the matrix A in any of the following equivalent ways:

$$\mathsf{A} = [A_{rc}] = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix}. \tag{A6.1}$$

The product of two matrices A and B is a new matrix

$$\mathsf{C} = \mathsf{A} \cdot \mathsf{B} \tag{A6.2}$$

each of whose elements is computed by the formula:

$$C_{rc} = \sum_{k=0}^{3} A_{rk} \cdot B_{kc}, \qquad r,c = 0, 1, 2, 3. \tag{A6.3}$$

Unfortunately our page isn't wide enough to display the complete product matrix C in one block; however, if we may be permitted to break it between the second and third columns we can show it all.

$$\mathsf{C} = [C_{rc}] = \left[\begin{matrix} A_{00}\cdot B_{00} + A_{01}\cdot B_{10} + A_{02}\cdot B_{20} + A_{03}\cdot B_{30} & A_{00}\cdot B_{01} + A_{01}\cdot B_{11} + A_{02}\cdot B_{21} + A_{03}\cdot B_{31} \\ A_{10}\cdot B_{00} + A_{11}\cdot B_{10} + A_{12}\cdot B_{20} + A_{13}\cdot B_{30} & A_{10}\cdot B_{01} + A_{11}\cdot B_{11} + A_{12}\cdot B_{21} + A_{13}\cdot B_{31} \\ A_{20}\cdot B_{00} + A_{21}\cdot B_{10} + A_{22}\cdot B_{20} + A_{23}\cdot B_{30} & A_{20}\cdot B_{01} + A_{21}\cdot B_{11} + A_{22}\cdot B_{21} + A_{23}\cdot B_{31} \\ A_{30}\cdot B_{00} + A_{31}\cdot B_{10} + A_{32}\cdot B_{20} + A_{33}\cdot B_{30} & A_{30}\cdot B_{01} + A_{31}\cdot B_{11} + A_{32}\cdot B_{21} + A_{33}\cdot B_{31} \end{matrix}\right.$$

(A6.4)

$$\left.\begin{matrix} A_{00}\cdot B_{02} + A_{01}\cdot B_{12} + A_{02}\cdot B_{22} + A_{03}\cdot B_{32} & A_{00}\cdot B_{03} + A_{01}\cdot B_{13} + A_{02}\cdot B_{23} + A_{03}\cdot B_{33} \\ A_{10}\cdot B_{02} + A_{11}\cdot B_{12} + A_{12}\cdot B_{22} + A_{13}\cdot B_{32} & A_{10}\cdot B_{03} + A_{11}\cdot B_{13} + A_{12}\cdot B_{23} + A_{13}\cdot B_{33} \\ A_{20}\cdot B_{02} + A_{21}\cdot B_{12} + A_{22}\cdot B_{22} + A_{23}\cdot B_{32} & A_{20}\cdot B_{03} + A_{21}\cdot B_{13} + A_{22}\cdot B_{23} + A_{23}\cdot B_{33} \\ A_{30}\cdot B_{02} + A_{31}\cdot B_{12} + A_{32}\cdot B_{22} + A_{33}\cdot B_{32} & A_{30}\cdot B_{03} + A_{31}\cdot B_{13} + A_{32}\cdot B_{23} + A_{33}\cdot B_{33} \end{matrix}\right].$$

We may think of a Stokes vector

$$\mathsf{L} = [L_r] = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix} \qquad \text{(A6.5)}$$

as a matrix in which all columns after the first column are zero. Viewed this way the product of a Mueller matrix by a Stokes vector

$$\mathsf{L}' = \mathsf{A}\cdot\mathsf{L} \qquad \text{(A6.6)}$$

follows the formula given above. Alternatively we may write a special formula for this product as

$$L_r' = \sum_{k=0}^{3} A_{rk}\cdot L_k \qquad r = 0,\ 1,\ 2,\ 3$$

or

$$\begin{aligned} L_0' &= A_{00}\cdot L_0 + A_{01}\cdot L_1 + A_{02}\cdot L_2 + A_{03}\cdot L_3 \\ L_1' &= A_{10}\cdot L_0 + A_{11}\cdot L_1 + A_{12}\cdot L_2 + A_{13}\cdot L_3 \\ L_2' &= A_{20}\cdot L_0 + A_{21}\cdot L_1 + A_{22}\cdot L_2 + A_{23}\cdot L_3 \\ L_3' &= A_{30}\cdot L_0 + A_{31}\cdot L_1 + A_{32}\cdot L_2 + A_{33}\cdot L_3\,. \end{aligned} \qquad \text{(A6.7)}$$

The meaning of the product of a matrix or vector by a scalar [which is any non-matrix, non-vector quantity such as $\frac{1}{2}$, $\tau(\lambda)$, or $\frac{1}{2}(L_0 + L_1\cdot\cos 2\beta + L_2\cdot\sin 2\beta)$] is that each element of the matrix or vector must be multiplied by the scalar. For example,

$$\tfrac{1}{2}\cdot\mathsf{A} = [\tfrac{1}{2}A_{rc}] = \begin{bmatrix} \frac{1}{2}A_{00} & \frac{1}{2}A_{01} & \frac{1}{2}A_{02} & \frac{1}{2}A_{03} \\ \frac{1}{2}A_{10} & \frac{1}{2}A_{11} & \frac{1}{2}A_{12} & \frac{1}{2}A_{13} \\ \frac{1}{2}A_{20} & \frac{1}{2}A_{21} & \frac{1}{2}A_{22} & \frac{1}{2}A_{23} \\ \frac{1}{2}A_{30} & \frac{1}{2}A_{31} & \frac{1}{2}A_{32} & \frac{1}{2}A_{33} \end{bmatrix}. \qquad \text{(A6.8)}$$

Similarly,

$$\tau(\lambda)\cdot\mathsf{L} = \begin{bmatrix} \tau(\lambda)\cdot L_0 \\ \tau(\lambda)\cdot L_1 \\ \tau(\lambda)\cdot L_2 \\ \tau(\lambda)\cdot L_3 \end{bmatrix}. \qquad \text{(A6.9)}$$

Appendix 7. The Measurement of Mueller Transmittance Matrices

by John B. Shumaker

In this appendix we show how Mueller filter transmittance matrices consisting of sixteen arbitrary matrix elements may be determined experimentally using an unpolarized source of radiation and a polarization-indifferent detector. The technique consists essentially of making unpolarized transmittance measurements of polarization filters singly, in pairs, and in triples, as a function of the angular orientations of the filters. In practice, in the presence of experimental measurement noise, difficulties may arise in the numerical solution of the equations which follow, but we ignore such problems here. The purpose of this appendix is merely to demonstrate in principle the measurability of an arbitrary Mueller matrix using only an unpolarized source and a polarization-indifferent radiometer with no previously characterized polarizing optical components.

The procedure simultaneously measures three polarization filters. We assume that each is mounted in such a way that it can be reproducibly inserted into and removed from the measurement radiation beam and that it can be rotated about the beam axis with its angular orientation read from a scale fixed to the filter mount. If the filters are labeled F, G, and H we take their Mueller transmittance matrices to be perfectly general:

$$\mathsf{F}(0°) = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix}$$

$$\mathsf{G}(0°) = \begin{bmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{10} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{31} & G_{32} & G_{33} \end{bmatrix} \qquad \text{(A7.1)}$$

and similarly for H. Their dependence upon the orientation angles ϕ_f, ϕ_g, and ϕ_h is given by eq. (6.39). These angles are just the orientation angles read from the scales attached to the filter mounts. We make no assumption about the relationship between the $\phi = 0°$ directions (at which the matrices written above apply) and the polarization axes of the filters, if any.

The simplest measurements to make are the transmittances of the individual filters as measured with our beam of unpolarized light and our polarization-indifferent detector. We shall denote these measured transmittances by simply writing a bar over the filter label: $\bar{F}$, $\bar{G}$, and $\bar{H}$. These transmittances give us directly

$$\bar{F} = F_{00}, \quad \bar{G} = G_{00}, \quad \text{and} \quad \bar{H} = H_{00}. \qquad \text{(A7.2)}$$

The next simplest set of measurements we can make is of the filters taken in pairs. Let us consider the pair F and H and pass the measurement beam through in the direction from F to H. The measured pair transmittance, which we will call $\overline{\overline{HF}}$, is just the (0,0) element of the matrix product

$$\mathsf{H}(\phi_h) \cdot \mathsf{F}(\phi_f).$$

That is

$$\overline{\overline{HF}} = (H_{00}\cdot F_{00} + H_{03}\cdot F_{30}) + (H_{01}\cdot F_{10} + H_{02}\cdot F_{20})\cdot C_{hf} + (H_{01}\cdot F_{20} - H_{02}\cdot F_{10})\cdot S_{hf} \qquad \text{(A7.3)}$$

where $C_{hf} = \cos 2(\phi_h-\phi_f)$, $S_{hf} = \sin 2(\phi_h-\phi_f)$, and the angular dependence comes from eq. (6.39). If this measurement is made at a number of angular differences, $\phi_h-\phi_f$, the results can be solved as simultaneous equations or least-squares fitted to obtain values for the coefficients (appearing in parentheses) of the constant and cosine and sine factors in eq. (A7.3). Let us call these quantities $\overline{\overline{HF}}_1$, $\overline{\overline{HF}}_2$, and $\overline{\overline{HF}}_3$:

$$\overline{\overline{HF}}_1 = H_{00}\cdot F_{00} + H_{03}\cdot F_{30} \qquad \text{(A7.4a)}$$

$$\overline{\overline{HF}}_2 = H_{01}\cdot F_{10} + H_{02}\cdot F_{20} \qquad \text{(A7.4b)}$$

$$\overline{\overline{HF}}_3 = H_{01}\cdot F_{20} - H_{02}\cdot F_{10}. \qquad \text{(A7.4c)}$$

We can also reverse either filter and repeat the measurements. Using eq. (6.40) we see that we will get two more sets of equations:

$$\overline{\overline{HF}}_1' = H_{00}\cdot F_{00} + H_{03}\cdot F_{03} \qquad \text{(A7.5a)}$$

$$\overline{\overline{HF}}_2' = H_{01}\cdot F_{01} - H_{02}\cdot F_{02} \qquad \text{(A7.5b)}$$

$$\overline{\overline{HF}}_3' = -H_{01}\cdot F_{02} - H_{02}\cdot F_{01} \qquad \text{(A7.5c)}$$

and

$$\overline{\overline{H'F}}_1 = H_{00}\cdot F_{00} + H_{30}\cdot F_{30} \qquad \text{(A7.6a)}$$

$$\overline{\overline{H'F}}_2 = H_{10}\cdot F_{10} - H_{20}\cdot F_{20} \qquad \text{(A7.6b)}$$

$$\overline{\overline{H'F}}_3 = H_{10}\cdot F_{20} + H_{20}\cdot F_{10} \qquad \text{(A7.6c)}$$

where we have used a prime (') to indicate a reversed filter. Using the other two pairs of filters F,G and G,H we can obtain similar equations in products of F and G matrix elements and products of G and H matrix elements. Since the (0,0) elements of F, G, and H are known [eq. (A7.2)], eq. (A7.5a) and its analogs in FG and GH can be solved for F_{03}, G_{03}, and H_{03}. Likewise from eq. (A7.6a) and its analogs F_{30}, G_{30}, and H_{30} can be obtained. The result for filter G is

$$G_{03}^2 = \frac{(\overline{FG}_1' - \bar{F}\cdot\bar{G})\cdot(\overline{GH}_1' - \bar{G}\cdot\bar{H})}{\overline{HF}_1' - \bar{H}\cdot\bar{F}} \tag{A7.7a}$$

$$G_{30}^2 = \frac{(\overline{F'G}_1 - \bar{F}\cdot\bar{G})\cdot(\overline{G'H}_1 - \bar{G}\cdot\bar{H})}{\overline{H'F}_1 - \bar{H}\cdot\bar{F}} . \tag{A7.7b}$$

The other solutions can be obtained by cyclic permutations of the symbols F, G, and H. Notice that when G_{03} (or G_{30}) is calculated we may take either the positive or negative sign for the square root. This ambiguity arises because we have not defined in our experiment what we mean by right-circular polarization. We are free to choose either direction to be associated with a positive value for the L_3 Stokes component. If consistency with an independent standard of handedness is necessary then a filter or beam of known handedness is required. Once one sign has been chosen then the signs of all the other (0,3) and (3,0) matrix elements are uniquely defined by equations (A7.4a), (A7.5a), (A7.6a) and their analogs.

Equations (A7.5b) and (A7.5c) and the similar equations written for the other two filter pairs constitute six equations in six unknowns. The solution of this system for filter G, for example, is

$$G_{01}^2 = \frac{1}{2}\,\frac{\gamma_1 + \gamma_2}{\overline{HF}_2'^2 + \overline{HF}_3'^2} \tag{A7.8a}$$

$$G_{02}^2 = \frac{1}{2}\,\frac{\gamma_1 + \gamma_2}{\overline{HF}_2'^2 + \overline{HF}_3'^2} \tag{A7.8b}$$

where

$$\gamma_1 = [(\overline{GH}_2'^2 + \overline{GH}_3'^2)\cdot(\overline{HF}_2'^2 + \overline{HF}_3'^2)\cdot(\overline{FG}_2'^2 + \overline{FG}_3'^2)]^{\frac{1}{2}}$$

and

$$\gamma_2 = \overline{HF}_3'\cdot(\overline{GH}_2'\cdot\overline{FG}_3' + \overline{GH}_3'\cdot\overline{FG}_2') + \overline{HF}_2'\cdot(\overline{GH}_2'\cdot\overline{FG}_2' - \overline{GH}_3'\cdot\overline{FG}_3') .$$

The solutions for F and H can be obtained by permuting the symbols. The solution of the six equations like eqs. (A7.6b) and (A7.6c) for the (1,0) and (2,0) matrix elements follows immediately by symmetry. Again we see that the sign of G_{01}, for example, can be chosen arbitrarily after taking the square root of eq. (A7.8a). As with G_{03} we are free to take either sign but not if we want to be consistent with other polarization optics. If, for example, G is primarily a linear polarizer which is more transparent to horizontally polarized light when $\phi_g = 0$ than to vertically polarized light then G_{01} should be chosen positive. A suitable source of partially horizontally polarized light is light specularly reflected from a glass surface oriented so as to produce a vertical plane of incidence. Similarly, if at $\phi_g = 0$ it is more transparent to +45° polarized light than to +135° polarized light then G_{02} should be taken positive. Once these two signs are established all the others are determined by eqs. (A7.4), (A7.5), (A7.6), etc.

Finally we measure the transmittance of all three filters together. Let us assume that the beam passes through them in the order F,G,H so that the measured transmittance, $\overline{HGF}$ is given by the (0,0) element of the matrix product

$$\mathsf{H}(\phi_h)\cdot\mathsf{G}(\phi_g)\cdot\mathsf{F}(\phi_f).$$

If we perform the matrix multiplication we find that this (0,0) element is given by

$$\begin{aligned}\overline{HGF} = {} & \overline{HGF}_1 + \overline{HGF}_2\cdot C_{gf} + \overline{HGF}_3\cdot S_{gf} + \overline{HGF}_4\cdot C_{hg} + \overline{HGF}_5\cdot S_{hg} \\ & + \overline{HGF}_6\cdot C_{gf}\cdot C_{hg} + \overline{HGF}_7\cdot C_{gf}\cdot S_{hg} + \overline{HGF}_8\cdot S_{gf}\cdot C_{hg} + \overline{HGF}_9\cdot S_{gf}\cdot S_{hg}\end{aligned} \tag{A7.9}$$

where

$$\begin{aligned}
\overline{HGF}_1 &= H_{00}\cdot G_{00}\cdot F_{00} + H_{00}\cdot G_{03}\cdot F_{30} + H_{03}\cdot G_{30}\cdot F_{00} + H_{03}\cdot G_{33}\cdot F_{30} \\
\overline{HGF}_2 &= H_{00}\cdot G_{01}\cdot F_{10} + H_{00}\cdot G_{02}\cdot F_{20} + H_{03}\cdot G_{31}\cdot F_{10} + H_{03}\cdot G_{32}\cdot F_{20} \\
\overline{HGF}_3 &= H_{00}\cdot G_{01}\cdot F_{20} - H_{00}\cdot G_{02}\cdot F_{10} + H_{03}\cdot G_{31}\cdot F_{20} - H_{03}\cdot G_{32}\cdot F_{10} \\
\overline{HGF}_4 &= H_{01}\cdot G_{10}\cdot F_{00} + H_{01}\cdot G_{13}\cdot F_{30} + H_{02}\cdot G_{20}\cdot F_{00} + H_{02}\cdot G_{23}\cdot F_{30} \\
\overline{HGF}_5 &= -H_{02}\cdot G_{10}\cdot F_{00} - H_{02}\cdot G_{13}\cdot F_{30} + H_{01}\cdot G_{20}\cdot F_{00} + H_{01}\cdot G_{23}\cdot F_{30} \\
\overline{HGF}_6 &= H_{01}\cdot G_{11}\cdot F_{10} + H_{01}\cdot G_{12}\cdot F_{20} + H_{02}\cdot G_{21}\cdot F_{10} + H_{02}\cdot G_{22}\cdot F_{20} \\
\overline{HGF}_7 &= -H_{02}\cdot G_{11}\cdot F_{10} - H_{02}\cdot G_{12}\cdot F_{20} + H_{01}\cdot G_{21}\cdot F_{10} + H_{01}\cdot G_{22}\cdot F_{20} \\
\overline{HGF}_8 &= H_{01}\cdot G_{11}\cdot F_{20} - H_{01}\cdot G_{12}\cdot F_{10} + H_{02}\cdot G_{21}\cdot F_{20} - H_{02}\cdot G_{22}\cdot F_{10} \\
\overline{HGF}_9 &= -H_{02}\cdot G_{11}\cdot F_{20} + H_{02}\cdot G_{12}\cdot F_{10} + H_{01}\cdot G_{21}\cdot F_{20} - H_{01}\cdot G_{22}\cdot F_{10}
\end{aligned} \tag{A7.10}$$

and $C_{gf} = \cos 2(\phi_g-\phi_f)$, $S_{gf} = \sin 2(\phi_g-\phi_f)$, etc. Again we assume that by least squares fitting or by solution of simultaneous equations we have obtained numerical values for the coefficients $\overline{HGF}_1$, $\overline{HGF}_2$, etc., of all the trigonometric factors in eq. (A7.9); that is, we assume that we know the left-hand sides of eqs. (A7.10). In the first of eqs. (A7.10) we now know everything except G_{33} so it can be solved for directly. The second and third of eqs. (A7.10) are a pair of equations in which the only remaining unknowns are G_{31} and G_{32}. Likewise the next two equations contain only two unknowns, G_{13} and G_{23}. The last four equations of the set (A7.10) contain the four unknowns G_{11}, G_{12}, G_{21}, and G_{22}. Thus, all the rest of the matrix elements of the Mueller matrix for the filter G can be determined from these nine coefficient values. Measurements of the three-filter train in the orders G, H, F and H, F, G will permit the remaining matrix elements of the other two filters to be determined. We won't attempt to write out explicit solutions for the unknowns appearing in the set of equations (A7.10). The procedure to obtain the solutions is obvious and straightforward, but the entire data reduction process for this general case is clearly best left to a computer, if possible.

We have assumed that the matrix elements in the first row and first column are non-zero. If zeros appear here some of the matrix elements may become indeterminate. For example, in eq. (A7.7a) if either F_{03} or H_{03} vanishes the denominator and one factor of the numerator will both vanish and G_{03} cannot be evaluated. The physical reason for this is clear. The experimental procedure, in effect, requires generating partially circularly polarized light with one filter and then analyzing it with a second filter. If the first filter fails to produce circularly polarized light, no information can be obtained about the circular polarization behavior of the second filter. Similar arguments apply to the other first column and first row matrix elements. The extreme case occurs when all the first column and first row elements of one filter, say G, vanish. Then G is totally opaque and obviously contributes nothing to the measurement of filters F and H. With only two filters, F_{00} and H_{00} can be measured as before, but equations (A7.4), (A7.5), and A7.6) provide only 9 equations in 12 unknowns and contain no information about the 18 other matrix elements which do not lie in the first row or column. When the equations become insoluble because of the occurrence of zero denominators one must either abandon the attempt to measure the inaccessible matrix elements or find other filters which don't have such limitations.

When the matrix elements for an optical component have been determined as outlined above they can be inserted in eq. (6.39) to obtain the orientation dependence of the Mueller matrix. In general, even for ideal optical components the result will not agree with, for example eq. (6.15), but this is only because the angle between the polarization axis of the device and the $\phi = 0°$ direction was assumed arbitrary. The direction of the polarization axis can be found, if desired, by finding the angle ϕ^o at which the experimental Mueller matrix most nearly resembles the matrix for the ideal component. For example, for an ideal linear polarizer at $\phi = 0°$ the (0,2) element vanishes so for a real polarizer we must find the angle ϕ^o at which the (0,2) element [from eq. (6.39)] of the experimental matrix vanishes. That is,

$$G_{01} \cdot \sin 2\phi^o + G_{02} \cdot \cos 2\phi^o = 0$$

or

$$\phi^o = -\tfrac{1}{2} \arctan(G_{02}/G_{01}).$$

This is the direction of the polarization axis and if the orientation angle for an ideal component is measured from this direction instead of from $\phi = 0°$ its experimentally determined Mueller matrix should agree with that given by eq. (6.15).