ON INTEGER SOLUTIONS OF $x^4 + y^4 - 2z^4 - 2w^4 = 0$

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ABSTRACT. In this article, we study the quartic Diophantine equation $x^4 + y^4 - 2z^4 - 2w^4 = 0$. We find non-trivial integer solutions. Furthermore, we show that when a solution has been found, a series of other solutions can be derived. We do so using two different techniques. The first is a geometric method due to Richmond, while the second involves elliptic curves.

1. INTRODUCTION

Diophantine equations have long been of interest to mathematicians. In this work, we consider the quartic surface

$$ax^4 + by^4 + cz^4 + dw^4 = 0, \quad a, b, c, d \in \mathbb{Z} \setminus \{0\}. \tag{1.1}$$

For arbitrary values of a, b, c, and d, there does not appear to be many results on finding integral solutions [5, 10, 14]. From Dickson's History [5], it appears that the first equation of the form (1.1) that has been extensively investigated is the classical one proposed by Euler, in which $a = 1$, $b = 1$, $c = -1$, and $d = -1$. See for example, [2, 12, 13, 18]. Bernstein [1] found there are 518 solutions with $0 \le x \le y \le 10^6$ and $0 \le z \le w \le 10^6$.

The other special case of (1.1) to receive much interest is when $a = 1$, $b = 1$, $c = 1$, and $d = -1$. Euler had conjectured there were no integer solutions, however Elkies found a solution in [6] despite early attempts by [13, 17] which had failed to find one. Subsequently a few other solutions have been found [1, 9]. We note other specific cases have been studied [3, 8]. There have also been computations to find the smallest integer solutions when $\max\{|a|, |b|, |c|, |d|\} \le 15$ [7].

In this work we study the particular equation with $a = 1, b = 1, c = -2$, and $d = -2$, namely

$$x^4 + y^4 - 2z^4 - 2w^4 = 0. \tag{1.2}$$

It is easy to see if (x_0, y_0, z_0, w_0) is a solution to (1.2), then so is (kx_0, ky_0, kz_0, kw_0) for any integer k. We call an integral solution (x, y, z, w) *primitive* if $0 \le x \le y$, $0 \le z \le w$, and in addition no integer $k > 1$ divides each of x, y, z, and w. It is trivial to see the first primitive solution is $(1, 1, 0, 1)$. The next primitive solution is $(19, 21, 7, 20)$. Searching for solutions with $0 \le x \le y \le 3000$ found the additional solutions $(181, 2077, 1247, 1620)$ and $(607, 1999, 951, 1640)$. Computer searches can be used to find all solutions below a given bound, however it is challenging to find an infinite family of solutions.

Our main result is the computation of new primitive solutions of (1.2). We use two different methods to find these solutions, each of which lead to an infinite number of primitive solutions. The first method is from Richmond [15]. Under

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the condition that $abcd$ is a square, he showed that if a rational solution to (1.1) is known then others can be found. The second method uses the theory of elliptic curves. We will show that every rational point on a certain elliptic curve E leads to an integral solution to (1.2). Since E has an infinite number of rational points (i.e., $\mathrm{rank}(E) = 1$), this will yield an infinite number of solutions.

2. Determination of Primitive Solutions

In this section we compute primitive integral solutions to the quartic Diophantine equation

$$x^4 + y^4 - 2z^4 - 2w^4 = 0.$$

We first show how starting with the solution $(19, 21, 7, 20)$, Richmond's method can be used to generate more primitive solutions. We then show how to find solutions from rational points on the elliptic curve $Y^2 = X^3 - 36X$.

2.1. **Richmond's method.** Richmond [15] considered the surface (1.1) with the additional constraint that $abcd$ is a square number. Suppose $P = (x_0, y_0, z_0, w_0)$ is a rational point on this surface. Richmond showed that the condition $abcd$ being square implies that a rational line ℓ can be drawn through P to meet the surface in three points at P. Hence if the line does not lie on the surface, then the fourth point of intersection, will also be rational. Thus starting with a single point P, other rational points can be found. We note that while Richmond's proof is almost entirely geometric, Mordell was able to reprove the same result in a different way [14].

For the curve (1.2) this work focuses on, we have $abcd = 4$. We can thus use Richmond's technique, with $P = (19, 21, 7, 20)$. It is easy to calculate the equation of the tangent plane at P:

$$19^3x + 21^3y - 2 \cdot 7^3z - 2 \cdot 20^3w = 0.$$

We likewise compute the inflectional tangent at P:

$$19^2x^2 + 21^2y^2 - 2 \cdot 7^2z^2 - 2 \cdot 20^2w^2 = 0.$$

Let Q be any point, and following Richmond, we write it in the form $Q = (19p, 21q, 7r, 20s)$, for some p, q, r, s. Then if Q lies on both the tangent and inflectional tangents at P, we have

$$19^4p + 21^4q - 2 \cdot 7^4r - 2 \cdot 20^4s = 0, \tag{2.1}$$

$$19^4p^2 + 21^4q^2 - 2 \cdot 7^4r^2 - 2 \cdot 20^4s^2 = 0. \tag{2.2}$$

Note that for any value of t, if we replace (p, q, r, s) by $(p+t, q+t, r+t, s+t)$ then (2.1) and (2.2) remain valid. We can thus assume $p+q+r+s = 0$, or $s = -p-q-r$. Solving for r in (2.1) we find $r = -(450321p + 514481q)/315198$. Substituting into (2.2), we obtain a quadratic equation in p and q:

$$1276746718401p^2 - 4052230076802pq + 1112497118401q^2 = 0.$$

The condition $abcd$ being square ensures that the quadratic factors:

$$(188391p - 57191q)(6777111p - 19452311q) = 0.$$

Taking the first factor, we set $q = 188391p/57191$ and hence $r = -389209p/57191$, $s = 143627p/57191$. This leads to the rational solution $(19p, 3956211p/57191, -2724463p/57191, 2872540p/57191)$. If we let $p = 57191$ we obtain the primitive integral solution

(1086629, 3956211, 2724463, 2872540). If instead we take the second factor, we end up with the primitive solution (142319331, 369593909, 252477340, 271973023).

We see that beginning with the rational point P, we have found two new primitive solutions to (1.2). Richmond's method can be applied repeatedly to obtain new solutions.

2.2. **Solutions from a congruent elliptic curve.** We assume a basic familiarity with elliptic curves (see, for example, [11]). Our second method uses birational transformations to relate the surface (1.2) to an elliptic curve. Let $x = z + t$ and $y = z - t$, where t is a (rational) parameter. Then (1.2) becomes

$$w^4 - t^4 = 6(tz)^2. \tag{2.3}$$

We need the following result of Cohen.

Proposition 2.1. [4, Prop. 6.5.6] *Let c be a nonzero integer. The equation $X^4 - Y^4 = cZ^2$ has a solution with $XYZ \neq 0$ if and only if $|c|$ is a congruent number. More precisely, if $X^4 - Y^4 = cZ^2$ with $XYZ \neq 0$ then $V^2 = U(U^2 - c^2)$ with*

$$(U, V) = (-cY^2/X^2, c^2YZ/X^3),$$

and conversely if $V^2 = U(U^2 - c^2)$ with $V \neq 0$ then $X^4 - Y^4 = cZ^2$, with

$$(X, Y, Z) = \left(U^2 + 2cU - c^2, U^2 - 2cU - c^2, 4V(U^2 + c^2)\right).$$

An integer c is congruent if it is the area of a right triangle with rational side lengths and area c. It is well-known that the elliptic curve $V^2 = U(U^2 - c^2)$ has a rational point (with $V \neq 0$) if and only if c is a congruent number [11]. For this reason we refer to $V^2 = U(U^2 - c^2)$ as a congruent elliptic curve.

Since the area of a 3-4-5 right triangle is 6, then 6 is a congruent number. Using Proposition 2.1 on the curve (2.3), we have mapped our surface into the congruent elliptic curve

$$E_6 : V^2 = U(U^2 - 36).$$

For our next result, recall that any rational point P on an elliptic curve $V^2 = U^3 + aU + b$ can be written in the form $P = \left(\frac{A}{B^2}, \frac{C}{B^3}\right)$, with $A, B, C \in \mathbb{Z}$.

Corollary 2.2. *Suppose $\left(\frac{A}{B^2}, \frac{C}{B^3}\right)$ is a rational point on the elliptic curve E_6, with $A, B, C \in \mathbb{Z}$. Let*

$$\begin{cases} x = 1296B^8 + 864B^6A + 144B^5C + 72B^4A^2 - 24B^2A^3 + 4BA^2C + A^4, \\ y = 1296B^8 + 864B^6A - 144B^5C + 72B^4A^2 - 24B^2A^3 - 4BA^2C + A^4, \\ z = 144B^5C + 4BA^2C, \\ w = 1296B^8 - 216B^4A^2 + A^4. \end{cases}$$

Then (x, y, z, w) is an integral solution to the Diophantine equation $x^4 + y^4 - 2z^4 - 2w^4 = 0$.

Proof. We have $w = U^2 + 12U - 36, t = U^2 - 12U - 36, tz = 4V(U^2 + 36)$ by the rational transformations used in Proposition 2.1. Since $x = z + t$ and $y = z - t$ we have

$$\begin{aligned}
x &= \frac{4U^2V + 144V + U^4 - 24U^3 + 72U^2 + 864U + 1296}{U^2 - 12U - 36},\\
y &= \frac{-4U^2V - 144V + U^4 - 24U^3 + 72U^2 + 864U + 1296}{U^2 - 12U - 36},\\
z &= \frac{4V(U^2+36)}{U^2 - 12U - 36},\\
w &= U^2 + 12U - 36.
\end{aligned}$$

Substituting in $U = \frac{A}{B^2}$ and $V = \frac{C}{B^3}$, we get

$$\begin{aligned}
x &= \frac{4CBA^2 + 144CB^5 + A^4 - 24A^3B^2 + 72A^2B^4 + 864AB^6 + 1296B^8}{B^4(A^2 - 12AB^2 - 36B^4)},\\
y &= \frac{-4CBA^2 - 144CB^5 + A^4 - 24A^3B^2 + 72A^2B^4 + 864AB^6 + 1296B^8}{B^4(A^2 - 12AB^2 - 36B^4)},\\
z &= \frac{4C(A^2 + 36B^4)}{B^3(A^2 - 12AB^2 - 36B^4)},\\
w &= \frac{A^2 + 12AB^2 - 36B^4}{B^4}.
\end{aligned}$$

Using the fact that (kx, ky, kz, kw) is a solution to (1.2) if (x, y, z, w) is, we can eliminate the denominators. The result now follows immediately. □

The elliptic curve E_6 is of rank 1, with the generator $P = (-3, 9)$ [16]. There are thus an infinite number of rational points on E_6. By Corollary 2.2, we see there will be infinitely many integer solutions of the Diophantine equation (1.2). By suitably changing the signs and swapping x and y (or z and w), we can make each solution primitive.

Computations show that this corollary yields new solutions. For example, the point $2P = (\frac{25}{4}, -\frac{35}{8})$ on E_6 leads to $(1661081, 988521, 336280, 1437599)$ on (1.2), which is smaller than the solutions obtained by using Richmond's method once. The point $3P = (-1587/1369, -321057/50653)$ gives the solution $(x, y, z, w) = (22394369951939, 59719152671941, 41056761311940, 43690772126393)$.

3. Conclusion

In this work, we have shown two different ways to find infinitely many integer solutions to the quartic Diophantine equation (1.2). While computer searches can find all solutions below a given bound, it is non-trivial to find infinitely many.

The equation (1.2) we have focused on is the $n = 4$ case of the more general equation

$$x^n + y^n - 2z^n - 2w^n = 0. \tag{3.1}$$

When $n = 2$, the identity

$$(z+w)^2 + (z-w)^2 = 2(z^2 + w^2)$$

gives an infinite parameterized family of solutions.

For $n = 3$, the strong version of Conjecture 6.4.26 of [4] implies that for any integer N, then $N = x^3 + y^3 + 2z^3$ for some integers x, y, z. So for any integer w, if we let $N = 2w^3$, there is a solution to (3.1) (assuming the conjecture). More concretely, we can show there are an infinite number of primitive solutions using the elliptic curve method of Section 2.2. If we let $z = y-t$ and $w = y+t$, then some simple algebra shows we can simplify to the elliptic curve $Y^2 = X^3 - 81$, where $X = 3x/y$ and $Y = 18t/y$. This is a rank 1 curve, with generator (13,46). Each rational point (X, Y) on the curve leads to the solution $(6X, 18, 18 - Y, 18 + Y)$. We can make these solutions integral by scaling.

We note that Manin conjectured that all rational solutions of the $n = 3$ case of (3.1) can be obtained from a finite number of solutions (x_i, y_i, z_i, w_i) by a succession of secant and tangent processes [4, Conj. 6.4.1]. Future work could involve finding these finite number of generating solutions. It would also be interesting to find integer solutions to (3.1), for $n \geq 5$. Preliminary computer searches have not found any non-trivial solutions.

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