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Separation of variables in an asymmetric cyclidic coordinate system

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A global analysis is presented of solutions for Laplace's equation on three-dimensional Euclidean space in one of the most general orthogonal asymmetric confocal cyclidic coordinate systems which admit solutions through separation of variables. We refer to this coordinate system as five-cyclide coordinates since the coordinate surfaces are given by two cyclides of genus zero which represent inversions of each other with respect to the unit sphere, a cyclide of genus one, and two disconnected cyclides of genus zero. This coordinate system is obtained by stereographic projection of sphero-conal coordinates on four-dimensional Euclidean space. The harmonics in this coordinate system are given by products of solutions of second-order Fuchsian ordinary differential equations with five elementary singularities. The Dirichlet problem for the global harmonics in this coordinate system is solved using multiparameter spectral theory in the regions bounded by the asymmetric confocal cyclidic coordinate surfaces. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4812321]

I. INTRODUCTION

In 1894, Maxime Bôcher's book "Ueber die Reihenentwickelungen der Potentialtheorie" was published.[2] It took its origin from lectures given by Felix Klein in Göttingen (see for instance, Refs. 7 and 8). In Bôcher's book, the author gives a list of 17 inequivalent coordinate systems in three dimensions in which the Laplace equation admits separated solutions of the form

$$U(x, y, z) = R(x, y, z)w_1(s_1)w_2(s_2)w_3(s_3), \tag{1}$$

where the modulation factor $R(x, y, z)$ (see p. 519 of Ref. 11) is a known and fixed function, and s_1, s_2, s_3 are curvilinear coordinates of x, y, z. The functions w_1, w_2, w_3 are solutions of second order ordinary differential equations. The symmetry group of Laplace's equation is the conformal group and equivalence between various separable coordinate systems is established by the existence of a conformal transformation which maps one separable coordinate system to another.

In general, the coordinate surfaces (called confocal cyclides) are given by the zero sets of polynomials in x, y, z of degree at most four which can be broken up into several different subclasses. For instance, eleven of these coordinate systems have coordinate surfaces which are given by confocal quadrics (Systems 1–11 on p. 164 of Ref. 9), nine are rotationally-invariant (Systems 2, 5–8 on p. 164 and Systems 14–17 on p. 210 of Ref. 9), four are cylindrical (Systems 1–4 of Ref. 9), and the five most general are of the asymmetric type, namely confocal ellipsoidal, paraboloidal, sphero-conal, and two cyclidic coordinate systems (Systems 9–11 on p. 164 and Systems 12 and 13 on p. 210 of Ref. 9). Bôcher[2] showed how to solve the Dirichlet problem for harmonic functions on regions bounded by such confocal cyclides. However, it is stated repeatedly in Bôcher's book that the presentation lacked convergence proofs, for instance, this is mentioned in the preface written by Felix Klein.

It is the purpose of this paper to supply the missing proofs for one of the asymmetric cyclidic coordinate systems which is listed as number 12 in Miller's list (see p. 210 of Ref. 9) (see also

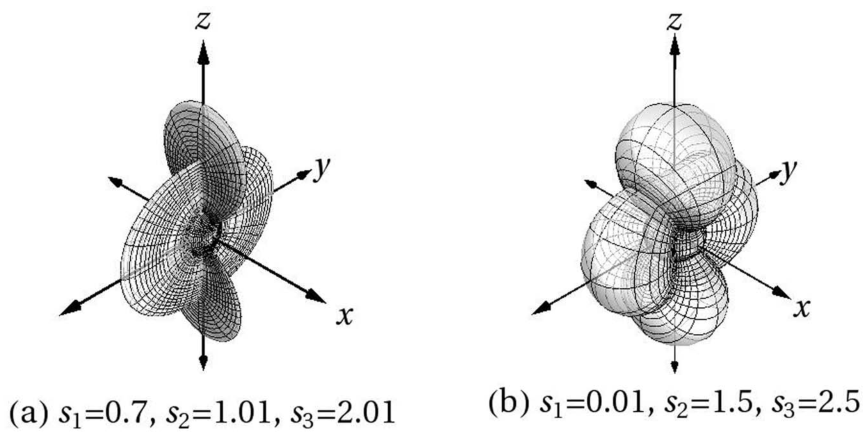

FIG. 1. Surfaces $s_{1,\,2,\,3} = const$ for $a_i = i$, where only the component of the cyclide $s_1 = const$ inside the ball $x^2 + y^2 + z^2 < 1$ is shown.

Table II of Ref. 3 and Ref. 4 for a more general setting). For lack of a better name we call it *5-cyclide coordinates*. This asymmetric orthogonal curvilinear coordinate system has coordinates $s_i \in \mathbf{R}$ ($i = 1, 2, 3$) with s_i in (a_0, a_1), (a_1, a_2) or (a_2, a_3), respectively, where $a_0 < a_1 < a_2 < a_3$ are given numbers. This coordinate system is described by coordinate surfaces $s_i = const$ which are five compact cyclides. The surfaces $s_1 = const$ for $s_1 \in (a_0, a_1)$ are two cyclides of genus zero representing inversions of each other with respect to the unit sphere. The surface $s_2 = const$ for $s_2 \in (a_1, a_2)$ represents a ring cyclide of genus one and the surfaces $s_3 = const$ for $s_3 \in (a_2, a_3)$ represent two disconnected cyclides of genus zero with reflection symmetry about the x, y-plane. The asymptotic behavior of this coordinate system as the size of these compact cyclides increases without limit is *6-sphere coordinates* (see p. 122 of Ref. 10), the inversion of Cartesian coordinates.

In our notation, the coordinate surfaces of this system are given by the variety

$$\frac{(x^2+y^2+z^2-1)^2}{s-a_0} + \frac{4x^2}{s-a_1} + \frac{4y^2}{s-a_2} + \frac{4z^2}{s-a_3} = 0, \tag{2}$$

where $s = s_i$ is either in (a_0, a_1), (a_1, a_2), or (a_2, a_3), respectively.

See Figures 1(a) and 1(b) for a graphical illustration of these triply orthogonal coordinate surfaces, where we have selected one of the confocal cyclides for $s_1 = const$. This is a very general coordinate system containing the parameters a_0, a_1, a_2, a_3 which generates many other coordinate systems by limiting processes. For example, rotationally invariant flat-ring coordinates (System 15 on p. 210 of Ref. 9) are obtained by setting $s_3 = a_2 \sin^2\phi + a_3 \cos^2\phi$ and letting $a_3 \to a_2$, and rotationally invariant bicyclidic coordinates (System 14 on p. 210 of Ref. 9) are obtained by setting $s_2 = a_1 \sin^2\phi + a_2 \cos^2\phi$ and letting $a_2 \to a_1$. Since the book by Bôcher is quite old and uses very geometrical methods, we will present our results independently of Bôcher's book. We supply convergence proofs based on general multiparameter spectral theory[1,13] which was created with such applications in mind. As far as we know this general theory has never before been applied to the Dirichlet problems considered by Bôcher.

We start with the observation that 5-cyclide coordinates are the stereographic image of sphero-conal coordinates in four dimensions (or, expressed in another way, of ellipsoidal coordinates on the hypersphere $\mathbb{S}^3$). We take the sphero-conal coordinate system as known but we present the needed facts in Sec. II. The well-known stereographic projection is dealt with in Sec. III which also explains the appearance of the factor R in (1). The 5-cyclide coordinate system is introduced in Sec. IV. The solution of the Dirichlet problem on regions bounded by surfaces (2) with $s \in (a_1, a_2)$ is presented in Sec. VI. Section V provides the needed convergence proofs based on multiparameter spectral theory. The remaining Secs. VII–X treat the Dirichlet problem on regions bounded by the surfaces (2) when $s \in (a_1, a_2)$ (ring cyclides) and $s \in (a_2, a_3)$.

II. SPHERO-CONAL COORDINATES ON $\mathbf{R}^{k+1}$

Let $k \in \mathbf{N}$. In order to introduce sphero-conal coordinates on $\mathbf{R}^{k+1}$, fix real numbers

$$a_0 < a_1 < a_2 < \cdots < a_k. \tag{3}$$

Let $(x_0, x_1, \ldots, x_k)$ be in the positive cone of $\mathbf{R}^{k+1}$

$$x_0 > 0, \ldots, x_k > 0. \tag{4}$$

Its sphero-conal coordinates $r, s_1, \ldots, s_k$ are determined in the intervals

$$r > 0,\ a_{i-1} < s_i < a_i, \quad i = 1, \ldots, k \tag{5}$$

by the equations

$$r^2 = \sum_{j=0}^{k} x_j^2 \tag{6}$$

and

$$\sum_{j=0}^{k} \frac{x_j^2}{s_i - a_j} = 0 \quad \text{for } i = 1, \ldots, k. \tag{7}$$

The latter equation determines $s_1, s_2, \ldots, s_k$ as the zeros of a polynomial of degree k with coefficients which are polynomials in $x_0^2, \ldots, x_k^2$.

In this way, we obtain a bijective (real-)analytic map from the positive cone in $\mathbf{R}^{k+1}$ to the set of points $(r, s_1, \ldots, s_k)$ satisfying (5). The inverse map is found by solving a linear system. It is also analytic, and it is given by

$$x_j^2 = r^2 \frac{\prod_{i=1}^{k}(s_i - a_j)}{\prod_{j\neq i=0}^{k}(a_i - a_j)}. \tag{8}$$

Sphero-conal coordinates are orthogonal, and its scale factors (metric coefficients) are given by $H_r = 1$, and

$$H_{s_i}^2 = \frac{1}{4}\sum_{j=0}^{k} \frac{x_j^2}{(s_i - a_j)^2} = -\frac{1}{4} r^2 \frac{\prod_{i\neq j=1}^{k}(s_i - s_j)}{\prod_{j=0}^{k}(s_i - a_j)}, \quad i = 1, 2, \ldots, k. \tag{9}$$

Consider the Laplace equation

$$\Delta U = \sum_{i=0}^{k} \frac{\partial^2 U}{\partial x_i^2} = 0 \tag{10}$$

for a function $U(x_0, x_1, \ldots, x_k)$. Using (9) we transform this equation to sphero-conal coordinates, and then we apply the method of separation of variables[12]

$$U(x_0, x_1, \ldots, x_k) = w_0(r) w_1(s_1) w_2(s_2) \ldots w_k(s_k). \tag{11}$$

For the variable r we obtain the Euler equation

$$w_0'' + \frac{k}{r} w_0' + \frac{4\lambda_0}{r^2} w_0 = 0, \tag{12}$$

while for each of the variables $s_1, s_2, \ldots, s_k$, we obtain the Fuchsian equation

$$\prod_{j=0}^{k}(s - a_j)\left[w'' + \frac{1}{2}\sum_{j=0}^{k}\frac{1}{s - a_j} w'\right] + \left[\sum_{i=0}^{k-1} \lambda_i s^{k-1-i}\right] w = 0. \tag{13}$$

More precisely, if $\lambda_0, \ldots, \lambda_{k-1}$ are any given numbers (separation constants), and if $w_0(r)$, $r > 0$, solves (12) and $w_i(s_i)$, $a_{i-1} < s_i < a_i$, solve (13) for each $i = 1, \ldots, k$, then U defined by (11) solves (10) in the positive cone of $\mathbf{R}^{k+1}$ (4).

Equation (13) has only regular points except for $k+2$ regular singular points at $s = a_0, a_1, \ldots, a_k$ and $s = \infty$. The exponents at each finite singularity $s = a_j$ are 0 and $\frac{1}{2}$. Therefore, for each choice of parameters $\lambda_0, \ldots, \lambda_{k-1}$, there is a nontrivial analytic solution at $s = a_j$ and another one of the form $w(s) = (s-a_j)^{1/2} v(s)$, where v is analytic at a_j. If ν, μ denote the exponents at $s = \infty$, then

$$\mu\nu = \lambda_0, \quad \mu + \nu = \frac{k-1}{2}. \tag{14}$$

The polynomial $\sum_{i=0}^{k-1} \lambda_i s^{k-1-i}$ appearing in (13) is known as van Vleck polynomial. If $k = 1$, then (13) is the hypergeometric differential equation (up to a linear substitution). If $k = 2$, then (13) is the Heun equation. We will use this equation for $k = 3$. According to Miller (see p. 209 of Ref. 9) (see also p. 71 of Ref. 3) in reference to the $k = 3$ case, "Very little is known about the solutions."

III. STEREOGRAPHIC PROJECTION

We consider the stereographic projection $P : \mathbb{S}^3 \setminus \{(1, 0, 0, 0)\} \to \mathbf{R}^3$ given by

$$P(x_0, x_1, x_2, x_3) = \frac{1}{1-x_0}(x_1, x_2, x_3).$$

The inverse map is

$$P^{-1}(x, y, z) = \frac{1}{x^2+y^2+z^2+1}(x^2+y^2+z^2-1, 2x, 2y, 2z).$$

We extend P^{-1} to a bijective map

$$Q : (0, \infty) \times \mathbf{R}^3 \to \mathbf{R}^4 \setminus \{(x_0, 0, 0, 0) : x_0 \geq 0\}$$

by defining

$$Q(r, x, y, z) := r P^{-1}(x, y, z).$$

If we set $(x_0, x_1, x_2, x_3) = Q(r, x, y, z)$, we may consider r, x, y, z as curvilinear coordinates on $\mathbf{R}^4$ with Cartesian coordinates x_0, x_1, x_2, x_3. We note that $x_0^2 + x_1^2 + x_2^2 + x_3^2 = r^2$ so r is just the distance between (x_0, x_1, x_2, x_3) and the origin. Moreover, (x, y, z) is the stereographic projection of the point $(x_0/r, x_1/r, x_2/r, x_3/r) \in \mathbb{S}^3$. It is easy to check that the coordinate system is orthogonal and scale factors are

$$h_r = 1, \quad h_x = h_y = h_z = 2rh, \quad \text{where } h := \frac{1}{x^2+y^2+z^2+1}.$$

Let $U(x_0, x_1, x_2, x_3) = V(r, x, y, z)$. Then

$$\Delta U = \frac{1}{8r^3h^3}\left((2rhV_x)_x + (2rhV_y)_y + (2rhV_z)_z + (8r^3h^3V_r)_r\right). \tag{15}$$

Suppose that U is homogeneous of degree α:

$$U(tx_0, tx_1, tx_2, tx_3) = t^\alpha U(x_0, x_1, x_2, x_3), \quad t > 0.$$

Then V can be written in the form

$$V(r, x, y, z) = r^\alpha w(x, y, z),$$

and (15) implies

$$\Delta U = \frac{r^{\alpha-2}}{4h^3}\left((hw_x)_x + (hw_y)_y + (hw_z)_z + 4\alpha(\alpha+2)h^3 w\right). \tag{16}$$

We now introduce the function

$$u(x, y, z) = w(x, y, z)(x^2+y^2+z^2+1)^{-1/2}.$$

Then a direct calculation changes (16) to

$$\Delta U = \frac{r^{\alpha-2}}{4h^{5/2}}\left(u_{xx}+u_{yy}+u_{zz}+(3+4\alpha(\alpha+2))h^2u\right). \tag{17}$$

If $3 + 4\alpha(\alpha + 2) = 0$, then U is harmonic if and only if u is harmonic. Noting that $3 + 4\alpha(\alpha + 2) = (2\alpha + 1)(2\alpha + 3)$, we obtain the following theorem.

Theorem 3.1. *Let D be an open subset of $\mathbb{S}^3$ not containing $(1, 0, 0, 0)$, let $E = \{(rx_0, rx_1, rx_2, rx_3)\colon r > 0, (x_0, x_1, x_2, x_3) \in D\}$, and let $F = P(D)$ be the stereographic image of D. Let the function $U\colon E \to \mathbf{R}$ be homogeneous of degree $-\frac{1}{2}$ or $-\frac{3}{2}$, and let $w : F \to \mathbf{R}$ satisfy $U = w \circ P$ on D. Then U is harmonic on E if and only if $w(x, y, z)(x^2 + y^2 + z^2 + 1)^{-1/2}$ is harmonic on F.*

IV. FIVE-CYCLIDE COORDINATE SYSTEM ON $\mathbf{R}^3$

We introduce sphero-conal coordinates

$$r > 0, \quad a_0 < s_1 < a_1 < s_2 < a_2 < s_3 < a_3,$$

on $\mathbf{R}^4$ as explained in Sec. II with $k = 3$. Then s_1, s_2, s_3 form a coordinate system for the intersection of the hypersphere $\mathbb{S}^3$ with the positive cone in $\mathbf{R}^4$. Using the stereographic projection P from Sec. III, we project these coordinates to $\mathbf{R}^3$. We obtain a coordinate system for the set

$$T = \{(x, y, z) : x, y, z > 0,\ x^2 + y^2 + z^2 > 1\}. \tag{18}$$

Explicitly,

$$x = \frac{x_1}{1-x_0}, \quad y = \frac{x_2}{1-x_0}, \quad z = \frac{x_3}{1-x_0}, \tag{19}$$

where

$$x_j^2 = \frac{\prod_{i=1}^3 (s_i - a_j)}{\prod_{j\neq i=0}^3 (a_i - a_j)}, \quad j = 0, 1, 2, 3. \tag{20}$$

Conversely, the coordinates s_1, s_2, s_3 of a point $(x, y, z) \in T$ are the solutions of

$$\frac{(x^2+y^2+z^2-1)^2}{s-a_0} + \frac{4x^2}{s-a_1} + \frac{4y^2}{s-a_2} + \frac{4z^2}{s-a_3} = 0. \tag{21}$$

Since sphero-conal coordinates are orthogonal and the stereographic projection preserves angles, 5-cyclide coordinates are orthogonal, too. This is the twelfth coordinate system in Miller (see p. 210 of Ref. 9). Miller uses a slightly different notation: $a_0 = 0$, $a_1 = 1$, $a_2 = b$, $a_3 = a$, and $s_1 = \rho$, $s_2 = \nu$, $s_3 = \mu$. Also, x, z are interchanged.

In order to calculate the scale factors for the 5-cyclide coordinate system, we proceed as follows. We start with

$$\frac{\partial x}{\partial s_i} = \frac{1}{1-x_0}\frac{\partial x_1}{\partial s_i} + \frac{x_1}{(1-x_0)^2}\frac{\partial x_0}{\partial s_i},$$

and similar formulas for the derivatives of y and z. Then using

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, \quad x_0\frac{\partial x_0}{\partial s_i} + x_1\frac{\partial x_1}{\partial s_i} + x_2\frac{\partial x_2}{\partial s_i} + x_3\frac{\partial x_3}{\partial s_i} = 0,$$

a short calculation gives

$$\frac{\partial x}{\partial s_i}\frac{\partial x}{\partial s_j} + \frac{\partial y}{\partial s_i}\frac{\partial y}{\partial s_j} + \frac{\partial z}{\partial s_i}\frac{\partial z}{\partial s_j} = \frac{1}{(1-x_0)^2}\sum_{\ell=0}^{3}\frac{\partial x_\ell}{\partial s_i}\frac{\partial x_\ell}{\partial s_j}.$$

This confirms that 5-cyclide coordinates are orthogonal and from (9) we obtain the squares of their scale factors

$$h_i^2 = \frac{1}{16}\left(\frac{(\rho^2-1)^2}{(s_i-a_0)^2} + \frac{4x^2}{(s_i-a_1)^2} + \frac{4y^2}{(s_i-a_2)^2} + \frac{4z^2}{(s_i-a_3)^2}\right), \tag{22}$$

where $\rho^2 = x^2 + y^2 + z^2$, or, equivalently,

$$h_1^2 = \frac{1}{16}(\rho^2+1)^2\frac{(s_3-s_1)(s_2-s_1)}{(s_1-a_0)(a_1-s_1)(a_2-s_1)(a_3-s_1)}, \tag{23}$$

$$h_2^2 = \frac{1}{16}(\rho^2+1)^2\frac{(s_2-s_1)(s_3-s_2)}{(s_2-a_0)(s_2-a_1)(a_2-s_2)(a_3-s_2)}, \tag{24}$$

$$h_3^2 = \frac{1}{16}(\rho^2+1)^2\frac{(s_3-s_1)(s_3-s_2)}{(s_3-a_0)(s_3-a_1)(s_3-a_2)(a_3-s_3)}. \tag{25}$$

We find harmonic functions by separation of variables in 5-cyclide coordinates as follows.

Theorem 4.1. *Let $w_1 : (a_0, a_1) \to \mathbf{C}$, $w_2 : (a_1, a_2) \to \mathbf{C}$, $w_3 : (a_2, a_3) \to \mathbf{C}$ be solutions of the Fuchsian equation*

$$\prod_{j=0}^{3}(s-a_j)\left[w'' + \frac{1}{2}\sum_{j=0}^{3}\frac{1}{s-a_j}w'\right] + \left(\frac{3}{16}s^2 + \lambda_1 s + \lambda_2\right)w = 0, \tag{26}$$

where λ_1, λ_2 are given (separation) constants. Then the function

$$u(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} w_1(s_1) w_2(s_2) w_3(s_3) \tag{27}$$

is a harmonic function on the set (18).

Proof. Using sphero-conal coordinates r, s_1, s_2, s_3 on $\mathbf{R}^4$, we define a function U in the positive cone of $\mathbf{R}^4$ by

$$U(x_0, x_1, x_2, x_3) = r^{-1/2} w_1(s_1) w_2(s_2) w_3(s_3).$$

The function $r^{-1/2}$ is a solution of (12) when $k = 3$, $\lambda_0 = \frac{3}{16}$. The results from Sec. II imply that U is harmonic, and, of course, U is homogeneous of degree $-\frac{1}{2}$. The function w defined on the set (18) by $U = w \circ P$ is given in 5-cyclide coordinates by

$$w(x, y, z) = w_1(s_1) w_2(s_2) w_3(s_3).$$

Therefore, Theorem 3.1 gives the statement of the theorem. □

Equation (26) has five regular singularities at $s = a_0, a_1, a_2, a_3, \infty$. The exponents at the finite singularities are 0 and $\frac{1}{2}$. Using (14), we find that the exponents at infinity are $\frac{1}{4}$ and $\frac{3}{4}$. So all five singularities are elementary in the sense of Ince.[5] Equation (26) is one of the standard equations in the classification of Ince (see p. 500 of Ref. 5).

We define the 5-cyclide coordinates s_1, s_2, s_3 for an arbitrary point $(x, y, z) \in \mathbf{R}^3$ as the zeros $s_1 \le s_2 \le s_3$ of the cubic equation

$$\prod_{j=0}^{3}(s-a_j)\left[\frac{(x^2+y^2+z^2-1)^2}{s-a_0} + \frac{4x^2}{s-a_1} + \frac{4y^2}{s-a_2} + \frac{4z^2}{s-a_3}\right] = 0. \tag{28}$$

For example, $s_j(0, 0, 0) = a_j$ for $j = 1, 2, 3$. Each function s_j: $\mathbf{R}^3 \to [a_{j-1}, a_j]$ is continuous. We observe that, in general, there are 16 different points in $\mathbf{R}^3$ which have the same coordinates s_1, s_2, s_3. If (x, y, z) is one of these points, the other ones are obtained by applying the group generated by inversion at $\mathbb{S}^2$

$$\sigma_0(x, y, z) = \rho^{-2}(x, y, z) \tag{29}$$

and reflections at the coordinate planes

$$\sigma_1(x, y, z) = (-x, y, z),\ \sigma_2(x, y, z) = (x, -y, z),\ \sigma_3(x, y, z) = (x, y, -z). \tag{30}$$

It is of interest to determine the sets where $s_j = a_{j-1}$ or $s_j = a_j$. We obtain

$$s_1 = a_0 \text{ iff } x^2 + y^2 + z^2 = 1, \tag{31}$$

$$s_1 = a_1 \text{ iff } x = 0 \text{ and } \frac{(\rho^2-1)^2}{a_1-a_0} + \frac{4y^2}{a_1-a_2} + \frac{4z^2}{a_1-a_3} \geq 0, \tag{32}$$

$$s_2 = a_1 \text{ iff } x = 0 \text{ and } \frac{(\rho^2-1)^2}{a_1-a_0} + \frac{4y^2}{a_1-a_2} + \frac{4z^2}{a_1-a_3} \leq 0, \tag{33}$$

$$s_2 = a_2 \text{ iff } y = 0 \text{ and } \frac{(\rho^2-1)^2}{a_2-a_0} + \frac{4x^2}{a_2-a_1} + \frac{4z^2}{a_2-a_3} \geq 0, \tag{34}$$

$$s_3 = a_2 \text{ iff } y = 0 \text{ and } \frac{(\rho^2-1)^2}{a_2-a_0} + \frac{4x^2}{a_2-a_1} + \frac{4z^2}{a_2-a_3} \leq 0, \tag{35}$$

$$s_3 = a_3 \text{ iff } z = 0. \tag{36}$$

We define the sets (consisting each of two closed curves)

$$\begin{aligned} A_1 &:= \{(x, y, z) \in \mathbf{R}^3 : s_1 = s_2 = a_1\} \\ &= \{(x, y, z) : x = 0, \frac{(\rho^2-1)^2}{a_1-a_0} + \frac{4y^2}{a_1-a_2} + \frac{4z^2}{a_1-a_3} = 0\}, \end{aligned} \tag{37}$$

see Figure 2, and

$$\begin{aligned} A_2 &:= \{(x, y, z) \in \mathbf{R}^3 : s_2 = s_3 = a_2\} \\ &= \{(x, y, z) : y = 0, \frac{(\rho^2-1)^2}{a_2-a_0} + \frac{4x^2}{a_2-a_1} + \frac{4z^2}{a_2-a_3} = 0\}, \end{aligned} \tag{38}$$

see Figure 3. Clearly, s_j is analytic at all points (x, y, z) at which s_j is a simple zero of the cubic equation (28). Therefore, s_1 is analytic on $\mathbf{R}^3\backslash A_1$, s_2 is analytic on $\mathbf{R}^3\backslash(A_1\cup A_2)$, and s_3 is analytic on $\mathbf{R}^3\backslash A_2$.

We may use (27) to define $u(x, y, z)$ for all $(x, y, z) \in \mathbf{R}^3$. Since the solutions w_1, w_2, w_3 of (26) have limits at the end points of their intervals of definition (because the exponents are 0 and $\frac{1}{2}$ there), we see that u is a continuous function on $\mathbf{R}^3$. The function $(x^2 + y^2 + z^2 + 1)^{1/2}u(x, y, z)$

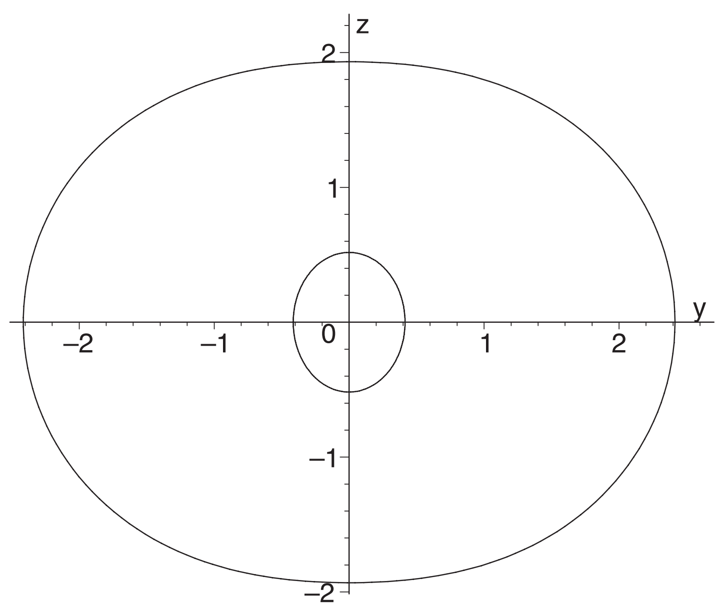

FIG. 2. The set A_1 in the y, z-plane for $a_i = i$.

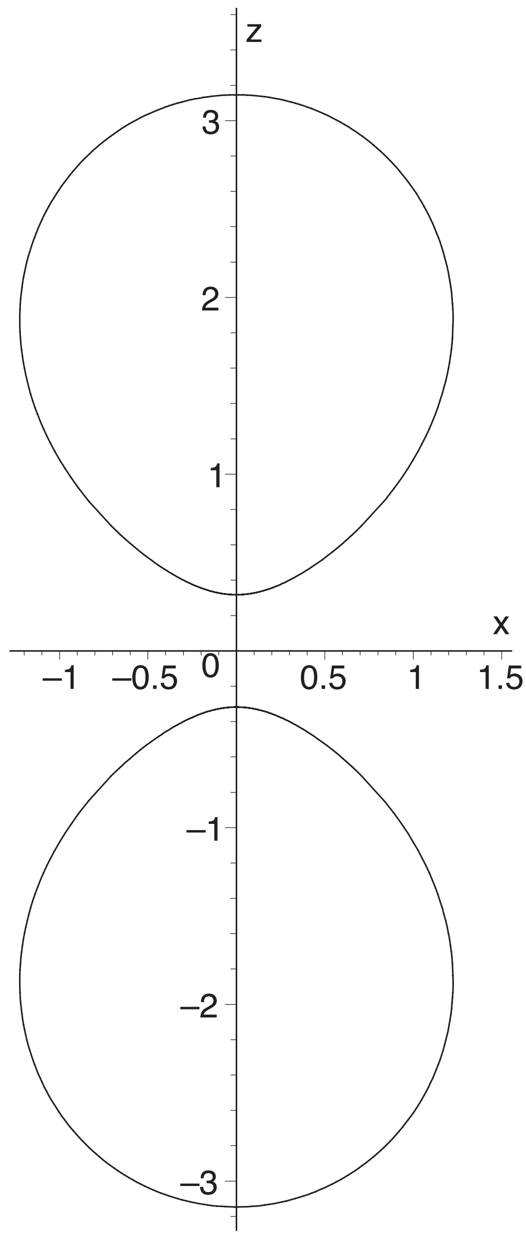

FIG. 3. The set A_2 in the x, z-plane for $a_i = i$.

is invariant under σ_i, $i = 0, 1, 2, 3$. In general, u is harmonic only away from the coordinate planes and the unit sphere. In fact, we observe that u is a bounded function which converges to 0 at infinity, so, by Liouville's theorem, u cannot be harmonic on all of $\mathbf{R}^3$ unless it is identically zero.

V. FIRST TWO-PARAMETER STURM-LIOUVILLE PROBLEM

We consider equation (26) on the intervals (a_1, a_2) and (a_2, a_3) and write it in formally self-adjoint form. Setting

$$\omega(s) := |(s - a_0)(s - a_1)(s - a_2)(s - a_3)|^{1/2}\,, \tag{39}$$

we obtain two Sturm-Liouville equations involving two parameters

$$(\omega(s_2)w_2')' + \frac{1}{\omega(s_2)}\left(\frac{3}{16}s_2^2 + \lambda_1 s_2 + \lambda_2\right)w_2 = 0, \quad a_1 < s_2 < a_2, \tag{40}$$

$$(\omega(s_3)w_3')' - \frac{1}{\omega(s_3)}\left(\frac{3}{16}s_3^2 + \lambda_1 s_3 + \lambda_2\right)w_3 = 0, \quad a_2 < s_3 < a_3. \tag{41}$$

In (40), w_2 is a function of s_2 and the derivatives are taken with respect to s_2. In (41), w_3 is a function of s_3 and the derivatives are taken with respect to s_3. We simplify the equations by substituting $t_j = \Omega(s_j)$, $u_j(t_j) = w_j(s_j)$, where $\Omega(s)$ is the elliptic integral (see, for instance, Ref. 8)

$$\Omega(s) := \int_{a_0}^{s} \frac{d\sigma}{\omega(\sigma)}. \tag{42}$$

This is an increasing absolutely continuous function Ω: $[a_0, a_3] \to [0, b_3]$, where $b_j := \Omega(a_j)$. Let ϕ: $[0, b_3] \to [a_0, a_3]$ be the inverse function of Ω. Then (40) and (41) become

$$u_2'' + \left(\frac{3}{16}\{\phi(t_2)\}^2 + \lambda_1\phi(t_2) + \lambda_2\right)u_2 = 0, \quad b_1 \le t_2 \le b_2, \tag{43}$$

$$u_3'' - \left(\frac{3}{16}\{\phi(t_3)\}^2 + \lambda_1\phi(t_3) + \lambda_2\right)u_3 = 0, \quad b_2 \le t_3 \le b_3. \tag{44}$$

We add the boundary conditions

$$u_2'(b_1) = u_2'(b_2) = u_3'(b_2) = u_3'(b_3) = 0. \tag{45}$$

Differential equations (43) and (44) together with boundary conditions (45) pose a two-parameter Sturm-Liouville eigenvalue problem. For the theory of such multiparameter problems, we refer to the studies[1,13] and the references therein. A pair (λ_1, λ_2) is called an *eigenvalue* if there exist (nontrivial) *eigenfunctions* $u_2(t_2)$ and $u_3(t_3)$ which satisfy (43)–(45). The two-parameter problem is *right-definite* in the sense that

$$\begin{vmatrix} \phi(t_2) & 1 \\ -\phi(t_3) & -1 \end{vmatrix} = \phi(t_3) - \phi(t_2) > 0 \quad \text{for } b_1 < t_2 < b_2 < t_3 < b_3.$$

However, this determinant is not positive on the closed rectangle $[b_1, b_2] \times [b_2, b_3]$. This lack of *uniform right-definiteness* make some proofs in this section a little longer than they would be otherwise.

We have the following Klein oscillation theorem (see Theorem 5.5.1 of Ref. 1).

Theorem 5.1. *For every $\mathbf{n} = (n_2, n_3) \in \mathbf{N}_0^2$, there exists a uniquely determined eigenvalue $(\lambda_{1,\mathbf{n}}, \lambda_{2,\mathbf{n}}) \in \mathbf{R}^2$ admitting an eigenfunction u_2 with exactly n_2 zeros in (b_1, b_2) and an eigenfunction u_3 with exactly n_3 zeros in (b_2, b_3).*

We state a result on the distribution of eigenvalues (compare with Chap. 8 of Ref. 1).

Theorem 5.2. *There are positive constants A_1, A_2, A_3, A_4 such that, for all $\mathbf{n} \in \mathbf{N}_0^2$,*

$$-A_1(n_2^2 + n_3^2 + 1) \le \lambda_{1,\mathbf{n}} \le -A_2(n_2^2 + n_3^2) + A_3, \tag{46}$$

$$|\lambda_{2,\mathbf{n}}| \le A_4(n_2^2 + n_3^2 + 1). \tag{47}$$

Proof. If a differential equation $u'' + q(t)u = 0$ with continuous q: $[a, b] \to \mathbf{R}$ admits a solution u satisfying $u'(a) = u'(b) = 0$ and having exactly m zeros in (a, b), then there is $t \in (a, b)$ such that $q(t) = \frac{\pi^2 m^2}{(b-a)^2}$. This is shown by comparing with the eigenvalue problem $u'' + \lambda u = 0$,

$u'(a) = u'(b) = 0$. Applying this fact, we find $t_2 \in (b_1, b_2)$ and $t_3 \in (b_2, b_3)$ such that

$$\frac{3}{16}\{\phi(t_2)\}^2 + \lambda_1\phi(t_2) + \lambda_2 = \frac{\pi^2 n_2^2}{(b_2 - b_1)^2}, \tag{48}$$

$$\frac{3}{16}\{\phi(t_3)\}^2 + \lambda_1\phi(t_3) + \lambda_2 = -\frac{\pi^2 n_3^2}{(b_3 - b_2)^2}, \tag{49}$$

where we abbreviated $\lambda_j = \lambda_{j,\mathbf{n}}$. By subtracting (48) from (49), we obtain

$$\frac{3}{16}\left(\{\phi(t_3)\}^2 - \{\phi(t_2)\}^2\right) + \lambda_1(\phi(t_3) - \phi(t_2)) = -\frac{\pi^2 n_2^2}{(b_2 - b_1)^2} - \frac{\pi^2 n_3^2}{(b_3 - b_2)^2} \le 0.$$

Dividing by $\phi(t_3) - \phi(t_2)$ and using $0 < \phi(t_3) - \phi(t_2) \le a_3 - a_1$, we obtain the second inequality in (46).

To prove the first inequality in (46), suppose that $\lambda_1 < -\frac{3}{8}a_3$. Then the van Vleck polynomial

$$Q(s) := \frac{3}{16}s^2 + \lambda_1 s + \lambda_2 \tag{50}$$

satisfies $Q'(s) = \frac{3}{8}s + \lambda_1 < 0$ for $s \le a_3$. Let $c \in (b_1, b_2)$ be determined by $\phi(c) = \frac{1}{2}(a_1 + a_2)$. If $Q(a_2) \ge 0$ then, for $t \in [b_1, c]$,

$$Q(\phi(t)) \ge Q(\frac{1}{2}(a_1 + a_2)) \ge \frac{1}{2}(a_2 - a_1)\left(-\lambda_1 - \frac{3}{8}a_3\right).$$

By Sturm's comparison theorem applied to Eq. (43), we get

$$(c - b_1)^2(a_2 - a_1)\left(-\lambda_1 - \frac{3}{8}a_3\right) \le 4\pi^2(n_2 + 1)^2,$$

which gives the desired inequality. If $Q(a_2) < 0$, we argue similarly working with (44) instead.

Finally, (47) follows from (46) and (48). □

Let $u_{2,\mathbf{n}}$ and $u_{3,\mathbf{n}}$ denote real-valued eigenfunctions corresponding to the eigenvalue $(\lambda_{1,\mathbf{n}}, \lambda_{2,\mathbf{n}})$. It is known (see Sec. 3.5 of Ref. 1) (and easy to prove) that the system of products $u_{2,\mathbf{n}}(t_2)u_{3,\mathbf{n}}(t_3)$, $\mathbf{n} \in \mathbf{N}_0^2$, is orthogonal in the Hilbert space H_1 consisting of measurable functions $f : (b_1, b_2) \times (b_2, b_3) \to \mathbf{C}$ satisfying

$$\int_{b_2}^{b_3}\int_{b_1}^{b_2}(\phi(t_3) - \phi(t_2))\,|f(t_2, t_3)|^2\, dt_2\, dt_3 < \infty$$

with inner product

$$\int_{b_2}^{b_3}\int_{b_1}^{b_2}(\phi(t_3) - \phi(t_2))\,f(t_2, t_3)\overline{g(t_2, t_3)}\, dt_2\, dt_3.$$

We normalize the eigenfunctions so that

$$\int_{b_2}^{b_3}\int_{b_1}^{b_2}(\phi(t_3) - \phi(t_2))\left\{u_{2,\mathbf{n}}(t_2)\right\}^2\left\{u_{3,\mathbf{n}}(t_3)\right\}^2\, dt_2\, dt_3 = 1. \tag{51}$$

We have the following completeness theorem (see Theorem 6.8.3 of Ref. 13).

Theorem 5.3. *The double sequence of functions*

$$u_{2,\mathbf{n}}(t_2)u_{3,\mathbf{n}}(t_3), \quad \mathbf{n} \in \mathbf{N}_0^2,$$

forms an orthonormal basis in the Hilbert space H_1.

The normalization (51) leads to a bound on the values of eigenfunctions.

Theorem 5.4. *There is a constant $B > 0$ such that, for all $\mathbf{n} \in \mathbf{N}_0^2$ and all $t_2 \in [b_1, b_2]$, $t_3 \in [b_2, b_3]$,*

$$|u_{2,\mathbf{n}}(t_2) u_{3,\mathbf{n}}(t_3)| \leq B(n_2^2 + n_3^2 + 1).$$

Proof. We abbreviate $u_j = u_{j,\mathbf{n}}$, $\lambda_j = \lambda_{j,\mathbf{n}}$. Condition (51) is a normalization for the product $u_2(t_2)u_3(t_3)$ but not for each factor separately, so we may assume that, additionally,

$$\int_{b_1}^{b_2} \{u_2(t_2)\}^2 \, dt_2 = 1. \tag{52}$$

Now (51) and (52) imply that

$$\int_{b_2}^{b_3} (\phi(t_3) - \phi(b_2)) \{u_3(t_3)\}^2 \, dt_3 \leq 1. \tag{53}$$

We multiply Eqs. (43) and (44) by u_2 and u_3, respectively, and integrate by parts to obtain

$$\int_{b_1}^{b_2} u_2'^2 = \frac{3}{16} \int_{b_1}^{b_2} \phi^2 u_2^2 + \lambda_1 \int_{b_1}^{b_2} \phi u_2^2 + \lambda_2 \int_{b_1}^{b_2} u_2^2, \tag{54}$$

$$\int_{b_2}^{b_3} u_3'^2 = -\frac{3}{16} \int_{b_2}^{b_3} \phi^2 u_3^2 - \lambda_1 \int_{b_2}^{b_3} \phi u_3^2 - \lambda_2 \int_{b_2}^{b_3} u_3^2. \tag{55}$$

It follows from (52) and (54) and Theorem 5.2 that there is a constant $B_1 > 0$ such that, for all $\mathbf{n} \in \mathbf{N}_0^2$,

$$\int_{b_1}^{b_2} u_2'^2 \leq B_1(n_2^2 + n_3^2 + 1). \tag{56}$$

Unfortunately, we cannot argue the same way for u_3 because we do not have an upper bound for $\int_{b_2}^{b_3} u_3^2$. Instead, we multiply (54) by $\int u_3^2$ and (55) by $\int u_2^2$ and add the equations. Then, noting (51), we find

$$\int_{b_1}^{b_2} u_2'^2 \int_{b_2}^{b_3} u_3^2 + \int_{b_1}^{b_2} u_2^2 \int_{b_2}^{b_3} u_3'^2 \leq -\lambda_1 + \frac{3}{8} \max_{t \in [b_1, b_3]} |\phi(t)|.$$

Using Theorem 5.2 and (52), we find a constant $B_2 > 0$ such that, for all $\mathbf{n} \in \mathbf{N}_0^2$,

$$\int_{b_2}^{b_3} u_3'^2 \leq B_2(n_2^2 + n_3^2 + 1). \tag{57}$$

We apply the following Lemma 5.5 (noting (52), (53), (56), (57)) and obtain the desired result. □

Lemma 5.5. Let u: $[a, b] \to \mathbf{R}$ be a continuously differentiable function, and let $a \leq c < d \leq b$. Then, for all $t \in [a, b]$,

$$(d - c) |u(t)|^2 \leq 2 \int_c^d |u(r)|^2 \, dr + 2(b - a)(d - c) \int_a^b \left|u'(r)\right|^2 \, dr.$$

Proof. For $s, t \in [a, b]$, we have

$$|u(t) - u(s)| = \left| \int_s^t u'(r) \, dr \right| \leq |t - s|^{1/2} \left(\int_a^b \left|u'(r)\right|^2 \, dr \right)^{1/2}.$$

This implies

$$|u(t)|^2 \leq 2|u(s)|^2 + 2|t - s| \int_a^b \left|u'(r)\right|^2 \, dr.$$

We integrate from $s = c$ to $s = d$ and obtain the desired inequality. □

Let $u_{1,\mathbf{n}}$ be the solution of

$$u_1'' - \left(\frac{3}{16}\{\phi(t_1)\}^2 + \lambda_{1,\mathbf{n}}\phi(t_1) + \lambda_{2,\mathbf{n}}\right)u_1 = 0, \quad b_0 \le t_1 \le b_1, \tag{58}$$

determined by the initial conditions

$$u_1(b_1) = 1, \quad u_1'(b_1) = 0.$$

The following estimate on $u_{1,\mathbf{n}}$ will be useful in Sec. VI.

Theorem 5.6. *We have* $u_{1,\mathbf{n}}(t_1) > 0$ *for all* $t_1 \in [b_0, b_1]$. *If* $0 = b_0 \le c_1 < c_2 < b_1$, *then there are constants* $C > 0$ *and* $0 < r < 1$ *such that, for all* $\mathbf{n} \in \mathbf{N}_0^2$ *and* $t_1 \in [c_2, b_1]$,

$$\frac{u_{1,\mathbf{n}}(t_1)}{u_{1,\mathbf{n}}(c_1)} \le C r^{n_2+n_3}.$$

Proof. We abbreviate $u_1 = u_{1,\mathbf{n}}$ and $\lambda_j = \lambda_{j,\mathbf{n}}$. By definition, u_1 satisfies the differential equation

$$u_1'' = Q(\phi(t_1))u_1, \quad t_1 \in [b_0, b_1],$$

where Q is given by (50). According to (48) and (49), there are $s_2 \in (a_1, a_2)$ and $s_3 \in (a_2, a_3)$ such that

$$Q(s_2) = \frac{\pi^2 n_2^2}{(b_2 - b_1)^2}, \quad Q(s_3) = -\frac{\pi^2 n_3^2}{(b_3 - b_2)^2}.$$

If $s \le s_2$ then $Q(s) \ge L(s)$, where $L(s)$ is the linear function with $L(s_j) = Q(s_j)$, $j = 2, 3$. It follows that $Q(s) \ge 0$ for $s \in [a_0, a_1]$ and

$$Q(\phi(t_1)) \ge C_1(n_2 + n_3)^2 \quad \text{for } t_1 \in [b_0, c_2], \tag{59}$$

where C_1 is a positive constant independent of $\mathbf{n}$. We now apply the following Lemma 5.7 (with $a = c_1$, $b = b_1$, $c = c_2$) to complete the proof. □

Lemma 5.7. Let u: $[a, b] \to \mathbf{R}$ *be a solution of the differential equation*

$$u''(t) = q(t)u(t), \quad t \in [a, b],$$

determined by the initial conditions $u(b) = 1$, $u'(b) = 0$, *where* q: $[a, b] \to \mathbf{R}$ *is a continuous function. Suppose that* $q(t) \ge 0$ *on* $[a, b]$ *and* $q(t) \ge \lambda^2$ *on* $[a, c]$ *for some* $\lambda > 0$ *and* $c \in (a, b]$. *Then* $u(t) > 0$ *for all* $t \in [a, b]$, *and*

$$\frac{u(t)}{u(a)} \le 2e^{-\lambda(c-a)} \text{ for all } t \in [c, b].$$

Proof. Since $q(t) \ge 0$, $u(t) > 0$ and $u'(t) \le 0$ for $t \in [a, b]$. The function $z = u'/u$ satisfies the Riccati equation

$$z' + z^2 = q(t),$$

and the initial condition $z(b) = 0$. It follows that

$$z(t) \le \lambda \tanh(\lambda(t - c)) \quad \text{for } t \in [a, c].$$

Integrating from $t = a$ to $t = c$ gives

$$\ln \frac{u(c)}{u(a)} \le -\ln\cosh \lambda(c - a) \le \ln(2e^{-\lambda(c-a)})$$

which yields the claim since u is nonincreasing. □

We now introduce a systematic notation for our eigenvalues and eigenfunctions. First of all, we note that the results of this section remain valid for other sets of boundary conditions. We will need eight sets of boundary conditions labeled by $\mathbf{p} = (p_1, p_2, p_3) \in \{0, 1\}^3$. These boundary conditions are

$$\begin{aligned} &u_2'(b_1) = 0 && \text{if } p_1 = 0, \quad && u_2(b_1) = 0 && \text{if } p_1 = 1,\\ &u_2'(b_2) = u_3'(b_2) = 0 && \text{if } p_2 = 0, && u_2(b_2) = u_3(b_2) = 0 && \text{if } p_2 = 1,\\ &u_3'(b_3) = 0 && \text{if } p_3 = 0, && u_3(b_3) = 0 && \text{if } p_3 = 1. \end{aligned} \tag{60}$$

The initial conditions for u_1 are

$$u_1(b_1) = 1, u_1'(b_1) = 0 \ \text{ if } p_1 = 0, \quad u_1(b_1) = 0, u_1'(b_1) = 1 \ \text{ if } p_1 = 1. \tag{61}$$

We denote the corresponding eigenvalues by $(\lambda^{(1)}_{1,\mathbf{n},\mathbf{p}}, \lambda^{(1)}_{2,\mathbf{n},\mathbf{p}})$. For the notation of eigenfunctions, we return to the s_i-variable connected to t_i by $t_i = \Omega(s_i)$. The eigenfunctions will be denoted by $E^{(1)}_{i,\mathbf{n},\mathbf{p}}(s_i) = u_{i,\mathbf{n}}(t_i)$, $i = 1, 2, 3$. The superscript (1) is used to distinguish from eigenvalues and eigenfunctions introduced in Secs. VII and IX. The subscript $\mathbf{n} = (n_2, n_3)$ indicates the number of zeros of $E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2)$, $E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ in (a_1, a_2), (a_2, a_3), respectively. The subscript $\mathbf{p}$ indicates the boundary conditions used to determine eigenvalues and eigenfunctions. By using the letter E for eigenfunctions, we follow Bôcher.[2] In our notation, we suppressed the dependence of eigenvalues and eigenfunctions on a_0, a_1, a_2, a_3.

Summarizing, for $i = 1, 2, 3$, $E^{(1)}_{i,\mathbf{n},\mathbf{p}}$ is a solution of (26) on (a_{i-1}, a_i) with $(\lambda_1, \lambda_2) = (\lambda^{(1)}_{1,\mathbf{n},\mathbf{p}}, \lambda^{(2)}_{2,\mathbf{n},\mathbf{p}})$. The solution $E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1)$ has exponent $\frac{1}{2}p_1$ at a_1 and it has no zeros in (a_0, a_1). The solution $E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2)$ has exponent $\frac{1}{2}p_1$ at a_1, exponent $\frac{1}{2}p_2$ at a_2, and its has n_2 zeros in (a_1, a_2). The solution $E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ has exponent $\frac{1}{2}p_2$ at a_2, exponent $\frac{1}{2}p_3$ at a_3, and it has n_3 zeros in (a_2, a_3).

VI. FIRST DIRICHLET PROBLEM

Consider the coordinate surface (21) for fixed $s = d_1 \in (a_0, a_1)$. See Figure 4 for a graphical depiction of the shape of this surface. Let $(x', y', z') \in \mathbb{S}^2$. The ray $(x, y, z) = t(x', y', z')$, $t > 0$, intersects the surface if

$$\frac{(t^2-1)^2}{d_1 - a_0} = ct^2, \tag{62}$$

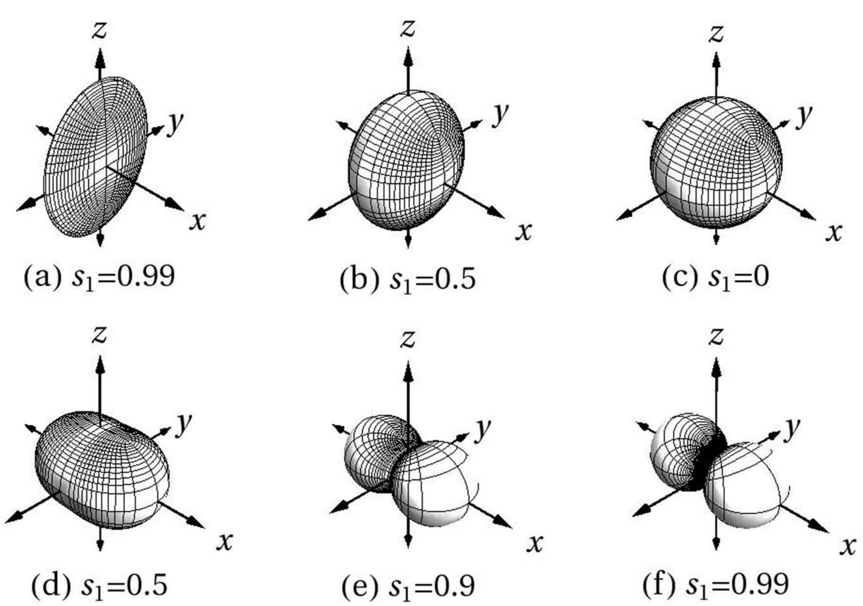

FIG. 4. Coordinate surfaces $s_1 = const$ for $a_i = i$ with (a), (b) inside $B_1(\mathbf{0})$; (d)–(f) outside $B_1(\mathbf{0})$; and (c) the unit sphere.

where

$$c = \frac{4x'^2}{a_1 - d_1} + \frac{4y'^2}{a_2 - d_1} + \frac{4z'^2}{a_3 - d_1} > 0.$$

Equation (62) has two positive solutions $t = t_1, t_2$ such that $0 < t_1 < 1 < t_2$ and $t_1 t_2 = 1$. Therefore, the coordinate surface $s_1 = d_1$ consists of two disjoint closed surfaces of genus zero. One lies inside the unit ball $B_1(\mathbf{0})$ centered at the origin and the other one is the image of it under the inversion (29). Let D_1 be the region interior to the first surface, that is,

$$D_1 = \{(x, y, z) \in B_1(\mathbf{0}) : s_1 > d_1\}, \tag{63}$$

or, equivalently,

$$D_1 = \{(x, y, z) \in B_1(\mathbf{0}) : \frac{(\rho^2 - 1)^2}{d_1 - a_0} + \frac{4x^2}{d_1 - a_1} + \frac{4y^2}{d_1 - a_2} + \frac{4z^2}{d_1 - a_3} > 0\}.$$

We showed that D_1 is star-shaped with respect to the origin. We now solve the Dirichlet problem for harmonic functions in D_1 by the method of separation of variables.

Let $\mathbf{p} = (p_1, p_2, p_3) \in \{0, 1\}^3$ and $\mathbf{n} = (n_2, n_3) \in \mathbf{N}_0^2$. Using the functions $E^{(1)}_{i,\mathbf{n},\mathbf{p}}$ introduced in Sec. V, we define the *internal 5-cyclidic harmonic of the first kind*

$$G^{(1)}_{\mathbf{n},\mathbf{p}}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3) \tag{64}$$

for $x, y, z \in B_1(\mathbf{0})$ with $x, y, z \geq 0$. We extend this function to $B_1(\mathbf{0})$ as a function of parity $\mathbf{p}$. We call a function f of parity $\mathbf{p}$ if

$$f(\sigma_i(x, y, z)) = (-1)^{p_i} f(x, y, z), \qquad \text{for } i = 1, 2, 3 \tag{65}$$

using the reflections (30).

Lemma 6.1. The function $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is harmonic on $B_1(\mathbf{0})$.

Proof. By Theorem 4.1, $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is harmonic on $B_1(\mathbf{0})$ away from the coordinate planes. Therefore, it is enough to show that $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is analytic on $B_1(\mathbf{0})$.

Consider first $\mathbf{p} = (0, 0, 0)$. Then (64) holds on $B_1(\mathbf{0})$. Since $s_1 \neq a_0$ on $B_1(\mathbf{0})$, $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is analytic on $B_1(\mathbf{0}) \setminus (A_1 \cup A_2)$ as a composition of analytic functions. In order to show that $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is also analytic at the points of $A_1 \cup A_2$, one may refer to a classical result on "singular curves" of harmonic functions (see Theorem XIII, p. 271 of Ref. 6), but we will argue more directly. Since A_1 and A_2 are disjoint sets, it is clear that $E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ is analytic at every point in $B_1(\mathbf{0}) \cap A_1$. In order to show that $E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2)$ is analytic at $(x', y', z') \in B_1(\mathbf{0}) \cap A_1$, we argue as follows. We may assume that there is an analytic function $w : (a_1, a_3) \to \mathbf{R}$ such that $E^{(1)}_{1,\mathbf{n},\mathbf{p}}$ and $E^{(1)}_{2,\mathbf{n},\mathbf{p}}$ are restrictions of this function to (a_1, a_2) and (a_2, a_3), respectively. Now $(s_1 - a_1) + (s_2 - a_1)$ and $(s_1 - a_1)(s_2 - a_1)$ are analytic functions of (x, y, z) in a neighborhood of (x', y', z'). Lemma 6.2 implies that $E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2)$ as a function of (x, y, z) is analytic at (x', y', z'). It follows that $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is analytic at every point in $B_1(\mathbf{0}) \cap A_1$. In the same way, we show that $G^{(1)}_{\mathbf{n},\mathbf{p}}$ is analytic at every point in $B_1(\mathbf{0}) \cap A_2$.

If $\mathbf{p} = (0, 0, 1)$, then we introduce the function

$$\chi := \begin{cases} \sqrt{a_3 - s_3} & \text{if } z \geq 0 \\ -\sqrt{a_3 - s_3} & \text{otherwise.} \end{cases}$$

It follows from (19) and (20) that χ is analytic on $\mathbf{R}^3 \setminus A_2$. Then

$$G^{(1)}_{\mathbf{n},\mathbf{p}}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) \chi(x, y, z) w_3(s_3)$$

on $B_1(\mathbf{0})$, where w_3 is analytic at $s_3 = a_3$. We then argue as above.

The other parity vectors $\mathbf{p}$ are treated similarly. □

Lemma 6.2. Let $f : (B_\epsilon)^2 \to \mathbf{C}$, $B_\epsilon = \{s \in \mathbf{C} : |s| < \epsilon\}$, be an analytic function which is symmetric: $f(s, t) = f(t, s)$. Let g, h: $(B_\delta)^3 \to B_\epsilon$ be functions such that $g + h$ and gh are analytic. Then the function $f(g(x, y, z), h(x, y, z))$ is analytic on $(B_\delta)^3$.

Substituting $t_j = \Omega(s_j)$, $j = 2, 3$, the Hilbert space H_1 from Sec. V transforms to the Hilbert space $\tilde{H}_1$ consisting of measurable functions $g : (a_1, a_2) \times (a_2, a_3) \to \mathbf{C}$ for which

$$\|g\|^2 := \int_{a_2}^{a_3} \int_{a_1}^{a_2} \frac{s_3 - s_2}{\omega(s_2)\omega(s_3)} |g(s_2, s_3)|^2 \, ds_2 \, ds_3 < \infty. \tag{66}$$

By Theorem 5.3, for $g \in \tilde{H}_1$ and fixed $\mathbf{p}$, we have the Fourier expansion

$$g(s_2, s_3) \sim \sum_{\mathbf{n}} c_{\mathbf{n},\mathbf{p}} E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3), \tag{67}$$

where the Fourier coefficients are given by

$$c_{\mathbf{n},\mathbf{p}} = \int_{a_2}^{a_3} \int_{a_1}^{a_2} \frac{s_3 - s_2}{\omega(s_2)\omega(s_3)} g(s_2, s_3) E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3) \, ds_2 \, ds_3. \tag{68}$$

We are now ready to solve the Dirichlet problem $\Delta u = 0$ in D_1, $u = e$ on ∂D_1 for D_1 given in (63). We will interpret the equation $u = e$ on ∂D_1 in the following weak sense: if u and e are expressed in terms of 5-cyclide coordinates s_1, s_2, s_3, and s_2, s_3, respectively, then u evaluated at $s_1 \in (d_1, a_1)$ converges to e in the Hilbert space $\tilde{H}_1$ as $s_1 \to d_1$.

Theorem 6.3. *Consider the region D_1 defined by (63) for some fixed $d_1 \in (a_0, a_1)$. Let e be a function defined on its boundary ∂D_1 of parity $\mathbf{p} \in \{0, 1\}^3$, and let $g(s_2, s_3)$ be the representation of*

$$f(x, y, z) := (x^2 + y^2 + z^2 + 1)^{1/2} e(x, y, z) \tag{69}$$

in 5-cyclide coordinates for $(x, y, z) \in \partial D_1$ with $x, y, z > 0$. Suppose $g \in \tilde{H}_1$ and expand g in the series (67). Then the function $u(x, y, z)$ given by

$$u(x, y, z) = \sum_{\mathbf{n}} \frac{c_{\mathbf{n},\mathbf{p}}}{E^{(1)}_{1,\mathbf{n},\mathbf{p}}(d_1)} G^{(1)}_{\mathbf{n},\mathbf{p}}(x, y, z) \tag{70}$$

is harmonic in D_1 and assumes the values e on the boundary of D_1 in the weak sense.

Proof. Let $d_1 < d < a_1$ and $s_1 \in [d, a_1]$. Using Theorems 5.4, 5.6, we estimate

$$\left| c_{\mathbf{n},\mathbf{p}} \frac{E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1)}{E^{(1)}_{1,\mathbf{n},\mathbf{p}}(d_1)} E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3) \right| \le |c_{\mathbf{n},\mathbf{p}}| C r^{n_2+n_3} B(n_2^2 + n_3^2 + 1),$$

where the constants $B, C > 0$ and $r \in (0, 1)$ are independent of $\mathbf{n}$ and $s_1 \in [d, a_1]$, $s_2 \in [a_1, a_2]$, $s_3 \in [a_2, a_3]$. Since $c_{\mathbf{n},\mathbf{p}}$ is a bounded double sequence, this proves that the series in (70) is absolutely and uniformly convergent on compact subsets of D_1. Consequently, by Lemma 6.1, $u(x, y, z)$ is harmonic in D_1. If we consider u for fixed $s_1 \in (d_1, a_1)$ and compute the norm $\|u - e\|$ in the Hilbert space $\tilde{H}_1$ by the Parseval equality, we obtain

$$\|u - e\|^2 \le \sum_{\mathbf{n}} |c_{\mathbf{n},\mathbf{p}}|^2 \left(1 - \frac{E^{(1)}_{1,\mathbf{n},\mathbf{p}}(s_1)}{E^{(1)}_{1,\mathbf{n},\mathbf{p}}(d_1)} \right)^2.$$

It is easy to see that the right-hand side converges to 0 as $s_1 \to d_1$. Taking into account that e and u have the same parity, it follows that u assumes the boundary values e in the weak sense. □

If e is a function on ∂D_1 without parity, we write the function f from (69) as a sum of eight functions

$$f = \sum_{\mathbf{p}} f_{\mathbf{p}},$$

where $f_{\mathbf{p}}$ is of parity $\mathbf{p}$. Then the solution of the corresponding Dirichlet problem is given by

$$u(x,y,z)=\sum_{\mathbf{n},\mathbf{p}}\frac{c_{\mathbf{n},\mathbf{p}}}{E^{(1)}_{1,\mathbf{n},\mathbf{p}}(d_1)}G^{(1)}_{\mathbf{n},\mathbf{p}}(x,y,z), \tag{71}$$

where

$$c_{\mathbf{n},\mathbf{p}}=\int_{a_2}^{a_3}\int_{a_1}^{a_2}\frac{s_3-s_2}{\omega(s_2)\omega(s_3)}g_{\mathbf{p}}(s_2,s_3)E^{(1)}_{2,\mathbf{n},\mathbf{p}}(s_2)E^{(1)}_{3,\mathbf{n},\mathbf{p}}(s_3)\,ds_2\,ds_3 \tag{72}$$

and $g_{\mathbf{p}}(s_2,s_3)$ is the representation of $f_{\mathbf{p}}$ in 5-cyclide coordinates.

We may write the coefficient $c_{\mathbf{n},\mathbf{p}}$ as an integral over the surface ∂D_1 itself. The surface element is $dS=h_2h_3\,ds_2\,ds_3$ with the scale factors h_2, h_3 given in (24) and (25). Using

$$\frac{h_2h_3}{h_1}=\frac{1}{4}(x^2+y^2+z^2+1)\frac{\omega(s_1)}{\omega(s_2)\omega(s_3)}(s_3-s_2),$$

we obtain from (72)

$$c_{\mathbf{n},\mathbf{p}}=\frac{1}{2\omega(d_1)E^{(1)}_{1,\mathbf{n},\mathbf{p}}(d_1)}\int_{\partial D_1}\frac{e}{h_1}G^{(1)}_{\mathbf{n},\mathbf{p}}\,dS, \tag{73}$$

where

$$h_1^2=\frac{1}{16}\left(\frac{(x^2+y^2+z^2-1)^2}{(d_1-a_0)^2}+\frac{4x^2}{(d_1-a_1)^2}+\frac{4y^2}{(d_1-a_2)^2}+\frac{4z^2}{(d_1-a_3)^2}\right).$$

VII. SECOND TWO-PARAMETER STURM-LIOUVILLE PROBLEM

We treat the two-parameter eigenvalue problem that appears when we wish to solve the Dirichlet problem in asymmetric ring cyclides. It is quite similar to the one considered in Sec. V; however, there are also some interesting differences. Consider Eq. (26) on the intervals (a_0,a_1) and (a_2,a_3). We obtain two Sturm-Liouville equations involving two parameters

$$(\omega(s_1)w_1')'-\frac{1}{\omega(s_1)}\left(\frac{3}{16}s_1^2+\lambda_1s_1+\lambda_2\right)w_1=0,\quad a_0<s_1<a_1, \tag{74}$$

$$(\omega(s_3)w_3')'-\frac{1}{\omega(s_3)}\left(\frac{3}{16}s_3^2+\lambda_1s_3+\lambda_2\right)w_3=0,\quad a_2<s_3<a_3. \tag{75}$$

We again simplify by substituting $t_j=\Omega(s_j)$, $u_j(t_j)=w_j(s_j)$. Then (74) and (75) become

$$u_1''-\left(\frac{3}{16}\{\phi(t_1)\}^2+\lambda_1\phi(t_1)+\lambda_2\right)u_1=0,\quad b_0\le t_1\le b_1, \tag{76}$$

$$u_3''-\left(\frac{3}{16}\{\phi(t_3)\}^2+\lambda_1\phi(t_3)+\lambda_2\right)u_3=0,\quad b_2\le t_3\le b_3. \tag{77}$$

We add boundary conditions

$$u_1'(b_0)=u_1'(b_1)=u_3'(b_2)=u_3'(b_3)=0. \tag{78}$$

Differential equations (76) and (77) together with boundary conditions (78) pose a two-parameter Sturm-Liouville eigenvalue problem. In contrast to Sec. V, we now have a uniformly right-definite problem:

$$-\begin{vmatrix}\phi(t_1) & 1\\ \phi(t_3) & 1\end{vmatrix}=\phi(t_3)-\phi(t_1)\ge a_2-a_1>0\quad \text{for } b_0\le t_1\le b_1\le t_3\le b_3.$$

We again have Klein's oscillation theorem.

Theorem 7.1. *For every* $\mathbf{n} = (n_1, n_3) \in \mathbf{N}_0^2$, *there exists a uniquely determined eigenvalue* $(\lambda_{1,\mathbf{n}}, \lambda_{2,\mathbf{n}}) \in \mathbf{R}^2$ *admitting an eigenfunction* u_1 *with exactly* n_1 *zeros in* (b_0, b_1) *and an eigenfunction* u_3 *with exactly* n_3 *zeros in* (b_2, b_3).

We state a result on the distribution of eigenvalues.

Theorem 7.2. *There are constants* $A_1, A_2, A_3 > 0$ *such that, for all* $\mathbf{n} \in \mathbf{N}_0^2$,

$$-A_1(n_3^2 + 1) \le \lambda_{1,\mathbf{n}} \le A_2(n_1^2 + 1), \tag{79}$$

$$|\lambda_{2,\mathbf{n}}| \le A_3(n_1^2 + n_3^2 + 1). \tag{80}$$

Proof. We abbreviate $\lambda_j = \lambda_{j,\mathbf{n}}$. Arguing as in the proof of Theorem 5.2, there are $t_1 \in [b_0, b_1]$ and $t_3 \in [b_2, b_3]$ such that

$$\frac{3}{16}\{\phi(t_1)\}^2 + \lambda_1\phi(t_1) + \lambda_2 = -\frac{\pi^2 n_1^2}{(b_1 - b_0)^2}, \tag{81}$$

$$\frac{3}{16}\{\phi(t_3)\}^2 + \lambda_1\phi(t_3) + \lambda_2 = -\frac{\pi^2 n_3^2}{(b_3 - b_2)^2}. \tag{82}$$

By subtracting (81) from (82), we obtain

$$\frac{3}{16}\left(\{\phi(t_3)\}^2 - \{\phi(t_1)\}^2\right) + \lambda_1(\phi(t_3) - \phi(t_1)) = \frac{\pi^2 n_1^2}{(b_1 - b_0)^2} - \frac{\pi^2 n_3^2}{(b_3 - b_2)^2},$$

which implies (79). Now (80) follows from (79) and (81). □

Let $u_{1,\mathbf{n}}$ and $u_{3,\mathbf{n}}$ denote eigenfunctions corresponding to the eigenvalue $(\lambda_{1,\mathbf{n}}, \lambda_{2,\mathbf{n}})$. The system of products $u_{1,\mathbf{n}}(t_1)u_{3,\mathbf{n}}(t_3)$, $\mathbf{n} \in \mathbf{N}_0^2$, is orthogonal in the Hilbert space H_2 consisting of measurable functions $f : (b_0, b_1) \times (b_2, b_3) \to \mathbf{C}$ satisfying

$$\int_{b_2}^{b_3}\int_{b_0}^{b_1} (\phi(t_3) - \phi(t_1))\,|f(t_1, t_3)|^2\, dt_1\, dt_3 < \infty$$

with inner product

$$\int_{b_2}^{b_3}\int_{b_0}^{b_1} (\phi(t_3) - \phi(t_1)) f(t_1, t_3)\overline{g(t_1, t_3)}\, dt_1\, dt_3.$$

We normalize the eigenfunctions so that

$$\int_{b_2}^{b_3}\int_{b_0}^{b_1} (\phi(t_3) - \phi(t_1))\left\{u_{1,\mathbf{n}}(t_1)\right\}^2\left\{u_{3,\mathbf{n}}(t_3)\right\}^2\, dt_1\, dt_3 = 1. \tag{83}$$

We have the following completeness theorem.

Theorem 7.3. *The double sequence of functions*

$$u_{1,\mathbf{n}}(t_1)u_{3,\mathbf{n}}(t_3), \quad \mathbf{n} \in \mathbf{N}_0^2,$$

forms an orthonormal basis in the Hilbert space H_2.

The normalization (83) leads to a bound on the values of eigenfunctions. Since we have uniform right-definiteness, the proof is simpler than the proof of Theorem 5.4.

Theorem 7.4. *There is a constant $B > 0$ such that, for all $\mathbf{n} \in \mathbf{N}_0^2$ and all $t_1 \in [b_0, b_1]$, $t_3 \in [b_2, b_3]$,*

$$|u_{1,\mathbf{n}}(t_1)u_{3,\mathbf{n}}(t_3)| \le B(n_1^2 + n_3^2 + 1).$$

Let $u_{2,\mathbf{n}}$ be the solution of

$$u_2'' + \left(\frac{3}{16}\{\phi(t_2)\}^2 + \lambda_{1,\mathbf{n}}\phi(t_2) + \lambda_{2,\mathbf{n}}\right)u_2 = 0, \quad b_1 \le t_2 \le b_2 \tag{84}$$

determined by initial conditions

$$u_2(b_1) = 1, \quad u_2'(b_1) = 0.$$

Theorem 7.5. *We have $u_{2,\mathbf{n}}(t_2) > 0$ for all $t_2 \in [b_1, b_2]$. If $b_1 < c_1 < c_2 < b_2$, then there are constants $C > 0$ and $0 < r < 1$ such that, for all $\mathbf{n} \in \mathbf{N}_0^2$ and $t_2 \in [b_1, c_1]$,*

$$\left|\frac{u_{2,\mathbf{n}}(t_2)}{u_{2,\mathbf{n}}(c_2)}\right| \le Cr^{n_1+n_3}.$$

Proof. We abbreviate $u_2 = u_{2,\mathbf{n}}$ and $\lambda_j = \lambda_{j,\mathbf{n}}$. We write (84) in the form

$$u_2'' + Q(\phi(t_2))u_2 = 0, \quad t_2 \in [b_1, b_2],$$

where Q is given by (50). According to (81) and (82), there are $s_1 \in (a_0, a_1)$ and $s_3 \in (a_2, a_3)$ such that

$$Q(s_1) = -\frac{\pi^2 n_1^2}{(b_1 - b_0)^2}, \quad Q(s_3) = -\frac{\pi^2 n_3^2}{(b_3 - b_2)^2}.$$

If $s \in [s_1, s_3]$, then $Q(s) \le L(s)$, where $L(s)$ is the linear function with $L(s_j) = Q(s_j)$, $j = 1, 3$. It follows that

$$Q(\phi(t_2)) \le -C(n_1 + n_3)^2 \quad \text{for } t_2 \in [c_1, c_2]. \tag{85}$$

We use a modification of Lemma 5.7 to complete the proof. □

The results of this section remain valid for other boundary conditions. This time we will need sixteen sets of boundary conditions labeled by $\mathbf{p} = (p_0, p_1, p_2, p_3) \in \{0, 1\}^4$. These boundary conditions are

$$\begin{aligned}
u_1'(b_0) &= 0 \text{ if } p_0 = 0, & u_1(b_0) &= 0 \text{ if } p_0 = 1,\\
u_1'(b_1) &= 0 \text{ if } p_1 = 0, & u_1(b_1) &= 0 \text{ if } p_1 = 1,\\
u_3'(b_2) &= 0 \text{ if } p_2 = 0, & u_3(b_2) &= 0 \text{ if } p_2 = 1,\\
u_3'(b_3) &= 0 \text{ if } p_3 = 0, & u_3(b_3) &= 0 \text{ if } p_3 = 1.
\end{aligned} \tag{86}$$

The initial conditions for u_2 are

$$u_2(b_1) = 1, u_2'(b_1) = 0 \text{ if } p_1 = 0, \quad u_2(b_1) = 0, u_2'(b_1) = 1 \text{ if } p_1 = 1. \tag{87}$$

We denote the corresponding eigenvalues by $(\lambda_{1,\mathbf{n},\mathbf{p}}^{(2)}, \lambda_{2,\mathbf{n},\mathbf{p}}^{(2)})$. The eigenfunctions will be denoted by $E_{i,\mathbf{n},\mathbf{p}}^{(2)}(s_i) = u_{i,\mathbf{n}}(t_i)$, $i = 1, 2, 3$.

Summarizing, for $i = 1, 2, 3$, $E_{i,\mathbf{n},\mathbf{p}}^{(2)}$ is a solution of (26) on (a_{i-1}, a_i) with $(\lambda_1, \lambda_2) = (\lambda_{1,\mathbf{n},\mathbf{p}}^{(2)}, \lambda_{2,\mathbf{n},\mathbf{p}}^{(2)})$. The solution $E_{1,\mathbf{n},\mathbf{p}}^{(2)}(s_1)$ has exponent $\frac{1}{2}p_0$ at a_0, exponent $\frac{1}{2}p_1$ at a_1, and it has n_1 zeros in (a_0, a_1). The solution $E_{2,\mathbf{n},\mathbf{p}}^{(2)}(s_2)$ has exponent $\frac{1}{2}p_1$ at a_1, and its has no zeros in (a_1, a_2). The solution $E_{3,\mathbf{n},\mathbf{p}}^{(2)}(s_3)$ has exponent $\frac{1}{2}p_2$ at a_2, exponent $\frac{1}{2}p_3$ at a_3, and it has n_3 zeros in (a_2, a_3).

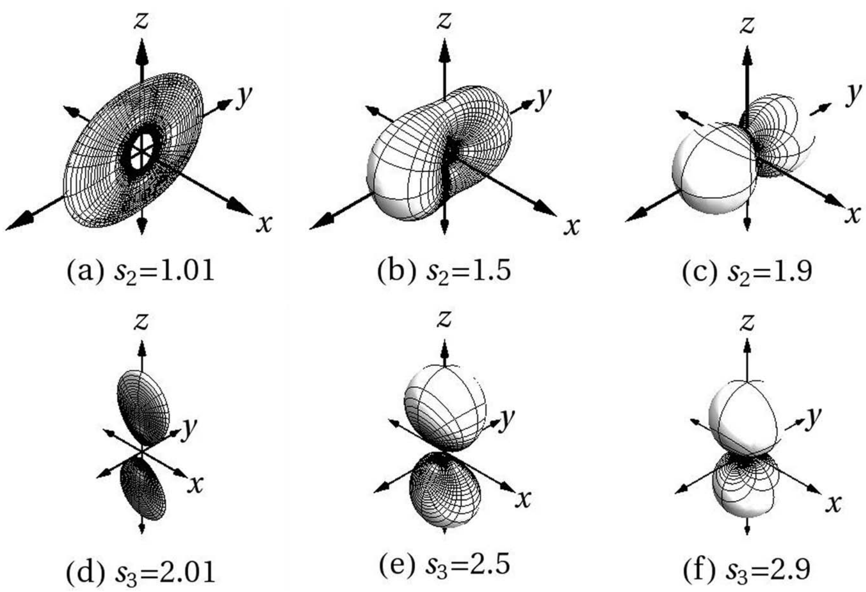

FIG. 5. Coordinate surfaces $s_{2,3} = const$ for $a_i = i$.

VIII. SECOND DIRICHLET PROBLEM

Consider the coordinate surface (21) for fixed $s = d_2 \in (a_1, a_2)$. See Figures 5(a)–5(c) for a graphical depiction of the shape of this surface. If $(x', y', z') \in \mathbb{S}_2$ then the ray $(x, y, z) = t(x', y', z')$, $t > 0$, is tangent to the surface if and only if (x', y', z') is on the surface. If (x', y', z') is in the elliptical cone

$$\frac{4x'^2}{d_2 - a_1} + \frac{4y'^2}{d_2 - a_2} + \frac{4z'^2}{d_2 - a_3} > 0,$$

then the ray does not intersect the surface. Otherwise we have two intersections $t = t_1, t_2$ and $t_1t_2 = 1$. It follows from these considerations that $s_2 = d_2$ describes a connected surface of genus one. The region interior to this surface is

$$D_2 = \{(x, y, z) \in \mathbf{R}^3 : s_2 < d_2\}, \tag{88}$$

or, equivalently,

$$D_2 = \{(x, y, z) : \frac{(x^2 + y^2 + z^2 - 1)^2}{d_2 - a_0} + \frac{4x^2}{d_2 - a_1} + \frac{4y^2}{d_2 - a_2} + \frac{4z^2}{d_2 - a_3} < 0\}.$$

In this section, we solve the Dirichlet problem for harmonic functions in D_2 by the method of separation of variables.

Let $\mathbf{p} = (p_0, p_1, p_2, p_3) \in \{0, 1\}^4$ and $\mathbf{n} = (n_1, n_3) \in \mathbf{N}_0^2$. Using the functions $E^{(2)}_{i,\mathbf{n},\mathbf{p}}$ introduced in Sec. VII, we define the *internal 5-cyclidic harmonic of the second kind*

$$G^{(2)}_{\mathbf{n},\mathbf{p}}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3) \tag{89}$$

for $x, y, z \in B_1(\mathbf{0})$ with $x, y, z \geq 0$. We extend the function

$$(x^2 + y^2 + z^2 + 1)^{1/2} G^{(2)}_{\mathbf{n},\mathbf{p}}(x, y, z)$$

to $\mathbf{R}^3$ as a function of parity $\mathbf{p}$. We call a function f of parity $\mathbf{p} = (p_0, p_1, p_2, p_3)$ if

$$f(\sigma_i(x, y, z)) = (-1)^{p_i} f(x, y, z), \qquad \text{for } i = 0, 1, 2, 3, \tag{90}$$

using inversion (29) and reflections (30).

We omit the proof of the following lemma which is similar to the proof of Lemma 6.1.

Lemma 8.1. The function $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is harmonic at all points $(x, y, z) \in \mathbf{R}^3$ at which $s_2 \neq a_2$; see (34).

Note that $s_2 < d_2 < a_2$ in D_2. Therefore, $G^{(2)}_{\mathbf{n},\mathbf{p}}$ is harmonic in an open set containing the closure of D_2. Geometrically speaking, the set $s_2 = a_2$ consists of the part of the plane $y = 0$ "outside" the two closed curves in Figure 3. The asymmetric ring cyclides D_2 passes through the $y = 0$ plane inside those two closed curves.

Substituting $t_j = \Omega(s_j)$, $j = 1, 3$, the Hilbert space H_2 from Sec. VII transforms to the Hilbert space $\tilde{H}_2$ consisting of measurable functions $g : (a_0, a_1) \times (a_2, a_3) \to \mathbf{C}$ for which

$$\|g\|^2 := \int_{a_2}^{a_3} \int_{a_0}^{a_1} \frac{s_3 - s_1}{\omega(s_1)\omega(s_3)} |g(s_1, s_3)|^2 \, ds_1 \, ds_3 < \infty. \tag{91}$$

By Theorem 7.3, for $g \in \tilde{H}_2$ and fixed $\mathbf{p}$, we have the Fourier expansion

$$g(s_1, s_3) \sim \sum_{\mathbf{n}} c_{\mathbf{n},\mathbf{p}} E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3), \tag{92}$$

where the Fourier coefficients are given by

$$c_{\mathbf{n},\mathbf{p}} = \int_{a_2}^{a_3} \int_{a_0}^{a_1} \frac{s_3 - s_1}{\omega(s_1)\omega(s_3)} g(s_1, s_3) E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3) \, ds_1 \, ds_3.$$

Theorem 8.2. *Consider the region D_2 defined by (88) for some fixed $d_2 \in (a_1, a_2)$. Let e be a function defined on its boundary ∂D_2, and set*

$$f(x, y, z) := (x^2 + y^2 + z^2 + 1)^{1/2} e(x, y, z). \tag{93}$$

Suppose that f has parity $\mathbf{p} \in \{0, 1\}^4$, and its representation $g(s_1, s_3)$ in 5-cyclide coordinates is in $\tilde{H}_2$. Expand g in the series (92). Then the function $u(x, y, z)$ given by

$$u(x, y, z) = \sum_{\mathbf{n}} \frac{c_{\mathbf{n},\mathbf{p}}}{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)} G^{(2)}_{\mathbf{n},\mathbf{p}}(x, y, z) \tag{94}$$

is harmonic in D_2 and assumes the values e on the boundary of D_2 in the weak sense.

Proof. The proof is similar to the proof of Theorem 6.3. It uses Theorems 7.4 and 7.5 to show that the series in (94) is absolutely and uniformly convergent on compact subsets of D_2. Consequently, by Lemma 8.1, $u(x, y, z)$ is harmonic in D_2. If we consider u for fixed $s_2 \in (a_1, d_2)$ and compute the norm $\|u - e\|$ in the Hilbert space $\tilde{H}_2$, we obtain

$$\|u - e\|^2 \leq \sum_{\mathbf{n}} |c_{\mathbf{n},\mathbf{p}}|^2 \left(1 - \frac{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(s_2)}{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)}\right)^2.$$

The right-hand side converges to 0 as $s_2 \to d_2$. Hence u assumes the boundary values e in the weak sense. □

If f is a function without parity, we write f as a sum of 16 functions

$$f = \sum_{\mathbf{p} \in \{0,1\}^4} f_{\mathbf{p}},$$

where $f_{\mathbf{p}}$ is of parity $\mathbf{p}$. Then the solution of the corresponding Dirichlet problem is given by

$$u(x, y, z) = \sum_{\mathbf{n},\mathbf{p}} \frac{c_{\mathbf{n},\mathbf{p}}}{E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)} G^{(2)}_{\mathbf{n},\mathbf{p}}(x, y, z), \tag{95}$$

where

$$c_{\mathbf{n},\mathbf{p}} = \int_{a_2}^{a_3} \int_{a_0}^{a_1} \frac{s_3 - s_1}{\omega(s_1)\omega(s_3)} g_{\mathbf{p}}(s_1, s_3) E^{(2)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(2)}_{3,\mathbf{n},\mathbf{p}}(s_3) \, ds_1 \, ds_3 \tag{96}$$

and $g_{\mathbf{p}}(s_1, s_3)$ is the representation of $f_{\mathbf{p}}$ in 5-cyclide coordinates. We may also write $c_{\mathbf{n},\mathbf{p}}$ as a surface integral

$$c_{\mathbf{n},\mathbf{p}} = \frac{1}{4\omega(d_2)E^{(2)}_{2,\mathbf{n},\mathbf{p}}(d_2)} \int_{\partial D_2} \frac{e}{h_2} G^{(2)}_{\mathbf{n},\mathbf{p}}\, dS, \tag{97}$$

where

$$h_2^2 = \frac{1}{16}\left(\frac{(x^2+y^2+z^2-1)^2}{(d_2-a_0)^2} + \frac{4x^2}{(d_2-a_1)^2} + \frac{4y^2}{(d_2-a_2)^2} + \frac{4z^2}{(d_2-a_3)^2}\right).$$

IX. THIRD TWO-PARAMETER STURM-LIOUVILLE PROBLEM

If we write (26) on the intervals (a_0, a_1) and (a_1, a_2) in formally self-adjoint form, we obtain

$$(\omega(s_1)w_1')' - \frac{1}{\omega(s_1)}\left(\frac{3}{16}s_1^2 + \lambda_1 s_1 + \lambda_2\right) w_1 = 0, \quad a_0 < s_1 < a_1, \tag{98}$$

$$(\omega(s_2)w_2')' + \frac{1}{\omega(s_2)}\left(\frac{3}{16}s_2^2 + \lambda_1 s_2 + \lambda_2\right) w_2 = 0, \quad a_1 < s_2 < a_2. \tag{99}$$

We simplify the equations by substituting $t_j = \Omega(s_j)$, $u_j(t_j) = w_j(s_j)$, where $\Omega(s)$ is the elliptic integral (42). Then (98) and (99) become

$$u_1'' - \left(\frac{3}{16}\{\phi(t_1)\}^2 + \lambda_1\phi(t_1) + \lambda_2\right) u_1 = 0, \quad b_0 \le t_1 \le b_1, \tag{100}$$

$$u_2'' + \left(\frac{3}{16}\{\phi(t_2)\}^2 + \lambda_1\phi(t_2) + \lambda_2\right) u_2 = 0, \quad b_1 \le t_2 \le b_2. \tag{101}$$

Of course, this system is very similar to the one considered in Sec. V. Therefore, we will be brief. For a given $\mathbf{p} = (p_0, p_1, p_2) \in \{0, 1\}^3$, we consider the boundary conditions

$$\begin{array}{llll} u_1'(b_0) = 0 & \text{if } p_0 = 0, & u_1(b_0) = 0 & \text{if } p_0 = 1,\\ u_1'(b_1) = u_2'(b_1) = 0 & \text{if } p_1 = 0, & u_1(b_1) = u_2(b_1) = 0 & \text{if } p_1 = 1,\\ u_2'(b_2) = 0 & \text{if } p_2 = 0, & u_2(b_2) = 0 & \text{if } p_2 = 1. \end{array} \tag{102}$$

The initial conditions for u_3 are

$$u_3(b_2) = 1, u_3'(b_2) = 0 \text{ if } p_2 = 0, \quad u_3(b_2) = 0, u_3'(b_2) = 1 \text{ if } p_2 = 1. \tag{103}$$

We denote the corresponding eigenvalues by $(\lambda^{(3)}_{1,\mathbf{n},\mathbf{p}}, \lambda^{(3)}_{2,\mathbf{n},\mathbf{p}})$, where $\mathbf{n} = (n_1, n_2) \in \mathbf{N}_0^2$. The eigenfunctions will be denoted by $E^{(3)}_{i,\mathbf{n},\mathbf{p}}(s_i) = u_{i,\mathbf{n}}(t_i)$, $i = 1, 2, 3$.

Summarizing, for $i = 1, 2, 3$, $E^{(3)}_{i,\mathbf{n},\mathbf{p}}$ is a solution of (26) on (a_{i-1}, a_i) with $(\lambda_1, \lambda_2) = (\lambda^{(3)}_{1,\mathbf{n},\mathbf{p}}, \lambda^{(3)}_{2,\mathbf{n},\mathbf{p}})$. The solution $E^{(3)}_{1,\mathbf{n},\mathbf{p}}(s_1)$ has exponent $\frac{1}{2}p_0$ at a_0, exponent $\frac{1}{2}p_1$ at a_1, and it has n_1 zeros in (a_0, a_1). The solution $E^{(3)}_{2,\mathbf{n},\mathbf{p}}(s_2)$ has exponent $\frac{1}{2}p_1$ at a_1, exponent $\frac{1}{2}p_2$ at a_2, and it has n_2 zeros in (a_1, a_2). The solution $E^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3)$ has exponent $\frac{1}{2}p_2$ at a_2, and it has no zeros in (a_2, a_3).

X. THIRD DIRICHLET PROBLEM

Consider the coordinate surface (21) for fixed $s = d_3 \in (a_2, a_3)$. See Figures 5(d)–5(f) for a graphical depiction of the shape of this surface. If $(x', y', z') \in \mathbb{S}_2$, then the ray $(x, y, z) = t(x', y', z')$, $t > 0$, is tangent to the surface if and only if (x', y', z') is on the surface. If (x', y', z') is in the elliptical cone

$$\frac{4x'^2}{d_2-a_1} + \frac{4y'^2}{d_2-a_2} + \frac{4z'^2}{d_2-a_3} < 0,$$

then the ray intersects the surface twice at $t = t_1, t_2$ with $t_1 t_2 = 1$. Otherwise, there is no intersection. Therefore, the coordinate surface $s_3 = d_3$ consists of two disjoint closed asymmetric surfaces of genus one separated by the plane $z = 0$, and they are mirror images of each other under the reflection σ_3.

We consider the region inside the surface $s_3 = d_3$ with $z > 0$

$$D_3 = \{(x, y, z) : z > 0, s_3 < d_3\}, \tag{104}$$

or, equivalently,

$$D_3 = \{(x, y, z) : z > 0, \frac{(x^2 + y^2 + z^2 - 1)^2}{d_3 - a_0} + \frac{4x^2}{d_3 - a_1} + \frac{4y^2}{d_3 - a_2} + \frac{4z^2}{d_3 - a_3} < 0\}.$$

Next, we solve the Dirichlet problem for harmonic functions in D_3 by the method of separation of variables.

Let $\mathbf{p} = (p_0, p_1, p_2) \in \{0, 1\}^3$ and $\mathbf{n} = (n_1, n_2) \in \mathbf{N}_0^2$. Using the functions $E^{(3)}_{i,\mathbf{n},\mathbf{p}}$ introduced in Sec. IX, we define the *internal 5-cyclidic harmonic of the third kind*

$$G^{(3)}_{\mathbf{n},\mathbf{p}}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E^{(3)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(3)}_{2,\mathbf{n},\mathbf{p}}(s_2) E^{(3)}_{3,\mathbf{n},\mathbf{p}}(s_3) \tag{105}$$

for $(x, y, z) \in B_1(\mathbf{0})$ with $x, y, z \geq 0$. We extend the function

$$(x^2 + y^2 + z^2 + 1)^{1/2} G^{(3)}_{\mathbf{n},\mathbf{p}}(x, y, z)$$

to the half-space $\{(x, y, z): z > 0\}$ as a function of parity $\mathbf{p}$. We call a function f of parity $\mathbf{p} = (p_0, p_1, p_2)$, if

$$f(\sigma_i(x, y, z)) = (-1)^{p_i} f(x, y, z), \qquad \text{for } i = 0, 1, 2 \tag{106}$$

using the inversion σ_0 and the reflections σ_1, σ_2. As before we have the following lemma.

Lemma 10.1. The function $G^{(3)}_{\mathbf{n},\mathbf{p}}$ is harmonic on $\{(x, y, z): z > 0\}$.

We have the Hilbert space $\tilde{H}_3$ consisting of measurable functions $g : (a_0, a_1) \times (a_1, a_2) \to \mathbf{C}$ for which

$$\|g\|^2 := \int_{a_1}^{a_2} \int_{a_0}^{a_1} \frac{s_2 - s_1}{\omega(s_1)\omega(s_1)} |g(s_1, s_2)|^2 \, ds_1 \, ds_2 < \infty. \tag{107}$$

For $g \in \tilde{H}_3$ and fixed $\mathbf{p}$, we have the Fourier expansion

$$g(s_1, s_2) \sim \sum_{\mathbf{n}} c_{\mathbf{n},\mathbf{p}} E^{(3)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(3)}_{2,\mathbf{n},\mathbf{p}}(s_2), \tag{108}$$

where the Fourier coefficients are given by

$$c_{\mathbf{n},\mathbf{p}} = \int_{a_1}^{a_2} \int_{a_0}^{a_1} \frac{s_2 - s_1}{\omega(s_1)\omega(s_2)} g(s_1, s_2) E^{(3)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(3)}_{2,\mathbf{n},\mathbf{p}}(s_2) \, ds_1 \, ds_2. \tag{109}$$

Theorem 10.2. *Consider the region D_3 defined by (104) for some fixed $d_3 \in (a_2, a_3)$. Let e be a function defined on its boundary ∂D_3, and set*

$$f(x, y, z) := (x^2 + y^2 + z^2 + 1)^{1/2} e(x, y, z). \tag{110}$$

Suppose that f has parity $\mathbf{p} \in \{0, 1\}^3$, and its representation $g(s_1, s_2)$ in 5-cyclide coordinates is in $\tilde{H}_3$. Expand g in the series (108). Then the function $u(x, y, z)$ given by

$$u(x, y, z) = \sum_{\mathbf{n}} \frac{c_{\mathbf{n},\mathbf{p}}}{E^{(3)}_{3,\mathbf{n},\mathbf{p}}(d_3)} G^{(3)}_{\mathbf{n},\mathbf{p}}(x, y, z) \tag{111}$$

is harmonic in D_3 and assumes the values e on the boundary of D_3 in the weak sense.

If f is a function without parity, we write f as a sum of eight functions

$$f = \sum_{\mathbf{p}\in\{0,1\}^3} f_{\mathbf{p}},$$

where $f_{\mathbf{p}}$ is of parity $\mathbf{p}$. Then the solution of the corresponding Dirichlet problem is given by

$$u(x, y, z) = \sum_{\mathbf{n},\mathbf{p}} \frac{c_{\mathbf{n},\mathbf{p}}}{E^{(3)}_{3,\mathbf{n},\mathbf{p}}(d_3)} G^{(3)}_{\mathbf{n},\mathbf{p}}(x, y, z), \tag{112}$$

where

$$c_{\mathbf{n},\mathbf{p}} = \int_{a_1}^{a_2}\int_{a_0}^{a_1} \frac{s_2 - s_1}{\omega(s_1)\omega(s_2)} g_{\mathbf{p}}(s_1, s_2) E^{(3)}_{1,\mathbf{n},\mathbf{p}}(s_1) E^{(3)}_{2,\mathbf{n},\mathbf{p}}(s_2)\, ds_1\, ds_2 \tag{113}$$

and $g_{\mathbf{p}}(s_1, s_2)$ is the representation of $f_{\mathbf{p}}$ in 5-cyclide coordinates. Alternatively, we have

$$c_{\mathbf{n},\mathbf{p}} = \frac{1}{2\omega(d_3)E^{(3)}_{3,\mathbf{n},\mathbf{p}}(d_3)} \int_{\partial D_3} \frac{e}{h_3} G^{(3)}_{\mathbf{n},\mathbf{p}}\, dS, \tag{114}$$

where

$$h_3^2 = \frac{1}{16}\left(\frac{(x^2 + y^2 + z^2 - 1)^2}{(d_3 - a_0)^2} + \frac{4x^2}{(d_3 - a_1)^2} + \frac{4y^2}{(d_3 - a_2)^2} + \frac{4z^2}{(d_3 - a_3)^2}\right).$$

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