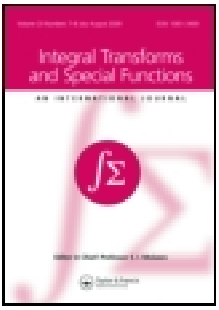

Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gitr20

On a generalization of the generating function for Gegenbauer polynomials

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Published online: 28 Jan 2013.

To cite this article: Howard S. Cohl (2013): On a generalization of the generating function for Gegenbauer polynomials, Integral Transforms and Special Functions, DOI:10.1080/10652469.2012.761613

To link to this article: http://dx.doi.org/10.1080/10652469.2012.761613

Integral Transforms and Special Functions, 2013
http://dx.doi.org/10.1080/10652469.2012.761613

On a generalization of the generating function for Gegenbauer polynomials

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(*Received 30 March 2012; final version received 19 December 2012*)

A generalization of the generating function for Gegenbauer polynomials is introduced whose coefficients are given in terms of associated Legendre functions of the second kind. We discuss how our expansion represents a generalization of several previously derived formulae such as Heine's formula and Heine's reciprocal square-root identity. We also show how this expansion can be used to compute hyperspherical harmonic expansions for power-law fundamental solutions of the polyharmonic equation.

Keywords: Euclidean space; polyharmonic equation; fundamental solution; Gegenbauer polynomials; associated Legendre functions

AMS Subject Classifications: 35A08; 35J05; 32Q45; 31C12; 33C05; 42A16

1. Introduction

Gegenbauer polynomials $C_n^\nu(x)$ are given as the coefficients of ρ^n for the generating function $(1+\rho^2-2\rho x)^{-\nu}$. The study of these polynomials was pioneered in a series of papers by Gegenbauer [1–5]. The main result, which this paper relies upon, is Theorem 2.1. This theorem gives a generalized expansion over Gegenbauer polynomials $C_n^\mu(x)$ of the algebraic function $z \mapsto (z-x)^{-\nu}$. Our proof is combinatoric in nature and has great potential for proving new expansion formulae which generalize generating functions. Our methodology can in principle be applied to any generating function for hypergeometric orthogonal polynomials of which there are many (see, for instance, [6,7]). The concept of the proof is to start with a generating function and use a connection formula to express the orthogonal polynomial as a finite series in polynomials of the same type with different parameters. The resulting formulae will then produce new expansions for the polynomials which result from a limiting process, e.g. Legendre polynomials and Chebyshev polynomials of the first and second kind. Connection formulae for classical orthogonal polynomials and their q-extensions are well known (see [8]). In this paper we applied this method of proof to the generating function for Gegenbauer polynomials.

This paper is organized as follows. In Section 2, we derive a complex generalization of the generating function for Gegenbauer polynomials. In Section 3, we discuss how our complex generalization reduces to previously derived expressions and leads to extensions in appropriate limits. In Section 4, we use our complex expansion to generalize a formula originally

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developed by Sack [9] on $\mathbf{R}^3$, to compute an expansion in terms of Gegenbauer polynomials for complex powers of the distance between two points on a d-dimensional Euclidean space for $d \geq 2$.

Throughout this paper we rely on the following definitions. For $a_1, a_2, a_3, \ldots \in \mathbf{C}$, if $i, j \in \mathbf{Z}$ and $j < i$ then $\sum_{n=i}^{j} a_n = 0$ and $\prod_{n=i}^{j} a_n = 1$, where $\mathbf{C}$ represents the complex numbers. The set of natural numbers is given by $\mathbf{N} := \{1, 2, 3, \ldots\}$, the set $\mathbf{N}_0 := \{0, 1, 2, \ldots\} = \mathbf{N} \cup \{0\}$ and the set $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$. The sets $\mathbf{Q}$ and $\mathbf{R}$ represent the rational and real numbers, respectively.

2. Generalization of the generating function for Gegenbauer polynomials

We present the following generalization of the generating function for Gegenbauer polynomials whose coefficients are given in terms of associated Legendre functions of the second kind.

THEOREM 2.1 *Let $\nu \in \boldsymbol{C} \setminus -\boldsymbol{N}_0$, $\mu \in (-1/2, \infty) \setminus \{0\}$ and $z \in \boldsymbol{C} \setminus (-\infty, 1]$ on any ellipse with foci at ± 1 with x in the interior of that ellipse. Then*

$$\frac{1}{(z-x)^\nu} = \frac{2^{\mu+1/2}\Gamma(\mu)\,\mathrm{e}^{\mathrm{i}\pi(\mu-\nu+1/2)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{(\nu-\mu)/2-1/4}} \sum_{n=0}^{\infty}(n+\mu)Q_{n+\mu-1/2}^{\nu-\mu-1/2}(z)C_n^\mu(x). \tag{1}$$

If one substitutes $z = (1+\rho^2)/(2\rho)$ in (1) with $0 < |\rho| < 1$, then one obtains an alternate expression with $x \in [-1, 1]$,

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \frac{\Gamma(\mu)\,\mathrm{e}^{\mathrm{i}\pi(\mu-\nu+1/2)}}{\sqrt{\pi}\,\Gamma(\nu)\rho^{\mu+1/2}(1-\rho^2)^{\nu-\mu-1/2}} \sum_{n=0}^{\infty}(n+\mu)Q_{n+\mu-1/2}^{\nu-\mu-1/2}\left(\frac{1+\rho^2}{2\rho}\right)C_n^\mu(x). \tag{2}$$

One can see that by replacing $\nu = \mu$ in (2), and using (8.6.11) in [10] that these formulae are generalizations of the generating function for Gegenbauer polynomials (first occurrence in [1])

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \sum_{n=0}^{\infty} C_n^\nu(x)\rho^n, \tag{3}$$

where $\rho \in \mathbf{C}$ with $|\rho| < 1$ and $\nu \in (-1/2, \infty) \setminus \{0\}$ (see, for instance, (18.12.4) in [11]). The Gegenbauer polynomials $C_n^\nu : \mathbf{C} \to \mathbf{C}$ can be defined by

$$C_n^\nu(x) := \frac{(2\nu)_n}{n!}\,{}_2F_1\left(-n, n+2\nu; \nu+\frac{1}{2}; \frac{1-x}{2}\right), \tag{4}$$

where $n \in \mathbf{N}_0$, $\nu \in (-1/2, \infty) \setminus \{0\}$, and ${}_2F_1 : \mathbf{C}^2 \times (\mathbf{C} \setminus -\mathbf{N}_0) \times \{z \in \mathbf{C} : |z| < 1\} \to \mathbf{C}$, the Gauss hypergeometric function, can be defined in terms of the following infinite series:

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{5}$$

(see (2.1.5) in [12]), and elsewhere by analytic continuation. The Pochhammer symbol (rising factorial) $(\cdot)_n : \mathbf{C} \to \mathbf{C}$ is defined by

$$(z)_n := \prod_{i=1}^{n}(z+i-1),$$

where $n \in \mathbf{N}_0$. For the Gegenbauer polynomials $C_n^\nu(x)$, we refer to n and ν as the degree and order, respectively.

Proof Consider the generating function for Gegenbauer polynomials (3). The connection relation which expresses a Gegenbauer polynomial with order ν as a sum over Gegenbauer polynomials with order μ is given by

$$C_n^\nu(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} \sum_{k=0}^{n} \frac{(\nu+k+\frac{1}{2})_{n-k}(2\nu+n)_k(\mu+\frac{1}{2})_k\Gamma(2\mu+k)}{(n-k)!(2\mu)_k\Gamma(2\mu+2k)}$$
$$\times {}_3F_2\left(\begin{matrix}-n+k, n+k+2\nu, \mu+k+\frac{1}{2}\\ \nu+k+\frac{1}{2}, 2\mu+2k+1\end{matrix}; 1\right) C_k^\mu(x). \tag{6}$$

This connection relation can be derived by starting with Theorem 9.1.1 in [8] combined with (see, for instance, (18.7.1) in [11])

$$C_n^\nu(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-1/2,\nu-1/2)}(x), \tag{7}$$

i.e. the Gegenbauer polynomials are given as symmetric Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)} : \mathbf{C} \to \mathbf{C}$ can be defined as (see, for instance, (18.5.7) in [11])

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right), \tag{8}$$

where $n \in \mathbf{N}_0$, and $\alpha, \beta > -1$ (see Table 18.3.1 in [11]). The generalized hypergeometric function ${}_3F_2 : \mathbf{C}^3 \times (\mathbf{C} \setminus -\mathbf{N}_0)^2 \times \{z \in \mathbf{C} : |z| < 1\} \to \mathbf{C}$ can be defined in terms of the following infinite series:

$${}_3F_2\left(\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n} \frac{z^n}{n!}.$$

If we replace the Gegenbauer polynomial in the generating function (3) using the connection relation (6), we obtain a double summation expression over k and n. By reversing the order of the summations (justification by Tannery's theorem) and shifting the n-index by k, we obtain after making some reductions and simplifications, the following double-summation representation:

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \frac{\sqrt{\pi}\Gamma(\mu)}{2^{2\nu-1}\Gamma(\nu)} \sum_{k=0}^{\infty} C_k^\mu(x) \frac{\rho^k}{2^{2k}} \frac{\mu+k}{\Gamma(\nu+k+\frac{1}{2})\Gamma(\mu+k+1)}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(2\nu+2k)}{n!} {}_3F_2\left(\begin{matrix}-n, n+2k+2\nu, \mu+k+\frac{1}{2}\\ \nu+k+\frac{1}{2}, 2\mu+2k+1\end{matrix}; 1\right). \tag{9}$$

The ${}_3F_2$ generalized hypergeometric function appearing the above formula may be simplified using Watson's sum

$${}_3F_2\left(\begin{matrix}a, b, c\\ \frac{1}{2}(a+b+1), 2c\end{matrix}; 1\right) = \frac{\sqrt{\pi}\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c+\frac{1}{2}(1-a-b)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)\Gamma\left(c+\frac{1}{2}(1-a)\right)\Gamma\left(c+\frac{1}{2}(1-b)\right)},$$

where $\mathrm{Re}(2c-a-b) > -1$ (see, for instance, (16.4.6) in [11]), therefore

$$\frac{1}{\Gamma(\nu+k+\frac{1}{2})\Gamma(\mu+k+1)} {}_3F_2\left(\begin{matrix}-n, n+2k+2\nu, \mu+k+\frac{1}{2}\\ \nu+k+\frac{1}{2}, 2\mu+2k+1\end{matrix}; 1\right)$$
$$= \frac{\sqrt{\pi}\Gamma(\mu-\nu+1)}{\Gamma((1-n)/2)\Gamma(\nu+k+(n+1)/2)\Gamma(\mu+k+1+n/2)\Gamma(\mu-\nu+1-n/2)}, \tag{10}$$

for $\mathrm{Re}(\mu-\nu) > -1$. By inserting (10) in (9), it follows that

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \frac{\pi\Gamma(\mu)\Gamma(\mu-\nu+1)}{2^{2\nu-1}\Gamma(\nu)}\sum_{k=0}^\infty(\mu+k)C_k^\mu(x)\frac{\rho^k}{2^{2k}}$$
$$\times\sum_{n=0}^\infty\frac{\rho^n\Gamma(2\nu+2k+n)(\Gamma(\mu-\nu+1-n/2))^{-1}}{n!\Gamma((1-n)/2)\Gamma(\nu+k+(n+1)/2)\Gamma(\mu+k+1+n/2)}.$$

It is straightforward to show using (5) and

$$\Gamma(z-n) = (-1)^n\frac{\Gamma(z)}{(-z+1)_n},$$

for $n\in\mathbf{N}_0$, $z\in\mathbf{C}\setminus-\mathbf{N}_0$, and the duplication formula (i.e. (5.5.5) in [11])

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

provided $2z\notin-\mathbf{N}_0$, that

$$\sum_{n=0}^\infty\frac{\rho^n\Gamma(2\nu+2k+n)}{n!\Gamma((1-n)/2)\Gamma(\nu+k+(n+1)/2)\Gamma(\mu+k+1+n/2)\Gamma(\mu-\nu+1-n/2)}$$
$$= \frac{2^{2\nu+2k-1}\Gamma(\nu+k)}{\pi\Gamma(\mu+k+1)\Gamma(\mu-\nu+1)}{}_2F_1(\nu+k,\nu-\mu;\mu+k+1;\rho^2),$$

so therefore

$$\frac{1}{(1+\rho^2-2\rho x)^\nu} = \frac{\Gamma(\mu)}{\Gamma(\nu)}\sum_{k=0}^\infty(\mu+k)C_k^\mu(x)\rho^k\frac{\Gamma(\nu+k)}{\Gamma(\mu+k+1)}{}_2F_1(\nu+k,\nu-\mu;\mu+k+1;\rho^2). \tag{11}$$

Finally, utilizing the quadratic transformation of the hypergeometric function

$${}_2F_1(a,b;a-b+1;z) = (1+z)^{-a}{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(z+1)^2}\right),$$

for $|z|<1$ (see (3.1.9) in [12]), combined with the definition of the associated Legendre function of the second kind $Q_\nu^\mu:\mathbf{C}\setminus(-\infty,1]\to\mathbf{C}$ in terms of the Gauss hypergeometric function

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi\mu}\Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})z^{\nu+\mu+1}}{}_2F_1\left(\frac{\nu+\mu+2}{2},\frac{\nu+\mu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right), \tag{12}$$

for $|z|>1$ and $\nu+\mu+1\notin-\mathbf{N}_0$ (cf. Section 14.21 and (14.3.7) in [11]), one can show that

$${}_2F_1\left(\nu+k,\nu-\mu;\mu+k+1;\rho^2\right)$$
$$= \frac{\Gamma(\mu+k+1)\,\mathrm{e}^{\mathrm{i}\pi(\mu-\nu+1/2)}}{\sqrt{\pi}\Gamma(\nu+k)\rho^{\mu+k+1/2}(1-\rho^2)^{\nu-\mu-1/2}}Q_{k+\mu-1/2}^{\nu-\mu-1/2}\left(\frac{1+\rho^2}{2\rho}\right),$$

which when used in (11) produces (2). Since the Gegenbauer polynomial is just a symmetric Jacobi polynomial (7), through Theorem 9.1.1 in [13] (expansion of an analytic function in a Jacobi series), since $f_z:\mathbf{C}\to\mathbf{C}$ defined by $f_z(x):=(z-x)^{-\nu}$ is analytic in $[-1,1]$, then the above expansion in Gegenbauer polynomials is convergent if the point $z\in\mathbf{C}$ lies on any ellipse with foci at ±1 and x can lie on any point interior to that ellipse. ■

3. Generalizations, extensions and applications

By considering in (2) the substitution $\nu = d/2 - 1$ and the map $\nu \mapsto -\nu/2$, one obtains the formula

$$\frac{\rho^{(d-1)/2}(1-\rho^2)^{\nu-(d-1)/2}}{(1+\rho^2-2\rho x)^\nu} = \frac{e^{-i\pi(\nu-(d-1)/2)}\Gamma((d-2)/2)}{2\sqrt{\pi}\Gamma(\nu)} \times \sum_{n=0}^{\infty}(2n+d-2)Q_{n+(d-3)/2}^{\nu-(d-1)/2}\left(\frac{1+\rho^2}{2\rho}\right)C_n^{d/2-1}(x).$$

This formula generalizes (9.9.2) in [12].

By taking the limit as $\mu \to 1/2$ in (1), one obtains a general result which is an expansion over Legendre polynomials, namely

$$\frac{1}{(z-x)^\nu} = \frac{e^{i\pi(1-\nu)}(z^2-1)^{(1-\nu)/2}}{\Gamma(\nu)}\sum_{n=0}^{\infty}(2n+1)Q_n^{\nu-1}(z)P_n(x), \tag{13}$$

using

$$P_n(x) = C_n^{1/2}(x), \tag{14}$$

which is clear by comparing (cf. (18.7.9) of [11] and (4) or (8))

$$P_n(x) := {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right), \tag{15}$$

and (4). If one takes $\nu = 1$ in (13) then one has an expansion of the Cauchy denominator which generalizes Heine's formula (see, for instance, [14, Ex. 13.1, 15, p. 78])

$$\frac{1}{z-x} = \sum_{n=0}^{\infty}(2n+1)Q_n(z)P_n(x).$$

By taking the limit as $\mu \to 1$ in (1), one obtains a general result which is an expansion over Chebyshev polynomials of the second kind, namely

$$\frac{1}{(z-x)^\nu} = \frac{2^{3/2}e^{i\pi(3/2-\nu)}}{\sqrt{\pi}\Gamma(\nu)(z^2-1)^{\nu/2-3/4}}\sum_{n=0}^{\infty}(n+1)Q_{n+1/2}^{\nu-3/2}(z)U_n(x), \tag{16}$$

using (18.7.4) in [11], $U_n(x) = C_n^1(x)$. If one considers the case $\nu = 1$ in (16) then the associated Legendre function of the second kind reduces to an elementary function through (8.6.11) in [10], namely

$$\frac{1}{z-x} = 2\sum_{n=0}^{\infty}\frac{U_n(x)}{(z+\sqrt{z^2-1})^{n+1}}.$$

By taking the limit as $\nu \to 1$ in (1), one produces the Gegenbauer expansion of the Cauchy denominator given in [16, (7.2)], namely

$$\frac{1}{z-x} = \frac{2^{\mu+1/2}}{\sqrt{\pi}}\Gamma(\mu)\, e^{i\pi(\mu-1/2)}(z^2-1)^{\mu/2-1/4}\sum_{n=0}^{\infty}(n+\mu)Q_{n+\mu-1/2}^{-\mu+1/2}(z)C_n^\mu(x).$$

Using (2.4) therein, the associated Legendre function of the second kind is converted to the Gegenbauer function of the second kind.

If one considers the limit as $\mu \to 0$ in (1) using

$$\lim_{\mu\to 0} \frac{n+\mu}{\mu} C_n^\mu(x) = \epsilon_n T_n(x)$$

(see, for instance, (6.4.13) in [12]), where $T_n : \mathbf{C} \to \mathbf{C}$ is the Chebyshev polynomial of the first kind defined as (see Section 5.7.2 in [17])

$$T_n(x) := {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right),$$

and $\epsilon_n = 2 - \delta_{n,0}$ is the Neumann factor, commonly appearing in Fourier cosine series, then one obtains

$$\frac{1}{(z-x)^\nu} = \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-\mathrm{i}\pi(\nu-1/2)}(z^2-1)^{-\nu/2+1/4}}{\Gamma(\nu)} \sum_{n=0}^\infty \epsilon_n T_n(x) Q_{n-1/2}^{\nu-1/2}(z). \tag{17}$$

The result (17) is a generalization of Heine's reciprocal square-root identity (see [18, p. 286, 19, (A5)]). Polynomials in $(z-x)$ also naturally arise by considering the limit $\nu \to n \in -\mathbf{N}_0$. This limit is given in (4.4) of Cohl and Dominici [20], namely

$$(z-x)^q = i(-1)^{q+1}\sqrt{\frac{2}{\pi}}(z^2-1)^{q/2+1/4} \sum_{n=0}^q \epsilon_n T_n(x) \frac{(-q)_n}{(q+n)!} Q_{n-1/2}^{q+1/2}(z), \tag{18}$$

for $q \in \mathbf{N}_0$. Note that all of the above formulae are restricted by the convergence criterion given by Theorem 9.1.1 in [13] (expansion of an analytic function in a Jacobi series), i.e. since the functions on the left-hand side are analytic in $[-1,1]$, then the expansion formulae are convergent if the point $z \in \mathbf{C}$ lies on any ellipse with foci at ± 1 then x can lie on any point interior to that ellipse. Except of course (18) which converges for all points $z, x \in \mathbf{C}$ since the function on the left-hand side is entire.

An interesting extension of the results presented in this paper, originally uploaded to `arXiv` in [21], has been obtained recently in [22], to obtain formulas such as

$$\sum_{n=0}^\infty \frac{n+\mu}{\mu} \mathrm{P}_{n+\mu-1/2}^{\nu-\mu}(t) C_n^\mu(x) = \frac{\sqrt{\pi}(1-t^2)^{(\nu-\mu)/2}}{2^{\mu-1/2}\Gamma(\mu+1)\Gamma\left(\frac{1}{2}-\nu\right)} \begin{cases} 0 & \text{if } -1 < x < t < 1, \\ (x-t)^{-\nu-1/2} & \text{if } -1 < t < x < 1, \end{cases}$$

and

$$\sum_{n=0}^\infty \frac{n+\mu}{\mu} \mathrm{Q}_{n+\mu-1/2}^{\nu-\mu}(t) C_n^\mu(x) = \frac{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})(1-t^2)^{(\nu-\mu)/2}}{2^{\mu+1/2}\Gamma(\mu+1)} \times \begin{cases} (t-x)^{-\nu-1/2} & \text{if } -1 < x < t < 1, \\ (x-t)^{-\nu-1/2}\cos[\pi(\nu+\frac{1}{2})] & \text{if } -1 < t < x < 1, \end{cases}$$

where $\mathrm{Re}\mu > -1/2$, $\mathrm{Re}\nu < 1/2$ and $\mathrm{P}_\nu^\mu, \mathrm{Q}_\nu^\mu : (-1,1) \to \mathbf{C}$ are Ferrers functions (associated Legendre functions on-the-cut) of the first and second kind. The Ferrers functions of the first and second kind can be defined using Olver *et al.* [11, (14.3.11–12)].

4. Expansion of a power-law fundamental solution of the polyharmonic equation

A fundamental solution for the polyharmonic equation on Euclidean space $\mathbf{R}^d$ is a function $\mathcal{G}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ which satisfies the equation

$$(-\Delta)^k \mathcal{G}_k^d(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'), \tag{19}$$

where $\Delta : C^p(\mathbf{R}^d) \to C^{p-2}(\mathbf{R}^d)$, $p \geq 2$, is the Laplacian operator on $\mathbf{R}^d$ defined by

$$\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2},$$

$\mathbf{x} = (x_1,\ldots,x_d), \mathbf{x}' = (x_1',\ldots,x_d') \in \mathbf{R}^d$, and δ is the Dirac delta function. Note that we introduce a minus sign into the equations where the Laplacian is used, such as in (19), to make the resulting operator positive. By Euclidean space $\mathbf{R}^d$, we mean the normed vector space given by the pair $(\mathbf{R}^d, \|\cdot\|)$, where $\|\cdot\| : \mathbf{R}^d \to [0,\infty)$ is the Euclidean norm on $\mathbf{R}^d$ defined by $\|\mathbf{x}\| := \sqrt{x_1^2+\cdots+x_d^2}$, with inner product $(\cdot,\cdot) : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ defined as

$$(\mathbf{x},\mathbf{x}') := \sum_{i=1}^{d} x_i x_i'. \tag{20}$$

Then $\mathbf{R}^d$ is a C^∞ Riemannian manifold with Riemannian metric induced from the inner product (20). Set $\mathbf{S}^{d-1} = \{\mathbf{x} \in \mathbf{R}^d : (\mathbf{x},\mathbf{x}) = 1\}$, then $\mathbf{S}^{d-1}$, the $(d-1)$-dimensional unit hypersphere, is a regular submanifold of $\mathbf{R}^d$ and a C^∞ Riemannian manifold with Riemannian metric induced from that on $\mathbf{R}^d$.

Theorem 4.1 *Let $d,k \in \boldsymbol{N}$. Define*

$$\mathcal{G}_k^d(\boldsymbol{x},\boldsymbol{x}') := \begin{cases} \dfrac{(-1)^{k+d/2+1}\|\boldsymbol{x}-\boldsymbol{x}'\|^{2k-d}}{(k-1)!(k-d/2)!2^{2k-1}\pi^{d/2}}(\log\|\boldsymbol{x}-\boldsymbol{x}'\| - \beta_{p,d}) & \text{if } d \text{ even}, k \geq d/2, \\ \dfrac{\Gamma(d/2-k)\|\boldsymbol{x}-\boldsymbol{x}'\|^{2k-d}}{(k-1)!2^{2k}\pi^{d/2}} & \text{otherwise}, \end{cases}$$

where $p = k - d/2$, $\beta_{p,d} \in \boldsymbol{Q}$ is defined as $\beta_{p,d} := \frac{1}{2}[H_p + H_{d/2+p-1} - H_{d/2-1}]$, with $H_j \in \boldsymbol{Q}$ being the jth harmonic number

$$H_j := \sum_{i=1}^{j} \frac{1}{i},$$

then $\mathcal{G}_k^d$ is a fundamental solution for $(-\Delta)^k$ on Euclidean space $\boldsymbol{R}^d$.

Proof See Cohl [23] and Boyling [24]. ■

Consider the following functions $\mathfrak{g}_k^d, \mathfrak{l}_k^d : (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in \mathbf{R}^d\} \to \mathbf{R}$ defined for d odd and for d even with $k \leq d/2 - 1$ as a power-law, namely

$$\mathfrak{g}_k^d(\mathbf{x},\mathbf{x}') := \|\mathbf{x}-\mathbf{x}'\|^{2k-d}, \tag{21}$$

and for d even, $k \geq d/2$, with logarithmic behaviour as

$$\mathfrak{l}_k^d(\mathbf{x},\mathbf{x}') := \|\mathbf{x}-\mathbf{x}'\|^{2p}(\log\|\mathbf{x}-\mathbf{x}'\| - \beta_{p,d}),$$

with $p = k - d/2$. By Theorem 4.1 we see that the functions $\mathfrak{g}_k^d$ and $\mathfrak{l}_k^d$ equal real non-zero constant multiples of $\mathcal{G}_k^d$ for appropriate parameters. Therefore by (19), $\mathfrak{g}_k^d$ and $\mathfrak{l}_k^d$ are fundamental solutions of the polyharmonic equation for appropriate parameters. In this paper, we only consider functions with power-law behaviour, although in future publications we will consider the logarithmic case (see [25] for the relevant Fourier expansions).

Now we consider the set of hyperspherical coordinate systems which parametrize points on $\mathbf{R}^d$. The Euclidean distance between two points represented in these coordinate systems is given by

$$\|\mathbf{x}-\mathbf{x}'\| = \sqrt{2rr'}[z-\cos\gamma]^{1/2},$$

where the toroidal parameter $z \in (1,\infty)$, (2.6) in [26], is given by

$$z := \frac{r^2+r'^2}{2rr'},$$

and the separation angle $\gamma \in [0,\pi]$ is given through

$$\cos\gamma = \frac{(\mathbf{x},\mathbf{x}')}{rr'}, \tag{22}$$

where $r, r' \in (0,\infty)$ are defined such that $r := \|\mathbf{x}\|$ and $r' := \|\mathbf{x}'\|$. We will use these quantities to derive Gegenbauer expansions of power-law fundamental solutions for powers of the Laplacian $\mathfrak{g}_k^d$ (21) represented in hyperspherical coordinates.

Corollary 4.2 *For $d \in \{3,4,5,\ldots\}$, $\nu \in \boldsymbol{C}$, $\boldsymbol{x},\boldsymbol{x}' \in \boldsymbol{R}^d$ with $r = \|\boldsymbol{x}\|$, $r' = \|\boldsymbol{x}'\|$, and $\cos\gamma = (\boldsymbol{x},\boldsymbol{x}')/(rr')$, the following formula holds:*

$$\|\boldsymbol{x}-\boldsymbol{x}'\|^\nu = \frac{e^{i\pi(\nu+d-1)/2}\Gamma((d-2)/2)}{2\sqrt{\pi}\Gamma(-\nu/2)}\frac{(r_>^2-r_<^2)^{(\nu+d-1)/2}}{(rr')^{(d-1)/2}} \times \sum_{n=0}^{\infty}(2n+d-2)\,Q_{n+(d-3)/2}^{(1-\nu-d)/2}\left(\frac{r^2+r'^2}{2rr'}\right)C_n^{d/2-1}(\cos\gamma), \tag{23}$$

where $r_{\lessgtr} := {\min \atop \max}\{r,r'\}$.

Note that (23) is seen to be a generalization of Laplace's expansion on $\mathbf{R}^3$ (see, for instance, [9])

$$\frac{1}{\|\mathbf{x}-\mathbf{x}'\|} = \sum_{n=0}^{\infty}\frac{r_<^n}{r_>^{n+1}}P_n(\cos\gamma),$$

which is demonstrated by utilizing (14) and simplifying the associated Legendre function of the second kind in (23) through $Q_{-1/2}^{1/2} : \mathbf{C}\setminus(-\infty,1] \to \mathbf{C}$ defined such that

$$Q_{-1/2}^{1/2}(z) = i\sqrt{\frac{\pi}{2}}(z^2-1)^{-1/4}$$

(cf. (8.6.10) in [10] and (12)).

The *addition theorem for hyperspherical harmonics*, which generalizes

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n}Y_{n,m}(\hat{\mathbf{x}})\overline{Y_{n,m}(\hat{\mathbf{x}}')}, \tag{24}$$

where $P_n(x)$ is the Legendre polynomial of degree $n \in \mathbf{N}_0$, for $d=3$, is given by

$$\sum_K Y_n^K(\hat{\mathbf{x}})\overline{Y_n^K(\hat{\mathbf{x}}')} = \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}(2n+d-2)C_n^{d/2-1}(\cos\gamma), \tag{25}$$

where K stands for a set of $(d-2)$-quantum numbers identifying degenerate harmonics for a given value of n and d, and γ is the separation angle (22). The functions $Y_n^K : \mathbf{S}^{d-1} \to \mathbf{C}$ are the normalized hyperspherical harmonics, and $Y_{n,k} : \mathbf{S}^2 \to \mathbf{C}$ are the normalized spherical harmonics for $d = 3$. Note that $\hat{\mathbf{x}}, \hat{\mathbf{x}}' \in \mathbf{S}^{d-1}$, are the vectors of unit length in the direction of $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^d$ respectively. For a proof of the addition theorem for hyperspherical harmonics (25), see Wen and Avery [27] and for a relevant discussion, see Section 10.2.1 in [28]. The correspondence between (24) and (25) arises from (4) and (15) namely (14).

One can use the addition theorem for hyperspherical harmonics to expand a fundamental solution of the polyharmonic equation on $\mathbf{R}^d$. Through the use of the addition theorem for hyperspherical harmonics we see that the Gegenbauer polynomials $C_n^{d/2-1}(\cos\gamma)$ are hyperspherical harmonics when regarded as a function of $\hat{\mathbf{x}}$ only (see [29]). Normalization of the hyperspherical harmonics is accomplished through the following integral:

$$\int_{\mathbf{S}^{d-1}} Y_n^K(\hat{\mathbf{x}}) \overline{Y_n^K(\hat{\mathbf{x}})}\, \mathrm{d}\Omega = 1,$$

where $d\Omega$ is the Riemannian volume measure on $\mathbf{S}^{d-1}$. The degeneracy, i.e. number of linearly independent solutions for a particular value of n and d, for the space of hyperspherical harmonics is given by

$$(2n + d - 2)\frac{(d - 3 + n)!}{n!(d - 2)!} \tag{26}$$

(see (9.2.11) in [29]). The total number of linearly independent solutions (26) can be determined by counting the total number of terms in the sum over K in (25). Note that this formula (26) reduces to the standard result in $d = 3$ with a degeneracy given by $2n + 1$ and in $d = 4$ with a degeneracy given by $(n + 1)^2$.

One can show the consistency of Corollary 4.2 with the result for $d = 2$ given by

$$\|\mathbf{x} - \mathbf{x}'\|^\nu = \frac{\mathrm{e}^{\mathrm{i}\pi(\nu+1)/2}}{\Gamma(-\nu/2)} \frac{(r_>^2 - r_<^2)^{(\nu+1)/2}}{\sqrt{\pi r r'}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} Q_{m-1/2}^{-(\nu+1)/2}\left(\frac{r^2 + r'^2}{2rr'}\right),$$

where $\nu \in \mathbf{C} \setminus \{0, 2, 4, \ldots\}$, by considering the limit as $\mu \to 0$ in (1) (see (17) above). These expansions are useful in that they allow one to perform azimuthal Fourier and Gegenbauer polynomial analysis for power-law fundamental solutions of the polyharmonic equation on $\mathbf{R}^d$.

5. Conclusion

In this paper, we introduced a generalization of the generating function for Gegenbauer polynomials which allows one to expand arbitrary powers of the distance between two points on d-dimensional Euclidean space $\mathbf{R}^d$ in terms of hyperspherical harmonics. This result has already found physical applications such as in [30], such as obtaining a solution of the momentum-space Schrödinger equation for bound states of the d-dimensional Coulomb problem. The Gegenbauer expansions presented in this paper can be used in conjunction with corresponding Fourier expansions [20] to generate infinite sequences of addition theorems for the Fourier coefficients (see [23]) of these expansions. In future publications, we will present some of these addition theorems as well as extensions related to Fourier and Gegenbauer expansions for logarithmic fundamental solutions of the polyharmonic equation.

Acknowledgements

Much thanks to T.H. Koornwinder for valuable discussions. Part of this work was conducted while H.S. Cohl was a National Research Council Research Postdoctoral Associate in the Applied and Computational Mathematics Division at the National Institute of Standards and Technology, Gaithersburg, Maryland, USA.

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