10.1098/rspa.2000.0656

THE ROYAL SOCIETY

Lattice sums and the two-dimensional, periodic Green's function for the Helmholtz equation

By Andrew Dienstfrey[1]†, Fengbo Hang[1] and Jingfang Huang[2]

[1] *Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA*

[2] *Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

Received 9 December 1999; revised 3 May 2000; accepted 31 May 2000

Many algorithms that are currently used for the solution of the Helmholtz equation in periodic domains require the evaluation of the Green's function, $\mathrm{G}(\boldsymbol{x}, \boldsymbol{x}_0)$. The fact that the natural representation of G via the method of images gives rise to a conditionally convergent series whose direct evaluation is prohibitive has inspired the search for more efficient procedures for evaluating this Green's function. Recently, the evaluation of G through the 'lattice-sum' representation has proven to be both accurate and fast. As a consequence, the computation of the requisite, also conditionally convergent, lattice sums has become an active area of research. We describe a new integral representation for these sums, and compare our results with other techniques for evaluating similar quantities.

Keywords: Helmholtz equation; lattice sums; plane-wave expansion

1. Introduction

In this paper we consider the evaluation of the Green's function for the Helmholtz equation in two dimensions with doubly periodic boundary conditions,

$$\left.\begin{aligned} (\nabla^2 + \beta^2)\mathrm{G}(\boldsymbol{x}, \boldsymbol{x}_0) &= \delta(\boldsymbol{x} - \boldsymbol{x}_0), \\ \mathrm{G}(\boldsymbol{x} + \boldsymbol{e}_1, \boldsymbol{x}_0) = \mathrm{G}(\boldsymbol{x}, \boldsymbol{x}_0), &\quad \boldsymbol{e}_1 = (1, 0), \\ \mathrm{G}(\boldsymbol{x} + \boldsymbol{e}_2, \boldsymbol{x}_0) = \mathrm{G}(\boldsymbol{x}, \boldsymbol{x}_0), &\quad \boldsymbol{e}_2 = (0, 1), \end{aligned}\right\} \tag{1.1}$$

where $\delta(\boldsymbol{y})$ is the Dirac delta function at the origin. Due to periodicity, we need only compute G on a fundamental cell $B_1 \times B_1$, where $B_1 = \{\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \mid |x|, |y| < 1/2\}$. Furthermore, it is a standard fact that G has a convolution structure so that, for any $(\boldsymbol{x}, \boldsymbol{x}_0) \in B_1 \times B_1$, we may consider

$$\begin{aligned} \mathrm{G}(\boldsymbol{x}, \boldsymbol{x}_0) &= G(\boldsymbol{x} - \boldsymbol{x}_0) \\ &= G(\boldsymbol{y}), \end{aligned}$$

with the new variable $\boldsymbol{y} \in 2B_1$. We will make use of this slight abuse of terminology and refer to G as the Green's function below. Finally, the spectrum for the periodic

† Present address: National Institute of Standards and Technology, Information Technology Laboratory, 325 Broadway, Boulder, CO 80303, USA.

Laplacian operator on B_1 is the set

$$\sigma = \{s \in \mathbb{R} \mid \exists m, n \in \mathbb{Z},\ s = (2\pi)\sqrt{m^2+n^2}\}. \tag{1.2}$$

We will assume that $\beta \in \mathbb{R}^+ \backslash \sigma$, in which case G defined by (1.1) exists and is unique.

As periodic Helmholtz equations occur in numerous applications, there is considerable interest in developing algorithms to compute G both accurately and efficiently. A natural idea is to proceed via the method of images. Keeping in mind that the free-space Green's function is given by $-\frac{1}{4}\mathrm{i}H_0(\beta|\boldsymbol{x}|)$ (H_0 is the zeroth-order Hankel function of the first-kind), we define the 'lattice' $\Lambda = \mathbb{Z}^2 \setminus \boldsymbol{0}$ and write G as the formal sum

$$G(\boldsymbol{x}) = -\tfrac{1}{4}\mathrm{i}H_0(\beta|\boldsymbol{x}|) - \tfrac{1}{4}\mathrm{i}\sum_{\boldsymbol{p}\in\Lambda} H_0(\beta|\boldsymbol{x}-\boldsymbol{p}|). \tag{1.3}$$

There are several problems with this representation. For one, $H_0(r) \sim \mathrm{e}^{\mathrm{i}(r-\pi/4)}r^{-1/2}$, hence the sum converges only conditionally. Therefore, for (1.3) to be sensible one must define a summation convention. Furthermore, with the convention specified, one may still anticipate that the convergence is so slow as to be computationally prohibitive.

Surveys of efforts directed at evaluating sums of the form (1.3) are presented in the review articles by Linton (1998), McPhedran *et al*. (1998) and Glasser & Zucker (1980). The evaluation of the two-dimensional, singly periodic analogue of such sums via Ewald's (1921) method was outlined in Linton (1998). This singly periodic case was also treated by Twersky using plane-wave expansions related to the ones we employ, and Euler's summation identity (Twersky 1961). Other approaches have focused on the application of various summation acceleration techniques, e.g. Kummer transformations, in which the principal parts of the asymptotic expansion of the conditionally convergent sums are subtracted from the summand term-by-term and added outside the summand analytically (Linton 1998; Nicorovici & McPhedran 1994*a*). These procedures are effective. However, they are algebraically very involved, and do not allow for significant gains in computational efficiency. Thus, other representations for both the singly and doubly periodic Green's function have been sought (Chin *et al*. 1994; Mathis & Peterson 1996; Nicorovici & McPhedran 1994*a*). Among these we isolate the so-called 'lattice-sum' representation for discussion below.

The lattice-sum representation for G is an immediate consequence of a separation of variables result for H_0 (for the original idea in the context of Laplace's equation see Rayleigh (1892)). Assuming for the moment that $\boldsymbol{x}$ and $\boldsymbol{p} \in \Lambda$ are well separated (see § 3 for a discussion of this point), by Graf's addition theorem:

$$H_0(\beta|\boldsymbol{x}-\boldsymbol{p}|) = \sum_{l=-\infty}^{l=\infty} J_l(\beta|\boldsymbol{x}|)\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{x}}} H_l(\beta|\boldsymbol{p}|)\mathrm{e}^{-\mathrm{i}l\theta_{\boldsymbol{p}}}. \tag{1.4}$$

Substituting (1.4) into (1.3), collecting like terms, we are led to

$$\left.\begin{aligned} G(\boldsymbol{x}) &= -\tfrac{1}{4}\mathrm{i}H_0(\beta|\boldsymbol{x}|) - \tfrac{1}{4}\mathrm{i}\sum_{l=-\infty}^{\infty} S_l(\beta)J_l(\beta|\boldsymbol{x}|)\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{x}}}, \\ S_l(\beta) &= \sum_{\boldsymbol{p}\in\Lambda} H_l(\beta|\boldsymbol{p}|)\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{p}}}. \end{aligned}\right\} \tag{1.5}$$

(The sign of the exponential in (1.5) is irrelevant as the lattice is symmetric.) Restricting to a square lattice, one may check that the four-fold symmetry implies that $S_l = 0$ for l not divisible by four. Rearranging terms we write

$$G(\boldsymbol{x}) = -\tfrac{1}{4}\mathrm{i}\biggl(H_0(\beta|\boldsymbol{x}|) + S_0(\beta)J_0(\beta|\boldsymbol{x}|) + 2\sum_{l=1}^{\infty} S_{4l}(\beta)J_{4l}(\beta|\boldsymbol{x}|)\cos(4l\theta_{\boldsymbol{x}})\biggr). \tag{1.6}$$

In applications, the summation (1.6) is truncated for $l < L$, leading to an evaluation procedure whose cost is proportional to L times the number of evaluation points. In practice this cost is significantly smaller than that necessary to obtain converged values of (1.3), even with the acceleration procedures (Linton 1998; Yasumoto & Yoshitomi 1999).

In a series of papers by McPhedran and co-workers (McPhedran & Dawes 1992; Nicorovici & McPhedran 1994*a*, *b*; Poulton *et al.* 1999), these sums were evaluated by recognizing an identity between the so-called 'spectral' and 'spatial' representations of G. We take a different approach. Our main result is presented in the following theorem.

Theorem 1.1. *Given $\beta \in \mathbb{R}^+ \setminus \sigma$, we define the index*

$$J_\beta = \max\{j \in \mathbb{Z} \mid j(2\pi/\beta) < 1\}. \tag{1.7}$$

Let $\boldsymbol{x} \in B_1$ be given by its polar coordinates $(|\boldsymbol{x}|, \theta_{\boldsymbol{x}})$, then $G(\boldsymbol{x})$ is given by the representation (1.6), where the lattice sums S_{4l} may be expressed as the sum of an 'evanescent' and a 'propagating' part, which, in turn, are given by

$$S_{4l}(\beta) = S_{4l}^{\mathrm{e}} + S_{4l}^{\mathrm{p}}, \tag{1.8}$$

$$\begin{aligned} S_{4l}^{\mathrm{e}} = -\frac{4\mathrm{i}}{\pi}\int_0^\infty \frac{\mathrm{e}^{-t}}{\sqrt{t^2+\beta^2}}\biggl(\frac{1+\cos\sqrt{t^2+\beta^2}}{1+\mathrm{e}^{-2t}-2\mathrm{e}^{-t}\cos\sqrt{t^2+\beta^2}}\biggr) \\ \times\biggl(\biggl(\frac{\sqrt{t^2+\beta^2}-t}{\beta}\biggr)^{4l} + \biggl(\frac{\sqrt{t^2+\beta^2}+t}{\beta}\biggr)^{4l}\biggr)\,\mathrm{d}t, \end{aligned} \tag{1.9}$$

$$\begin{aligned} S_{4l}^{\mathrm{p}} = -\delta_{l,0} + 2\mathrm{i}\sqrt{2}(-1)^l \sum_{j=0}^{J_\beta} \frac{\epsilon_j}{\sqrt{\beta^2-2j^2\pi^2}} \\ \times\cot(\tfrac{1}{4}\sqrt{2}\sqrt{\beta^2-2j^2\pi^2} - \tfrac{1}{2}j\pi)\cos\biggl(4l\arcsin\biggl(\frac{\sqrt{2}j\pi}{\beta}\biggr)\biggr). \end{aligned} \tag{1.10}$$

Note that in the summation (1.10), ϵ_j is the Neumann symbol ($\epsilon_0 = 1$, $\epsilon_j = 2$, $j \geqslant 1$) and $\delta_{l,0}$ is the standard Kronecker delta symbol.

The new feature of theorem 1.1 is the integral-sum representation for the coefficients S_{4l} (see (1.8)–(1.10)). In the future, we intend to extend theorem 1.1 to arbitrary, two-dimensional lattices, and to derive analogous integral-sum representations for the corresponding lattice constants. Such formulae would have significant practical and theoretical interest. However, as the analysis and algebra for the square array are already considerable, we present only this case below. (See also § 3 for further generalizations.) Clearly, the integration corresponding to the evanescent contribution (1.9) must be performed numerically. However, as the integrand is exponentially decaying, this quadrature poses no problem. Finally, inspection of formulae (1.9) and (1.10) reveals that, given a square array, there are symmetries which allow for further simplification of (1.6). Following McPhedran *et al*. (1998), we designate the real and imaginary parts of the lattice sum, $S_l = S_l^J(\beta) + \mathrm{i}S_l^Y(\beta)$. The corresponding splitting of (1.9), (1.10) gives

$$S_{4l}^J(\beta) = -\delta_{l,0},$$

and $S_{4l}^Y(\beta)$ is given by the remaining integral and sum. Rearranging (1.6) we observe

$$G(\boldsymbol{x}) = -\tfrac{1}{4}\left(Y_0(\beta|\boldsymbol{x}|) + S_0^Y(\beta)J_0(\beta|\boldsymbol{x}|) + 2\sum_{l=1}^{\infty} S_{4l}^Y(\beta)J_{4l}(\beta|\boldsymbol{x}|)\cos(4l\theta_{\boldsymbol{x}})\right). \quad (1.11)$$

We refer to both S_l and S_l^Y as the 'lattice sums' below.

We conclude this section with a brief outline of the rest of the paper. In § 2 we derive theorem 1.1 via a series of propositions and computations. The algebra is lengthy; however, we feel it important to give as complete a sketch as possible and thus include many of the details. In this section we also address some of the more subtle technical points. In the interest of clarity we reserve the proofs for Appendix A.

These technical considerations aside, the idea of the theorem is quite elementary. We begin by defining a summation convention for the lattice sums, as they too are only conditionally convergent. For any integer $N \geqslant 1$, we define $\Lambda^N = \{(m,n) \mid (m,n) \in \Lambda,\ |m|,|n| \leqslant N-1\}$. We consider $G^N(\boldsymbol{x})$ as the partial summation (1.3) restricted to Λ^N. Again, employing the addition theorem (1.4) and collecting terms we are led to the corresponding partial lattice sums:

$$S_{4l}^N(\beta) = \sum_{\boldsymbol{p}\in\Lambda^N} H_{4l}(\beta|\boldsymbol{p}|)\mathrm{e}^{\mathrm{i}4l\theta_{\boldsymbol{p}}}. \quad (1.12)$$

In § 2 *a* we derive a standard plane-wave expansion for $H_l(|\boldsymbol{x}|)\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{x}}}$ from the classical plane-wave expansion for H_0 and a differential identity relating Hankel functions of different orders. Each expansion receives contributions from an exponentially decaying and an oscillatory integrand, which we term the evanescent and propagating parts, respectively. In both parts of this expansion, the centres $\{\boldsymbol{p} \in \Lambda^N\}$ in (1.12) appear in the exponents of the integrands. Thus, the sum over centres maps to a geometric series that admits explicit summation. The evanescent integral must be evaluated numerically. However, as it contains exponentially decaying terms, this poses no problem. This is precisely the integral (1.9) above.

The propagating contribution is more involved. As these integrals contain a mixture of highly oscillatory terms and principal-value-type singularities, the large-N limit must be taken with care. We evaluate this limit explicitly with the aid of proposition 2.5, proved below. The result is surprising in that the limits do not exist in the classical sense, but rather are themselves oscillatory, i.e.

$$S_{4l}^{\mathrm{p},N} \sim \sum_{j=-J_\beta}^{J_\beta} A_{4l}^j + \sum_{k=-K_\beta}^{K_\beta} B_{4l}^k \mathrm{e}^{\mathrm{i}\omega_k N}, \quad \omega_k \in \mathbb{R}^+ \setminus 2\pi\mathbb{Z}$$

(see theorem 2.2). This motivates consideration of the weak limit of the summation (definition 2.6). It is a simple matter to see that this standard limiting procedure annihilates the oscillatory terms.

Now, however, it is no longer obvious that the function defined by substituting these weak limits in the lattice sum representation (1.6) yields the desired periodic Green's function. Although we can prove that this weak-limiting procedure is valid, the details are tedious and are omitted. Instead, we rely on numerical demonstration of this fact in § 3. In the same section we compare the computation of the lattice sums via the integral-sum formulae (1.9), (1.10) with previous results existing in the literature.

We mention that the approach outlined above is similar in spirit to the computation of one-dimensional lattice sums performed by Yasumoto & Yoshitomi (1999), and also the computation of two-dimensional and three-dimensional lattice sums for the harmonic equation by one of the authors (Huang 1999).

2. Theory

(*a*) *Plane-wave representation for* $H_l(\beta r)\mathrm{e}^{\mathrm{i}l\theta}$

In this section we present integral representations for Hankel functions of arbitrary order. These representations derive from a plane-wave expansion of H_0, and a differentiation identity relating H_l to H_0.

The subject of plane-wave representations for $H_0(\beta r)$ is classical (Morse & Feshbach 1953). These representations are typically derived via contour integration and Cauchy's theorem. Hence, it is natural that the contours employed depend on the location of the point (x, y). For our purposes it is convenient to divide $\mathbb{R}^2$ into four overlapping regions—north, south, east, west—corresponding to points $(x, y) \in \mathbb{R}^2$ with $y > 0$, $y < 0$, $x > 0$, $x < 0$, respectively. Making the necessary arguments and performing some elementary algebra one arrives at the following,

$$H_0(\beta r) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(y\sin\theta - x\cos\theta)}\,\mathrm{d}\theta + \dfrac{1}{\mathrm{i}\pi}\displaystyle\int_0^{\infty} \dfrac{\mathrm{e}^{-ty}}{\rho_\beta(t)}(\mathrm{e}^{\mathrm{i}\rho_\beta(t)x} + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)x})\,\mathrm{d}t & \text{north}, \\ \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(-y\sin\theta - x\cos\theta)}\,\mathrm{d}\theta + \dfrac{1}{\mathrm{i}\pi}\displaystyle\int_0^{\infty} \dfrac{\mathrm{e}^{ty}}{\rho_\beta(t)}(\mathrm{e}^{\mathrm{i}\rho_\beta(t)x} + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)x})\,\mathrm{d}t & \text{south}, \\ \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(-y\cos\theta + x\sin\theta)}\,\mathrm{d}\theta + \dfrac{1}{\mathrm{i}\pi}\displaystyle\int_0^{\infty} \dfrac{\mathrm{e}^{-tx}}{\rho_\beta(t)}(\mathrm{e}^{\mathrm{i}\rho_\beta(t)y} + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)y})\,\mathrm{d}t & \text{east}, \\ \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(-y\cos\theta - x\sin\theta)}\,\mathrm{d}\theta + \dfrac{1}{\mathrm{i}\pi}\displaystyle\int_0^{\infty} \dfrac{\mathrm{e}^{tx}}{\rho_\beta(t)}(\mathrm{e}^{\mathrm{i}\rho_\beta(t)y} + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)y})\,\mathrm{d}t & \text{west}, \end{cases}$$

where, for convenience, we have defined

$$\rho_\beta(t) = \sqrt{t^2 + \beta^2}.$$

We next require equivalent representations for the higher-order Hankel functions H_l. This may be achieved using the following differential identity

$$H_l(\beta|\boldsymbol{x}|)\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{x}}} = \left(\frac{-1}{\beta}\right)^l \left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)^l H_0(\beta|\boldsymbol{x}|). \tag{2.1}$$

Readers are referred to Hobson (1931) or Greengard *et al*. (1998) for a detailed discussion.

Theorem 2.1. *For a point (x, y) with polar coordinates (r, θ) in $\mathbb{R}^2$, we have the following integral representations, $H_l(\beta r)\mathrm{e}^{\mathrm{i}l\theta}$, which is equal to*

north:

$$\frac{\mathrm{i}^l}{\pi}\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(y\sin\theta - x\cos\theta)}\mathrm{e}^{-\mathrm{i}l\theta}\,\mathrm{d}\theta + \frac{(-\mathrm{i})^l}{\mathrm{i}\pi}\int_0^{\infty}\frac{\mathrm{e}^{-ty}}{\rho_\beta(t)}\left(\mathrm{e}^{\mathrm{i}\rho_\beta(t)x}\left(\frac{\rho_\beta(t)-t}{\beta}\right)^l + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)x}\left(\frac{-\rho_\beta(t)-t}{\beta}\right)^l\right)\mathrm{d}t; \quad (2.2)$$

south:

$$\frac{\mathrm{i}^l}{\pi}\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(-y\sin\theta - x\cos\theta)}\mathrm{e}^{\mathrm{i}l\theta}\,\mathrm{d}\theta + \frac{(-\mathrm{i})^l}{\mathrm{i}\pi}\int_0^{\infty}\frac{\mathrm{e}^{ty}}{\rho_\beta(t)}\left(\mathrm{e}^{\mathrm{i}\rho_\beta(t)x}\left(\frac{\rho_\beta(t)+t}{\beta}\right)^l + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)x}\left(\frac{-\rho_\beta(t)+t}{\beta}\right)^l\right)\mathrm{d}t'; \quad (2.3)$$

east:

$$\frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(x\sin\theta - y\cos\theta)}\mathrm{e}^{\mathrm{i}l\theta}\,\mathrm{d}\theta + \frac{(-1)^l}{\mathrm{i}\pi}\int_0^{\infty}\frac{\mathrm{e}^{-tx}}{\rho_\beta(t)}\left(\mathrm{e}^{\mathrm{i}\rho_\beta(t)y}\left(\frac{-\rho_\beta(t)-t}{\beta}\right)^l + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)y}\left(\frac{\rho_\beta(t)-t}{\beta}\right)^l\right)\mathrm{d}t; \quad (2.4)$$

west:

$$\frac{(-1)^l}{\pi}\int_0^{\pi} \mathrm{e}^{\mathrm{i}\beta(-x\sin\theta - y\cos\theta)}\mathrm{e}^{-\mathrm{i}l\theta}\,\mathrm{d}\theta + \frac{(-1)^l}{\mathrm{i}\pi}\int_0^{\infty}\frac{\mathrm{e}^{tx}}{\rho_\beta(t)}\left(\mathrm{e}^{\mathrm{i}\rho_\beta(t)y}\left(\frac{-\rho_\beta(t)+t}{\beta}\right)^l + \mathrm{e}^{-\mathrm{i}\rho_\beta(t)y}\left(\frac{\rho_\beta(t)+t}{\beta}\right)^l\right)\mathrm{d}t. \quad (2.5)$$

(*b*) *Integral representations of the lattice sums*

In this section we derive theorem 1.1. The idea is that after substituting the integral representations (2.2)–(2.5) into the partial summation (1.12), and summing the resulting geometric series, one obtains explicit integral representations for the partial lattice sums S^N_{4l}. Inspection of these integrals leads to the natural decomposition into evanescent and propagating parts,

$$S^N_{4l} = S^{\mathrm{e},N}_{4l} + S^{\mathrm{p},N}_{4l},$$

corresponding to the exponentially decaying and oscillatory integrals, respectively.

Taking the suitable large-N limit of the evanescent integral is straightforward, giving the integral (1.9). As mentioned previously, evaluation of the propagating contribution is more technical. With the aid of proposition 2.5, we derive the following theorem.

Theorem 2.2. *Given $\beta \in \mathbb{R}^+ \setminus \sigma$, we define the indices*

$$\left.\begin{aligned} J_\beta &= \max\{j \in \mathbb{Z} \mid j(2\pi/\beta) < 1\}, \\ K_\beta &= \max\{k \in \mathbb{Z} \mid k(2\pi/\beta) < \sqrt{2}/2\}, \end{aligned}\right\} \tag{2.6}$$

and the corresponding angles

$$\left.\begin{aligned} \sin(\theta_j^{(0)}) &= \frac{1}{\sqrt{2}}\left(\frac{2\pi}{\beta}\right) j, \quad 0 \leqslant j < J_\beta, \quad \{\theta_j^{(0)}\} \in [0, \tfrac{1}{4}\pi], \\ \cos(\theta_k^{(1)}) &= \left(\frac{2\pi}{\beta}\right) k, \qquad 0 \leqslant k < K_\beta, \quad \{\theta_k^{(1)}\} \in [\pi, \tfrac{1}{2}\pi]. \end{aligned}\right\} \tag{2.7}$$

Then, for large N, the propagating contribution to the lattice sums defined by the partial summation (1.12) can be expressed by the following

$$\begin{aligned} \lim_{N\to\infty} S_{4l}^{\mathrm{p},N} \sim -\delta_{l,0} + \mathrm{i}(-1)^l \left(\frac{2\sqrt{2}}{\beta}\right) \sum_{j=0}^{J_\beta} \epsilon_j \frac{\cos(4l\theta_j^{(0)})}{\cos\theta_j^{(0)}} \cot(\tfrac{1}{2}\beta\cos(\theta_j^{(0)} + \tfrac{1}{4}\pi)) \\ - \mathrm{i}\left(\frac{4}{\beta}\right) \sum_{k=0}^{K_\beta} \epsilon_k \frac{\cos(4l\theta_k^{(1)})}{\sin\theta_k^{(1)}} \left(\frac{\mathrm{e}^{-(\mathrm{i}\beta/2)\sin\theta_k^{(1)}}}{\sin(\frac{1}{2}\beta\sin\theta_k^{(1)})}\right) \mathrm{e}^{N\mathrm{i}\beta\sin\theta_k^{(1)}}, \end{aligned} \tag{2.8}$$

where $\delta_{l,0}$ and ϵ_j are the standard Kronecker delta and Neumann symbols as in theorem 1.1.

We draw attention to the oscillatory terms in (2.8). It is clear that the lattice sums defined by our partial summation do not converge in the classical sense due to these oscillations. Considering Cesaro summation of S_{4l} and the corresponding weak limit (definition 2.6), we have the following immediate corollary.

Corollary 2.3. *Given β, J_β, $\theta_j^{(0)}$ as above,*

$$\begin{aligned} \operatorname*{wk\,lim}_{N\to\infty} S_{4l}^{\mathrm{p},N} &= -\delta_{l,0} + \mathrm{i}(-1)^l \left(\frac{2\sqrt{2}}{\beta}\right) \sum_{j=0}^{J_\beta} \epsilon_j \frac{\cos(4l\theta_j^{(0)})}{\cos\theta_j^{(0)}} \cot(\tfrac{1}{2}\beta\cos(\theta_j^{(0)} + \tfrac{1}{4}\pi)) \\ &= -\delta_{l,0} + \mathrm{i}2\sqrt{2}(-1)^l \sum_{j=0}^{J_\beta} \frac{\epsilon_j}{\sqrt{\beta^2 - 2j^2\pi^2}} \\ &\qquad \times \cot(\tfrac{1}{4}\sqrt{2}\sqrt{\beta^2 - 2j^2\pi^2} - \tfrac{1}{2}j\pi) \cos\left(4l \arcsin\left(\frac{\sqrt{2}j\pi}{\beta}\right)\right), \end{aligned}$$

where the last identity follows from evaluating $\sin\theta_j^{(0)}$, $\cos\theta_j^{(0)}$ using the definition (2.7).

Theorem 1.1 then follows from the substitution of the weak limits given by corollary 2.3 in place of the oscillatory sums of theorem 2.2.

Before performing these computations, we draw attention to some general facts and conventions. In considering sums over the truncated lattice Λ^N, we group the terms as follows ($\boldsymbol{x} = (x, y)$):

$$\sum_{\boldsymbol{x} \in \Lambda^N} f(x,y) = \sum_{n=1}^{N-1} \sum_{m=-n}^{n} f(n,m) + f(-n,m) + \sum_{n=1}^{N-1} \sum_{m=-n+1}^{n-1} f(m,n) + f(m,-n). \tag{2.9}$$

Employing the terminology from the previous section, we see that the first two sums lie strictly in the east and west regions of $\mathbb{R}^2$, while the second two are in the north and south, respectively. (The fact that the lattice points along the diagonal lines $x = \pm y$ are preferentially assigned to the east and west summations has no significance.)

Substituting the plane-wave expansions for $H_l(\boldsymbol{p})\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{p}}}$ into the summation gives rise to certain geometric series. We record the following geometric summation for reference as it appears many times in the expressions below

$$\sum_{n=1}^{N-1} p^n \left(\sum_{m=-n}^{n} + \sum_{m=-n+1}^{n-1} q^m \right) = \left(\frac{1+q}{1-q} \right) \left[\frac{pq^{-1}}{1-pq^{-1}} - \frac{pq}{1-pq} - \frac{(pq^{-1})^N}{1-pq^{-1}} + \frac{(pq)^N}{1-pq} \right]. \tag{2.10}$$

As a final convention, we note that the partial summation of geometric series will give rise to functions that appear to have simple poles in the domain of integration. These singularities are removable in the sense that the terms in (2.10) are differences that cancel to first order at any apparent pole. However, we wish to consider the terms separately, in which case the integration is to be taken in the principal-value sense, e.g. if $f(s)$ has a simple pole singularity at $s = s_0 \in (a, b)$, then by integration of f we mean

$$\int_a^b f(s)\,\mathrm{d}s \equiv \lim_{\epsilon \to 0} \int_a^{s_0-\epsilon} + \int_{s_0+\epsilon}^{b} f(s)\,\mathrm{d}s.$$

As a matter of convenience, we do not write a special integration symbol to denote this operator. As the location of the poles below will be independent of N, we leave it to the reader to check that our interpretation is consistent, i.e. the integration of the difference of terms with a first-order cancellation is equal to the difference of principal-value integrals.

Returning to the lattice sums, we consider the evanescent term first. Upon grouping the terms (2.9), substituting the appropriate plane-wave expansions for the Hankel functions (2.2)–(2.5), applying the geometric summation formulae (2.10), and performing some algebra we obtain

$$\rho_\beta(t) = \sqrt{t^2 + \beta^2},$$

$$S_{4l}^{\mathrm{e},N} = \frac{2}{\pi \mathrm{i}} \int_0^\infty \frac{1}{\rho_\beta(t)} \sum_{n=1}^{N-1} \mathrm{e}^{-tn} \left(\sum_{m=-n}^{n} + \sum_{m=-n+1}^{n-1} \mathrm{e}^{\mathrm{i}\rho_\beta(t)m} \right) \times \left(\left(\frac{\rho_\beta(t)+t}{\beta} \right)^{4l} + \left(\frac{\rho_\beta(t)-t}{\beta} \right)^{4l} \right) \mathrm{d}t$$

$$= +\frac{2}{\pi \mathrm{i}} \int_0^\infty \frac{\mathrm{e}^{-t}}{\rho_\beta(t)} \left(\frac{1+\mathrm{e}^{-\mathrm{i}\rho_\beta(t)}}{1-\mathrm{e}^{-\mathrm{i}\rho_\beta(t)}} \right) \left[\frac{\mathrm{e}^{-\mathrm{i}\rho_\beta(t)}}{1-\mathrm{e}^{-t-\mathrm{i}\rho_\beta(t)}} - \frac{\mathrm{e}^{\mathrm{i}\rho_\beta(t)}}{1-\mathrm{e}^{-t+\mathrm{i}\rho_\beta(t)}} \right] \times \left(\left(\frac{\rho_\beta(t)+t}{\beta} \right)^{4l} + \left(\frac{\rho_\beta(t)-t}{\beta} \right)^{4l} \right) \mathrm{d}t$$

$$- \frac{2}{\pi \mathrm{i}} \int_0^\infty \frac{\mathrm{e}^{-Nt}}{\rho_\beta(t)} \left(\frac{1+\mathrm{e}^{-\mathrm{i}\rho_\beta(t)}}{1-\mathrm{e}^{-\mathrm{i}\rho_\beta(t)}} \right) \left[\frac{\mathrm{e}^{-\mathrm{i}\rho_\beta(t)N}}{1-\mathrm{e}^{-t-\mathrm{i}\rho_\beta(t)}} - \frac{\mathrm{e}^{\mathrm{i}\rho_\beta(t)N}}{1-\mathrm{e}^{-t+\mathrm{i}\rho_\beta(t)}} \right] \times \left(\left(\frac{\rho_\beta(t)+t}{\beta} \right)^{4l} + \left(\frac{\rho_\beta(t)-t}{\beta} \right)^{4l} \right) \mathrm{d}t.$$

We first note that for $\beta = 2\pi m$, $m \geqslant 0$, both integrands in the final expression above exhibit a dipole singularity as $t \to 0$. However, as such β are in σ (1.2), these values are excluded from the analysis. Turning to the N-dependent term, even with the restriction $\beta \neq 2\pi m$, there remain simple poles of the integrand for $\{t_n = \sqrt{(2\pi n)^2 - \beta^2},\ n > 2\pi/\beta,\ n \in \mathbb{Z}^+\}$. Clearly, however, the principal-value limit is uniform with respect to N. Therefore, we apply the 'dominated convergence theorem' and observe that this term goes to zero in the large-N limit. Performing further algebraic simplifications we arrive at the following expression for the evanescent contribution:

$$S_{4l}^{\mathrm{e}} = \frac{4}{\mathrm{i}\pi} \int_0^\infty \frac{\mathrm{e}^{-t}}{\rho_\beta(t)} \left(\frac{1+\cos \rho_\beta(t)}{1+\mathrm{e}^{-2t} - 2\mathrm{e}^{-t} \cos \rho_\beta(t)} \right) \left(\left(\frac{\rho_\beta(t)-t}{\beta} \right)^{4l} + \left(\frac{\rho_\beta(t)+t}{\beta} \right)^{4l} \right) \mathrm{d}t.$$

We next consider the propagating part. Before taking the large-N limit, the analysis is similar to that for the evanescent terms. Without further ado we write

$$X = \mathrm{e}^{\mathrm{i}\beta \cos(\theta)},$$
$$Y = \mathrm{e}^{\mathrm{i}\beta \sin(\theta)},$$

$$S_{4l}^{\mathrm{p},N} = \frac{2}{\pi} \int_0^\pi \cos(4l\theta) \left(\sum_{n=1}^{N-1} Y^n \left(\sum_{m=-n}^{n} + \sum_{m=-n+1}^{n-1} X^m \right) \right) \mathrm{d}\theta$$

$$= +\frac{2}{\pi} \int_0^\pi \cos(4l\theta) \left(\frac{1+X}{1-X} \right) \left[\frac{YX^{-1}}{1-YX^{-1}} - \frac{YX}{1-YX} \right] \mathrm{d}\theta - \frac{2}{\pi} \int_0^\pi \cos(4l\theta) \left(\frac{1+X}{1-X} \right) \left[\frac{(YX^{-1})^N}{1-YX^{-1}} - \frac{(YX)^N}{1-YX} \right] \mathrm{d}\theta$$

$$= +\frac{4}{\pi} \int_0^\pi \cos(4l\theta) \left(\frac{1+X}{1-X} \right) \left[\frac{YX^{-1}}{1-YX^{-1}} \right] \mathrm{d}\theta \tag{2.11}$$

$$- \frac{4}{\pi} \int_0^\pi \cos(4l\theta) \left(\frac{1+X}{1-X} \right) \left[\frac{(YX^{-1})^N}{1-YX^{-1}} \right] \mathrm{d}\theta, \tag{2.12}$$

where, in the final equality, we have made the change of variables $\theta' = \pi - \theta$, which takes $X \to X^{-1}$ while leaving fixed Y, $\cos(4l\theta)$, and the domain of integration.

We dispense with the N-independent term (2.11) in the following proposition.

Proposition 2.4. *Given $\beta \in \mathbb{R}$, $l \in \mathbb{Z}$:*

$$\frac{4}{\pi} \int_0^\pi \cos(4l\theta) \left(\frac{1+\mathrm{e}^{\mathrm{i}\beta\cos(\theta)}}{1-\mathrm{e}^{\mathrm{i}\beta\cos(\theta)}} \right) \left[\frac{\mathrm{e}^{\mathrm{i}\beta(\sin(\theta)-\cos(\theta))}}{1-\mathrm{e}^{\mathrm{i}\beta(\sin(\theta)-\cos(\theta))}} \right] \mathrm{d}\theta = -\delta_{l,0}. \tag{2.13}$$

Proof. See Appendix A. ■

Unlike the evanescent contributions, the N-dependence in (2.12) contributes to an oscillatory integrand, hence the large-N limit requires some care. One observes that there are many values of $\theta \in [0, \pi]$ that will cause the integrands to become singular. As we demonstrate below, a subset of these singularities gives non-zero contributions to the lattice sums in the limit of infinite oscillations. Adding to the complexity of the problem, we will see that the integrals do not converge to a number, but rather have well-defined limits of the form

$$S_{4l}^{\mathrm{p},N} \sim \sum_{j=-J_\beta}^{J_\beta} A_{4l}^{j} + \sum_{k=-K_\beta}^{K_\beta} B_{4l}^{k} \mathrm{e}^{\mathrm{i}\omega_k N}, \quad \omega_k = \beta \sin\theta_k^{(1)}. \tag{2.14}$$

Note that $\omega_k \in \mathbb{R}^+$ and $\omega_k \neq 2\pi n$ as $\beta \notin \sigma$ (see (1.2)). Thus this term is always oscillatory. We introduce the weak-convergence ideas below so as to eliminate the oscillations in (2.14).

The following result concerning asymptotics of oscillatory integrals facilitates the evaluation of (2.12). We reserve the proof for Appendix A.

Proposition 2.5. *Given functions $f(s)$, $g(s)$, $h(s)$ analytic in the interval $s \in [s_0, s_1]$ satisfying the following conditions:*

(1) *h is real and $h' \neq 0$ on $[s_0, s_1]$,*

(2) *g has a single simple zero at $s = a$ in the open interval (s_0, s_1);*

then we have the following limits:

$$\lim_{N\to\infty} \int_{s_0}^{s_1} f(s)\mathrm{e}^{\mathrm{i}Nh(s)}\,\mathrm{d}s = 0, \tag{2.15}$$

$$\lim_{N\to\infty} \int_{s_0}^{s_1} \frac{f(s)\mathrm{e}^{\mathrm{i}Nh(s)}}{g(s)}\,\mathrm{d}s \sim \pi\mathrm{i}\frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)}\,\mathrm{sgn}(h'(a)), \tag{2.16}$$

where $\mathrm{sgn}(x) = x/|x| = \pm 1$ *for $x > 0$ or $x < 0$, respectively.*

Proof. See Appendix A. ■

Returning to the problem, we wish to apply proposition 2.5 to evaluate

$$\lim_{N\to\infty} \left(-\frac{4}{\pi}\right) \int_0^\pi \cos(4l\theta) \left(\frac{1+\mathrm{e}^{\mathrm{i}\beta\cos(\theta)}}{1-\mathrm{e}^{\mathrm{i}\beta\cos(\theta)}}\right) \left[\frac{\mathrm{e}^{\mathrm{i}N\beta(\sin(\theta)-\cos(\theta))}}{1-\mathrm{e}^{\mathrm{i}\beta(\sin(\theta)-\cos(\theta))}}\right] \mathrm{d}\theta.$$

For β fixed and $\theta \in [0, \pi]$, there are two possible types of singularities corresponding to $\beta(\sin\theta - \cos\theta) = 2\pi n$ or $\beta\cos\theta = 2\pi n$. Regarding the former, we will find it more convenient to make the change of variables $\theta = \theta - \pi/4$, which takes $\sin\theta - \cos\theta \to \sqrt{2}\sin\theta'$. Dropping the primes, we are led to consider the points

$$\left.\begin{aligned} \sin(\theta_j^{(0)}) &= \frac{1}{\sqrt{2}}\left(\frac{2\pi}{\beta}\right)j, \quad & \theta_j^{(0)} &\in [-\tfrac{1}{4}\pi, \tfrac{3}{4}\pi], \\ \cos(\theta_k^{(1)}) &= \left(\frac{2\pi}{\beta}\right)k, & \theta_k^{(1)} &\in [0, \pi], \end{aligned}\right\} \quad j, k \in \mathbb{Z}. \tag{2.17}$$

Inspection of (2.17) leads us to define the indices

$$\left.\begin{aligned} J_\beta^* &= \max\{j \mid j(2\pi/\sqrt{2}\beta) < 1\}, \\ K_\beta^* &= \max\{k \mid k(2\pi/\beta) < 1\}. \end{aligned}\right\}$$

As we will see below, there is cancellation between some but *never all* of the pairs of these singularities, e.g. for $J_\beta < j \leqslant J_\beta^*$ (J_β is defined below), the contribution from $\theta_{-j}^{(0)}$ will cancel that from $\theta_j^{(0)}$.

Turning to the set $\{\theta_k^{(1)}\}$, one may verify that the hypotheses of proposition 2.5 are met. Furthermore, in evaluating the limit it is sufficient to restrict the integration (2.12) to small neighbourhoods of these singular points. We compute

$$\begin{aligned} &\lim_{N\to\infty}\left(-\frac{4}{\pi}\right)\int_{\theta_k^{(1)}-\epsilon}^{\theta_k^{(1)}+\epsilon} \cos(4l\theta)\left(\frac{1+\mathrm{e}^{\mathrm{i}\beta\cos(\theta)}}{1-\mathrm{e}^{\mathrm{i}\beta\cos(\theta)}}\right)\left[\frac{\mathrm{e}^{\mathrm{i}N\beta(\sin(\theta)-\cos(\theta))}}{1-\mathrm{e}^{\mathrm{i}\beta(\sin(\theta)-\cos(\theta))}}\right]\mathrm{d}\theta \\ &\quad = \pi\mathrm{i}\left(-\frac{4}{\pi}\right)\cos(4l\theta_k^{(1)})\frac{2}{(1-\mathrm{e}^{\mathrm{i}\beta\sin\theta_k^{(1)}})}\frac{1}{\mathrm{i}\beta\sin\theta_k^{(1)}}\mathrm{e}^{\mathrm{i}N\beta\sin\theta_k^{(1)}}\,\mathrm{sgn}(\cos\theta_k^{(1)}+\sin\theta_k^{(1)}), \end{aligned} \tag{2.18}$$

where we have made repeated use of the fact that

$$\exp(\mathrm{i}\beta\cos\theta_k^{(1)}) = 1.$$

We next make use of the symmetry of the set $\{\theta_k^{(1)}\}$ about $\theta_0^{(1)} = \pi/2$. Clearly, for $0 \leqslant k < K_\beta^*$, one observes that

$$\left\{\begin{aligned} \theta_{-k}^{(1)} &= \tfrac{1}{2}\pi - \Delta\theta_k^{(1)} \\ \theta_k^{(1)} &= \tfrac{1}{2}\pi + \Delta\theta_k^{(1)} \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \sin(\theta_{-k}^{(1)}) &= \sin(\theta_k^{(1)}) \\ \cos(4l\theta_{-k}^{(1)}) &= \cos(4l\theta_k^{(1)}) \end{aligned}\right\},$$

yet

$$\begin{aligned} \mathrm{sgn}(\sin\theta_{-k}^{(1)}+\cos\theta_{-k}^{(1)}) &= \mathrm{sgn}(\sin\theta_k^{(1)}+\cos\theta_k^{(1)}), &\quad |\theta_k^{(1)}-\tfrac{1}{2}\pi| < \tfrac{1}{4}\pi, \\ \mathrm{sgn}(\sin\theta_{-k}^{(1)}+\cos\theta_{-k}^{(1)}) &= -\,\mathrm{sgn}(\sin\theta_k^{(1)}+\cos\theta_k^{(1)}), &\quad |\theta_k^{(1)}-\tfrac{1}{2}\pi| > \tfrac{1}{4}\pi, \end{aligned}$$

as the cosine contribution is dominant. Looking at (2.18), we see that the sum of these two terms will add in the former case and cancel in the later. Therefore, we define $K_\beta = \max\{k|k(2\pi/\beta) < \sqrt{2}/2\}$; this is the index appearing in (2.8).

We next examine the singularities $\{\theta_j^{(0)}\}$. We perform a change of variables to bring the points into a more symmetric position. Considering $\theta' = \theta - \frac{1}{4}\pi$, one observes that (again we employ the shorthand $X = \exp(\mathrm{i}\beta\cos(\theta))$, $Y = \exp(\mathrm{i}\beta\sin(\theta))$):

$$\begin{aligned} &-\frac{4}{\pi}\int_0^{\pi}\cos(4l\theta)\left(\frac{1+X}{1-X}\right)\frac{(YX^{-1})^N}{1-YX^{-1}}\,\mathrm{d}\theta \\ &\qquad = (-1)^{l+1}\frac{4}{\pi}\int_{-\pi/4}^{3\pi/4}\cos(4l\theta)\left(\frac{1+(XY^{-1})^{1/\sqrt{2}}}{1-(XY^{-1})^{1/\sqrt{2}}}\right)\frac{Y^{\sqrt{2}N}}{1-Y^{\sqrt{2}}}\,\mathrm{d}\theta. \end{aligned}$$

In this variable, the singularities are clearly $\{\theta_j^{(0)}\}$ defined in (2.17). Applying proposition 2.5 once again we see that

$$
\begin{aligned}
&(-1)^{l+1}\left(\frac{4}{\pi}\right)\lim_{N\to\infty}\int_{\theta_j^{(0)}-\epsilon}^{\theta_j^{(0)}+\epsilon}\cos(4l\theta)\left(\frac{1+\mathrm{e}^{\mathrm{i}\beta(\cos(\theta)-\sin(\theta))/\sqrt{2}}}{1-\mathrm{e}^{\mathrm{i}\beta(\cos(\theta)-\sin(\theta))/\sqrt{2}}}\right)\left[\frac{\mathrm{e}^{N\mathrm{i}\beta\sin(\theta)\sqrt{2}}}{1-\mathrm{e}^{\mathrm{i}\beta\sin(\theta)\sqrt{2}}}\right]\mathrm{d}\theta \\
&\quad=(-1)^{l+1}\left(\frac{4}{\pi}\right)\pi\mathrm{i}\cos(4l\theta_j^{(0)})\left(\frac{1+\mathrm{e}^{\mathrm{i}\beta(\cos(\theta_j^{(0)}+(\pi/4)))}}{1-\mathrm{e}^{\mathrm{i}\beta(\cos(\theta_j^{(0)}+(\pi/4)))}}\right)\frac{1}{-\sqrt{2}\mathrm{i}\beta\cos\theta_j^{(0)}}\operatorname{sgn}(\cos\theta_j^{(0)}).
\end{aligned}
\tag{2.19}
$$

Note that the set $\{\theta_j^{(0)}\}$ may be considered as two subsets $-\pi/4<\theta_j^{(0)}<\pi/4$ symmetric about $\theta=0$, and $\pi/4<\theta_j^{(0)}<3\pi/4$ symmetric about $\theta=\pi/2$. We observe that the sum of the contributions from the latter cancel due to the change in sign of $\operatorname{sgn}(\cos\theta_j^{(0)})$. Therefore, as before, we need only consider contributions from the singularities $\{\theta_j^{(0)}\mid |j|<J_\beta\}$, where $J_\beta=\max\{j\mid j(2\pi/\beta)<1\}$.

We collect the results, (2.18), (2.19), perform more algebra, and arrive at the statement of theorem 2.2:

$$
\begin{aligned}
\lim_{N\to\infty}S_{4l}^{\mathrm{p},N}\sim{}&-\delta_{l,0}+\mathrm{i}(-1)^l\left(\frac{2\sqrt{2}}{\beta}\right)\sum_{j=0}^{J_\beta}\epsilon_j\frac{\cos(4l\theta_j^{(0)})}{\cos\theta_j^{(0)}}\cot(\tfrac{1}{2}\beta\cos(\theta_j^{(0)}+\tfrac{1}{4}\pi)) \\
&-\mathrm{i}\left(\frac{4}{\beta}\right)\sum_{k=0}^{K_\beta}\epsilon_k\frac{\cos(4l\theta_k^{(1)})}{\sin\theta_k^{(1)}}\left(\frac{\mathrm{e}^{-(\mathrm{i}\beta/2)\sin\theta_k^{(1)}}}{\sin((\beta/2)\sin\theta_k^{(1)})}\right)\mathrm{e}^{N\mathrm{i}\beta\sin\theta_k^{(1)}}.
\end{aligned}
$$

We will discard the oscillating terms above. Formally, this corresponds to interpreting S_{4l} as a weak-limit of S_{4l}^N, in the following sense.

Definition 2.6. Given a sequence $\{a_n\}$, we define the weak limit by

$$\operatorname*{wk\,lim}_{n\to\infty}a_n=\lim_{n\to\infty}\frac{\sum_{i=1}^n a_i}{n}.$$

The above is nothing more than the standard definition of Cesaro summation of a series. It is an easy matter to verify (or see Zygmond 1959) the following proposition.

Proposition 2.7. *Given a sequence* $\{b_n\}$ *with* $\lim\{b_n\}=b$ *and* $\omega\in\mathbb{R}\setminus 2\pi\mathbb{Z}$, *then*

$$
\begin{aligned}
\operatorname*{wk\,lim}_{n\to\infty}b_n&=b,\\
\operatorname*{wk\,lim}_{n\to\infty}b_n\mathrm{e}^{\mathrm{i}\omega n}&=0.
\end{aligned}
$$

Applying this proposition to the result for the lattice sums (theorem 2.2) gives the statement of corollary 2.3.

3. Numerical results

As a first test of our result, we compare the values of the lattice sums computed via theorem 1.1 with the results of computations using a dual-lattice summation identity described by Chin *et al.* (1994). As the values of S_{4l} have been tabulated previously in the literature, we do not report an entire table here but, rather, only a

Table 1. *Numerical values of $S_0(\beta)$, $S_{12}(\beta)$ and $S_{24}(\beta)$ given by our method, and reported in Chin et al. (1994)*

β	S_0: theorem 1.1	S_0: Chin *et al.* (1994)
2	1.39962×10^0	1.39963×10^0
10.9548	-0.14189×10^0	-0.14187×10^0
20	6.46161×10^0	6.46161×10^0
	S_{12}: theorem 1.1	S_{12}: Chin *et al.* (1994)
2	-5.47552×10^7	-5.47550×10^7
10.9548	-2.16949×10^0	-2.16950×10^0
20	-4.53737×10^0	-4.53738×10^0
	S_{24}: theorem 1.1	S_{24}: Chin *et al.* (1994)
2	-3.43888×10^{22}	-3.43888×10^{22}
10.9548	-2.38213×10^5	-2.38225×10^5
20	-3.65184×10^0	-3.65184×10^0

general 'cross-section' (table 1). The formulae (1.9), (1.10) for the lattice sums were implemented in Fortran. As the summation (1.10) is finite, the only potentially intensive computation would be the quadrature (1.9). Over a range of wavenumbers and indices (β and $4l$), we found these integrals to converge with a relative error $O(10^{-15})$ employing N-point Gauss–Laguerre quadrature rules with $N < 100$. Naturally, more sophisticated quadrature rules could be developed that would reduce this burden even further. Therefore, assuming minimal overhead for the evaluation of standard special functions, we conclude that our technique requires $O(10^2)$ operations for the computation of each lattice sum. As a rough comparison of efficiency, Chin *et al.* (1994) report their results of summations over a 201×201 square lattice, i.e. $O(10^4)$ operations for each sum. It was brought to our attention by an anonymous referee that the form of the integrand may be simplified considerably by a judicious change of variables; letting $t = \beta \sinh \alpha$, we see that

$$S^{\mathrm{e}}_{4l} = -\frac{8\mathrm{i}}{\pi} \int_0^\infty \mathrm{e}^{-\beta \sinh \alpha} \left(\frac{1 + \cos(\beta \cosh \alpha)}{1 - 2\mathrm{e}^{-\beta \sinh \alpha} \cos(\beta \cosh \alpha) + \mathrm{e}^{-2\beta \sinh \alpha}} \right) \cosh 4l\alpha \, \mathrm{d}\alpha.$$

We note the significant aesthetic improvement of the above formulae over (1.9). However, we have not attempted any systematic study of the computational efficiency of one over the other.

With regards to the substitution of the lattice sums defined by (1.5) with the weak limits of the partial summation (1.12), we have a rigorous proof that this weak limit of G^N converges to G. However, the idea is standard and the estimates technical. Instead, we adopt an empirical approach. For β not an eigenvalue of the periodic Laplace equation on B_1, G defined in (1.1) exists and is unique. Subtracting the singular term, $H_0(\beta|\boldsymbol{x}|)$, from G, using the fact that $\{J_l(|\boldsymbol{x}|)\mathrm{e}^{\mathrm{i}l\theta_{\boldsymbol{x}}}\}$ form a basis of smooth solutions to the Helmholtz equation, and employing the symmetries of the problem, one knows *a priori* that a representation of the form (1.6) also exists and is unique. Therefore, we need only check that the Bessel expansion in theorem 1.1 converges, and is periodic.

The convergence of the sum representation of the periodic Green's function requires brief discussion. It is a general fact that such an expansion converges only when there exists separation between the evaluation points and the images of the singular term at the origin, ie. $|\boldsymbol{x}| \leqslant c < 1$. This is a common situation for readers familiar with multipole-type codes. The remedy is simply to include more of the singular source terms explicitly in the formulae (1.5) and modify the lattice sums so as to reflect this, e.g.

$$G(\boldsymbol{x}) = -\tfrac{1}{4}\mathrm{i}\Bigg(\sum_{\boldsymbol{p}\in\mathcal{N}} H_0(\beta|\boldsymbol{x}-\boldsymbol{p}|) + \tilde{S}_0(\beta)J_0(\beta|\boldsymbol{x}|) + 2\sum_{l=1}^{\infty}\tilde{S}_{4l}J_{4l}(\beta|\boldsymbol{x}|)\cos(4l\theta_{\boldsymbol{x}})\Bigg), \tag{3.1}$$

where $\mathcal{N}$ denotes the origin and its nearest neighbours

$$\mathcal{N} = \{(-\infty,\infty),(\prime,\infty),(\infty,\infty),(-\infty,\prime),(\prime,\prime),(\infty,\prime),(-\infty,-\infty),(\prime,-\infty),(\infty,-\infty)\}$$

and $\tilde{S}_{4l}(\beta)$ are defined by

$$\tilde{S}_{4l}(\beta) = \sum_{\boldsymbol{p}\in\mathbb{Z}^2\setminus\mathcal{N}} H_{4l}(\beta|\boldsymbol{p}|)\mathrm{e}^{\mathrm{i}4l\phi_{\boldsymbol{p}}} = S_{4l} - \sum_{\mathcal{N}\setminus\boldsymbol{0}} H_{4l}(\beta|\boldsymbol{p}|)\mathrm{e}^{\mathrm{i}4l\phi_{\boldsymbol{p}}}.$$

We refer the reader to Greengard & Rokhlin (1987) and Berman & Greengard (1994) for a detailed discussion of this type of acceleration procedure applied to fast multipole periodic Laplace solvers. This modification is an analytical necessity but neither adds nor subtracts from the question of convergence of our formulae to the true Green's function. We implement this modification below.

As to the periodicity, we return to the conventional Green's function $\mathrm{G}(\boldsymbol{x},\boldsymbol{x}_0)$ (1.1) and ascertain that its representation via (3.1) is periodic under arbitrary placement of the 'source' point $\boldsymbol{x}_0 \in [-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$. As expression (3.1) is invariant under rotation by $\pi/2$, we need only test for periodicity in the y variable. We truncate the summation to $4L$ and define

$$\mathrm{err}_L(\boldsymbol{x}_0) = \max_{\substack{\boldsymbol{x}=(x,-1/2)\\ -1/2\leqslant x\leqslant 1/2}} \{|\mathrm{G}(\boldsymbol{x}+\boldsymbol{e}_2;\boldsymbol{x}_0) - \mathrm{G}(\boldsymbol{x};\boldsymbol{x}_0)|\}.$$

Clearly, this difference should be zero. In figure 1 we plot $\log_{10}\mathrm{err}$ to determine the accuracy of our results for values of $L = 4$ and 8. In both cases the error is least for $\boldsymbol{x}_0$ near the origin and increases as $\boldsymbol{x}_0$ approaches the boundary of the fundamental cell. We see that the error goes down as L increases, and that for $L = 8$ the representation gives approximately seven-digit accuracy. Therefore, we conclude that the weak-limiting procedure converges to the true Green's function as claimed.

To our knowledge, the representation of $S_{4l}(\beta)$ as a closed-form integral and sum (1.9), (1.10) is new. We are in the process of exploring the analytical consequences implied by this representation. As a simple first step, one may simply enquire as to the relative strength of the two contributions to the total sum. In figure 2 we

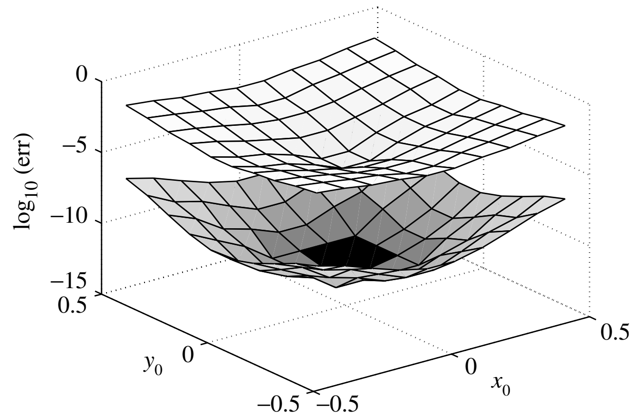

Figure 1. Plot of convergence ($L = 4$ and 8, top and bottom surfaces, respectively) and periodicity of G given by (3.1) ($\boldsymbol{x}_0 = (x_0, y_0)$, $\beta = 10.9548$).

plot the values of $S_l^{\mathrm{p}}(\beta)$ and $S_l^{\mathrm{e}}(\beta)$ as functions of β for $l = 0$. The eigenvalues $\beta = 2\pi\sqrt{m^2+n^2}$ appear clearly as poles of S_0. Inspection of (1.9) reveals that for $\beta = 2\pi n$, the evanescent integral (1.9) has a second-order singularity at $t = 0$, which gives rise to a pole contribution to the lattice sum. The behaviour of the propagating part is more subtle. As J_β (see theorem 1.1) is defined by

$$J_\beta = \max\{j \in \mathbb{Z} \mid j(2\pi/\beta) < 1\},$$

one may check that

$$\lim_{\beta \to (2\pi n)_-} S^{\mathrm{p}}(\beta) < \infty,$$
$$\lim_{\beta \to (2\pi n)_+} S^{\mathrm{p}}(\beta) = \infty,$$

as, in the latter case, one may let $j = n$ in the summation (1.10). In contrast, for $\beta = 2\pi\sqrt{m^2+n^2}$, the singularity is due only to the cotangent term in the propagating sum (1.10), and the evanescent integral is clearly finite. For all other values of β, the two contributions are comparable.

Finally, we address possible extensions of our analysis. One of the many physical problems that gives rise to a two-dimensional, periodic Helmholtz equation is electromagnetic scattering by an array of infinite cylinders. In such an experiment, strict periodicity is equivalent to the requirement that the incident radiation is normal to the cylindrical axis. Scattering by off-axis radiation is accommodated by replacing the periodicity condition on G with a phase-shift or 'quasi-periodicity' factor,

$$G(\boldsymbol{x} + \boldsymbol{e}_i) = G(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{e}_i \boldsymbol{k}_\perp},$$

where $\boldsymbol{k}_\perp$ is the x, y-plane component of the incident light, and $\boldsymbol{e}_i$ is a translation by a lattice generator. At the level of the lattice sums, this exponential factor appears inside the summand (1.5). We are considering incorporating this term into our analysis. Furthermore, one may consider more general lattices. Although the Green's function representation under a general lattice must be modified in the case of large 'aspect ratio', the computation of the sums proceeds in an entirely analogous manner. We are currently considering this extension.

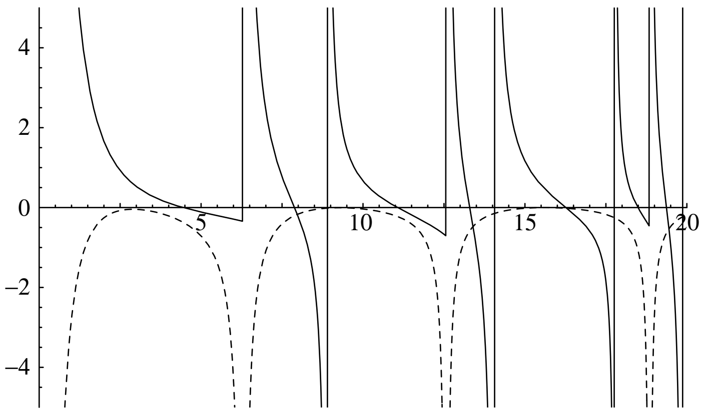

Figure 2. Comparison of the magnitudes of S_0^{p} (solid line) and S_0^{e} (dashed line) as functions of β.

The authors thank L. Greengard and R. C. McPhedran for their valuable discussions concerning all aspects of this work. Furthermore, the comments of anonymous referees were helpful in clarifying the presentation of our results. A.D. was supported by the DARPA/NSF Virtual Integrated Prototyping (VIP) initiative. J.H. was partly supported by grants from DARPA Composite CAD Program, the DARPA MURI Program and the Semiconductor Research Corporation.

Appendix A.

(a) *Derivation of proposition 2.4*

We prove (2.13) of proposition 2.4. Although not obvious at first glance, we choose to work with the integral prior to (2.11) as it is real-valued:

$$
\begin{aligned}
&\frac{4}{\pi}\int_0^{\pi}\left(\frac{1+\mathrm{e}^{\mathrm{i}\beta\cos(t)}}{1-\mathrm{e}^{\mathrm{i}\beta\cos(t)}}\right)\left[\frac{\mathrm{e}^{\mathrm{i}\beta(\sin(t)-\cos(t))}}{1-\mathrm{e}^{\mathrm{i}\beta(\sin(t)-\cos(t))}}\right]\cos(4lt)\,\mathrm{d}t\\
&\quad=-\frac{2}{\pi}\int_0^{\pi}\left(\frac{1+\mathrm{e}^{\mathrm{i}\beta\cos(t)}}{1-\mathrm{e}^{\mathrm{i}\beta\cos(t)}}\right)\left[\frac{\mathrm{e}^{\mathrm{i}\beta(\sin(t)-\cos(t))}}{1-\mathrm{e}^{\mathrm{i}\beta(\sin(t)-\cos(t))}}-\frac{\mathrm{e}^{\mathrm{i}\beta(\sin(t)+\cos(t))}}{1-\mathrm{e}^{\mathrm{i}\beta(\sin(t)+\cos(t))}}\right]\cos(4lt)\,\mathrm{d}t\\
&\quad=\frac{2}{\pi}\int_0^{\pi}\frac{1+\cos(\beta\cos(t))}{\cos(\beta\cos(t))-\cos(\beta\sin(t))}\cos(4lt)\,\mathrm{d}t.
\end{aligned}
$$

We define $f(t;\beta)$:

$$
f(t;\beta)=\frac{1+\cos(\beta\cos(t))}{\cos(\beta\cos(t))-\cos(\beta\sin(t))}.
$$

We first show that the integral is independent of β:

$$
\frac{\partial f}{\partial\beta}=\frac{(\cos(\beta\sin(t))+1)\cos(t)\sin(\beta\cos(t))}{(\cos(\beta\cos(t))-\cos(\beta\sin(t)))^2}-\frac{(1+\cos(\beta\cos(t)))\sin(t)\sin(\beta\sin(t))}{(\cos(\beta\cos(t))-\cos(\beta\sin(t)))^2}.
$$

Performing the change of variable to shift the integration, one finds

$$\begin{aligned}
\int_0^\pi \frac{\partial f}{\partial \beta}\cos(4lt)\,\mathrm{d}t
&= +\int_{-\pi/2}^{\pi/2} \frac{(\cos(\beta\cos(t))+1)\sin(t)\sin(\beta\sin(t))}{(\cos(\beta\sin(t))-\cos(\beta\cos(t)))^2}\cos(4lt)\,\mathrm{d}t \\
&\quad -\int_{-\pi/2}^{\pi/2} \frac{(\cos(\beta\sin(t))+1)\cos(t)\sin(\beta\cos(t))}{(\cos(\beta\sin(t))-\cos(\beta\cos(t)))^2}\cos(4lt)\,\mathrm{d}t \\
&= +\int_{-\pi/2}^{0} + \int_0^{\pi/2} \frac{(\cos(\beta\cos(t))+1)\sin(t)\sin(\beta\sin(t))}{(\cos(\beta\sin(t))-\cos(\beta\cos(t)))^2}\cos(4lt)\,\mathrm{d}t \\
&\quad -\int_{-\pi/2}^{0} - \int_0^{\pi/2} \frac{(\cos(\beta\sin(t))+1)\cos(t)\sin(\beta\cos(t))}{(\cos(\beta\sin(t))-\cos(\beta\cos(t)))^2}\cos(4lt)\,\mathrm{d}t \\
&= 0,
\end{aligned}$$

as the integrals 'cross-cancel', i.e. changing $t' = t - \pi/2$ in the last integral cancels the first, and similarly for the second and third.

Next we evaluate the integral in the limit of small β. First note that, arguing as above, one observes that

$$\int_0^\pi \frac{1}{\cos(\beta\cos(t))-\cos(\beta\sin(t))}\cos(4lt)\,\mathrm{d}t = 0.$$

Thus we may ignore constant terms appearing in the numerator. Assuming $\beta \to 0$, we obtain

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi f(t;\beta)\cos(4lt)\,\mathrm{d}t
&= -\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1+\cos(\beta\sin(t))}{\cos(\beta\cos(t))-\cos(\beta\sin(t))}\cos(4lt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\beta^2\sin^2(t)+O(\beta^4)}{\cos(\beta\cos(t))-\cos(\beta\sin(t))}\cos(4lt)\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{-\sin^2(t)}{\cos^2(t)-\sin^2(t)}\cos(4lt)\,\mathrm{d}t + O(\beta^2) \\
&= -\frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2}\cos(4lt)\,\mathrm{d}t - \int_{-\pi/2}^{\pi/2}\frac{1}{\cos^2(t)-\sin^2(t)}\cos(4lt)\,\mathrm{d}t\right) \\
&= \begin{cases} -1, & l = 0, \\ 0, & l = \pm 1, \pm 2, \dots, \end{cases}
\end{aligned}$$

as, once again, one may show that the second integral in the penultimate line evaluates to 0 by splitting the domain of integration and performing a change of variables. This is the desired identity.

(*b*) *Proof of proposition 2.5*

Formula (2.15) is simply a corollary of the Riemann–Lebesgue lemma concerning asymptotics of Fourier coefficients. The idea is that the non-vanishing condition on the derivative of h allows one to change the scale of the integration. Specifically, as $h'(s) \neq 0$ by the 'inverse function theorem', one may find an analytic inverse function $s = s(h)$. Pulling back the integration we see that

$$\begin{aligned}\lim_{N\to\infty} \int_{s_0}^{s_1} f(s)\mathrm{e}^{\mathrm{i}Nh(s)}\,\mathrm{d}s &= \lim_{N\to\infty} \int_{h(s_0)}^{h(s_1)} f(s(h))\mathrm{e}^{\mathrm{i}Nh} s'(h)\,\mathrm{d}h \\ &= \lim_{N\to\infty} \int_{h(s_0)}^{h(s_1)} \tilde{f}(h)\mathrm{e}^{\mathrm{i}Nh}\,\mathrm{d}h \\ &= 0.\end{aligned}$$

To derive the second formula (2.16), we first consider smooth $f(s)$ and the simpler case,

$$\begin{aligned}\lim_{N\to\infty} \int_{-\epsilon}^{+\epsilon} \frac{f(s)}{s}\mathrm{e}^{\mathrm{i}Ns}\,\mathrm{d}s &= \lim_{N\to\infty} \int_{-\epsilon}^{+\epsilon} \frac{f(0)+f'(0)s+O(s^2)}{s}\mathrm{e}^{\mathrm{i}Ns}\,\mathrm{d}s \\ &= f(0) \lim_{N\to\infty} \int_{-\epsilon}^{+\epsilon} \frac{1}{s}\mathrm{e}^{\mathrm{i}Ns}\,\mathrm{d}s \qquad \text{(A 1)} \\ &= f(0) \int_{-\infty}^{\infty} \frac{1}{s}\mathrm{e}^{\mathrm{i}s}\,\mathrm{d}s \\ &= \pi\mathrm{i}f(0). \qquad \text{(A 2)}\end{aligned}$$

Equation (A 1) follows from the fact that the remaining terms of the Taylor expansion give rise to a smooth integrand that, in the limit, evaluates to 0. Equality (A 2) is standard.

We turn to (2.16) of proposition 2.5. First, one may multiply by a smooth 'bump function' supported in a neighbourhood of the singularity $s = a$, apply (2.15), and thereby see that it is sufficient to consider the restriction of the integral to any neighbourhood of a. Making use of analyticity, performing algebra, and applying (A 2), we see that

$$\begin{aligned}\lim_{N\to\infty} \int_{s_0}^{s_1} \frac{f(s)}{g(s)}\mathrm{e}^{\mathrm{i}Nh(s)}\,\mathrm{d}s &= \lim_{N\to\infty} \int_{a-\epsilon}^{a+\epsilon} \frac{f(s)}{g(s)}\mathrm{e}^{\mathrm{i}Nh(s)}\,\mathrm{d}s \\ &= \lim_{N\to\infty} \int_{-\epsilon}^{+\epsilon} \frac{f(a)+f'(a)s+O(s^2)}{g'(a)s+O(s^2)}\mathrm{e}^{\mathrm{i}Nh(a+s)}\,\mathrm{d}s \\ &= \lim_{N\to\infty} \frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)} \int_{-\epsilon}^{+\epsilon} \frac{1}{s}\mathrm{e}^{\mathrm{i}N(h(a+s)-h(a))}\,\mathrm{d}s \\ &= \lim_{N\to\infty} \frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)} \int_{h(a-\epsilon)-h(a)}^{h(a+\epsilon)-h(a)} \frac{1}{s(h)}\mathrm{e}^{\mathrm{i}Nh}s'(h)\,\mathrm{d}h \\ &= \lim_{N\to\infty} \frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)} \int_{\epsilon_1}^{\epsilon_2} \frac{1}{s(h)}\mathrm{e}^{\mathrm{i}Nh}s'(h)\,\mathrm{d}h,\end{aligned}$$

where $\epsilon_1 < 0 < \epsilon_2$ if $h'(a) > 0$, and $\epsilon_2 < 0 < \epsilon_1$ otherwise. We let $\tilde{\epsilon} = \min\{|\epsilon_1|, |\epsilon_2|\}$ and rearrange the integration limits according to the sign of this derivative. Finally we see

$$\begin{aligned}\lim_{N\to\infty}\int_{s_0}^{s_1}\frac{f(s)}{g(s)}\mathrm{e}^{\mathrm{i}Nh(s)}\,\mathrm{d}s &= \operatorname{sgn}(h'(a))\lim_{N\to\infty}\frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)}\frac{1}{s'(0)}\int_{-\tilde{\epsilon}}^{\tilde{\epsilon}}\frac{1}{h}\mathrm{e}^{\mathrm{i}Nh}s'(h)\,\mathrm{d}h\\ &= \operatorname{sgn}(h'(a))\lim_{N\to\infty}\frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)}\frac{1}{s'(0)}\int_{-\tilde{\epsilon}}^{\tilde{\epsilon}}\frac{1}{h}\mathrm{e}^{\mathrm{i}Nh}s'(h)\,\mathrm{d}h\\ &\sim \pi\mathrm{i}\frac{f(a)}{g'(a)}\mathrm{e}^{\mathrm{i}Nh(a)}\operatorname{sgn}(h'(a)).\end{aligned}$$

References

Berman, C. L. & Greengard, L. 1994 A renormalization method for the evaluation of lattice sums. *J. Math. Phys.* **35**, 6036.

Chin, S. K., Nicorovici, N. A. & McPhedran, R. C. 1994 Green's function and lattice sums for electromagnetic scattering by a square array of cylinders. *Phys. Rev.* E **49**, 4590.

Ewald, P. 1921 Die Berechnung optischer und elektrostatischer Gitterpotentiale. *Ann. Phys.* **64**, 253–287.

Glasser, M. L. & Zucker, I. J. 1980 *Theoretical chemistry: advances and perspectives*, p. 67. Academic.

Greengard, L. & Rokhlin, V. 1987 A fast algorithm for particle simulations. *J. Comp. Phys.* **73**, 325–334.

Greengard, L., Huang, J. F., Rokhlin, V. & Wandzura, S. 1998 Accelerating fast multipole methods for the Helmholtz equation at low frequencies. *IEEE Comp. Sci. Engng* **5**, 32–38.

Hobson, E. W. 1931 *The theory of spherical and ellipsoidal harmonics*. Cambridge University Press.

Huang, J. F. 1999 Integral representations of harmonic lattice sums. *J. Math. Phys.* **40**, 5240.

Linton, C. M. 1998 The Green's function for the two dimensional Helmholtz equation in periodic domains. *J. Engng Math.* **33**, 377.

McPhedran, R. C. & Dawes, D. H. 1992 Lattice sums for an electromagnetic scattering problem. *J. Electromag. Waves Appl.* **6**, 1327.

McPhedran, R. C., Nicorovici, N. A., Botten, L. C. & Bao Ke-Da 1998 Green's function, lattice sums and Rayleigh's identity for a dynamic scattering problem. In *Wave propagation in complex media* (ed. G. Papanicolaou). IMA Volumes in Mathematics and its Applications, vol. 96. Springer.

Mathis, A. W. & Peterson, A. F. 1996 A comparison of acceleration procedures for the two-dimensional periodic Green's function. *IEEE Trans. Antennas Propagation* **44**, 567.

Morse, P. M. & Feshbach, H. 1953 *Methods of theoretical physics*. New York: McGraw-Hill.

Nicorovici, N. A. & McPhedran, R. C. 1994*a* Efficient calculation of the Green's function for electromagnetic scattering by gratings. *Phys. Rev.* E **49**, 4563.

Nicorovici, N. A. & McPhedran, R. C. 1994*b* Lattice sums for off-axis electromagnetic scattering by gratings. *Phys. Rev.* E **50**, 3143.

Poulton, C. G., Botten, L. C., McPhedran, R. C. & Movchan, A. B. 1999 Source-neutral Green's functions for periodic problems in electrostatics, and their equivalents in electromagnetism. *Proc. R. Soc. Lond.* A **455**, 1107.

Rayleigh, Lord 1892 *Phil. Mag.* **34**, 481.

Twersky, V. 1961 Elementary function representations of Schlomilch series. *Arch. Ration. Mech. Analysis* **8**, 323–332.

Yasumoto, K. & Yoshitomi, K. 1999 Efficient calculation of lattice sums for free space periodic Green's function. *IEEE Trans. Antennas Propagation* **47**, 1050.

Zygmond, A. 1959 *Trignometric series*, vol. 1. Cambridge University Press.