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Uncertainties of Frequency Response Estimates Derived from Responses To Uncertain Step-Like Inputs

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Abstract - The frequency response of a linear time-invariant system can be estimated from the measurement of its response to an ideal step input. However, an ideal step is unrealizable, and various other error sources affect the accuracy of such estimates. This paper investigates the effect of using an uncertain (inexactly known), step-like test signal. An approach is developed here for determining the systematic uncertainties of the frequency response estimate of a device under test (DUT), when it is estimated from the response of the DUT to the uncertain, step-like test signal. The time-domain uncertainties of the test signal, and those of the DUT response, are converted to the frequency domain and processed, resulting in uncertainties for the frequency response of the DUT. Also, a mathematical proof is provided for the "envelope-modulation" method of calculating the systematic uncertainties of a frequency response estimate of a device, as derived from the uncertain response of the device to an ideal step.

I. INTRODUCTION

Previous research has shown that the frequency responses of filters, digital oscilloscopes, analog-to-digital converters (ADCs), and linear systems in general can be estimated effectively from discrete-time step response measurements (e.g., [1-5]). Earlier studies have analyzed error sources that can affect the frequency response estimates, such as noise, jitter, aliasing, and derivative estimation errors. In addition to these error sources, it is important to know the uncertainties of the estimated frequency content of the step-like test signal. If known or bounded, these uncertainties can be used to determine the resulting uncertainties in frequency response estimates of DUTs that are measured using the step-like test signal. In a recent paper [6], we examined the first part of this problem, namely, determining the uncertainties of the estimate of the *equivalent frequency response* of a step-like signal generator, based on the systematic uncertainties of a time-domain measurement of its output (Fig. 1). By "equivalent frequency response," we mean the frequency response of the linear network that would produce the step-like signal in response to an ideal input step. The present work outlines a method for solving the second part of the problem, computing the systematic uncertainties of the estimated frequency response of a DUT when the uncertain step-like signal is the test input. By "uncertain," we mean that the signal is not known with complete certainty: the measurement system that is available can only measure or estimate the signal to within some uncertainties.

In [6], we developed two methods of determining the systematic uncertainties of the equivalent frequency response of a step-like signal generator. Of these, the so-called "envelope-modulation" method warrants further attention because it was offered without proof, but it produces smaller frequency response uncertainties. A mathematical proof of the envelope-modulation method is provided in this paper, in the Appendix. Because of its smaller uncertainties, that method is preferred, and is used in the present work.

The main contribution of this work is outlining the application of the envelope-modulation method to the practical problem of finding the uncertainties of the estimate of the frequency response of a DUT from its response to an uncertain, step-like signal produced by a given signal generator. This application was sketched out at the end of [6]. The step-like-signal generator is modelled as an ideal step generator in series with a network having frequency response G(f). The output of the step-like-signal generator is carefully measured, and the systematic measurement uncertainties are estimated. By differentiation and Discrete Fourier Transform (DFT), and use of the envelope-modulation method, an estimate of G(f) with systematic uncertainties is produced. Then the generator is connected to the input of the DUT whose frequency response, H(f), we wish to know (Fig. 2a). The DUT might be, for example, a filter, amplifier, attenuator, oscilloscope, or an ADC. The response of the DUT to the step-like input is carefully measured, and the systematic measurement uncertainties are again estimated. By differentiation, DFT, and using the envelope-modulation method, an estimate of an equivalent frequency response ($G(f)\cdot H(f)$) with systematic uncertainties is produced. Using that result, and the estimate of G(f), an estimate of H(f) is reconstructed (i.e, deconvolved [7], from the time-domain perspective). The uncertainties are processed appropriately, and the result is an estimate of the DUT frequency response H(f) with systematic magnitude and phase uncertainties (Fig. 2b).

II. BACKGROUND

A step-like-signal generator can be modeled as an ideal step generator in series with a linear network, as shown in Fig. 1a. The network has frequency response G(f). The network represents the physical realization of the generator, and is a useful model only for certain purposes: while its output can always be represented as the result of such a combination, it is unlikely that the signal generator is in fact a linear system.

The output of the step-like-signal generator, $s_g(t)$, is carefully measured using measurement system A to be $s_{gm}[n] = s_{gm}(nT)$, where T is the sampling period and there are M points in the data record (Fig. 1a). T and M are set according to the allowable aliasing error[1,4], the required resolution of the frequency response estimate (Δf=1/MT), and the requirement that the step-like signal be virtually completely settled at the beginning and end of the recorded time epoch, to avoid spectral leakage errors [5]; this is discussed further in Section III.

The systematic uncertainties of $s_{gm}[n]$, $u_{sgm+}[n]$ and $u_{sgm-}[n]$, are assumed to be determined correctly (Fig. 1b), such that they enclose the measurement error, $e_{sg}[n]$, with acceptable confidence but are not too large

$$s_{gm}[n] + u_{sgm-}[n] \le s_g(nT) \le s_{gm}[n] + u_{sgm+}[n], \tag{1}$$

$$e_{sg}[n] = s_{gm}[n] - s_g(nT), \tag{2}$$

$$u_{sgm-}[n] \le -e_{sg}[n] \le u_{sgm+}[n], \tag{3}$$

where $u_{sgm-}[n] \le 0$ and $u_{sgm+}[n] \ge 0$. Note the negative sign in front of $e_{sg}[n]$ in (3) due to the way it is defined in (2). Determining $u_{sgm+}[n]$ and $u_{sgm-}[n]$ properly is a significant problem (e.g., [8]) that we do not venture to solve here. To minimize the uncertainties, it is assumed that $s_{gm}[n]$ has been corrected as much as possible for any known linear, nonlinear, impedance, or timebase errors (e.g., [7,9-11]) of measurement system A. These corrections are important, because small time-domain uncertainties can produce significant frequency-domain uncertainty, as shown in [6]. The uncertainties $u_{sgm+}[n]$ and $u_{sgm-}[n]$ then enclose any remaining systematic errors of the measurement system, resulting for example from imperfect correction for its impulse response, or uncorrectible nonlinearity. Uncertainties due to aliasing, derivative estimation, and random error sources should be handled separately, by stochastic or other means [1,4,6].

The signal $s_{gm}[n]$ is differentiated in discrete time[1,4,12], yielding $g_m[n]$. The DFT of $g_m[n]$, normalized by T, is the estimate of G(f):

$$G_m(f_k) = T\sum_{n=0}^{M-1} g_m[n]\cdot\exp(-j2\pi kn/M), \quad \text{for } k = 0,\pm1,\pm2,\ldots,\pm(M/2). \tag{4}$$

Note that $G_m(f_k)$ is only available at discrete frequencies f_k:

$$f_k = \frac{k}{MT}. \tag{5}$$

The systematic magnitude and phase uncertainties of $G_m(f_k)$, $U_{\|Gm}(f_k)$ and $U_{\phi Gm}(f_k)$, respectively, are produced from $u_{sgm+}[n]$ and $u_{sgm-}[n]$ by the envelope-modulation method. Note that $U_{\|Gm}(f_k)$ and $U_{\phi Gm}(f_k)$ are both symmetric uncertainties: the negative uncertainty for each is simply the negative of the corresponding positive uncertainty.

In the next section, we extend these ideas, which were presented in [6], in order to estimate the frequency response of a DUT, using the non-ideal but available step-like signal $s_g(t)$ as the test input. Note that in order to produce a valid estimate of the DUT frequency response, the step-like-signal generator should have significantly higher performance (i.e., at least four times higher bandwidth, shorter risetime, and shorter settling time) than that of the DUT.

III. DUT FREQUENCY RESPONSE ESTIMATION

To estimate the DUT frequency response, H(f), the output of the step-like signal generator is connected to the input of the DUT (Fig. 2a). The resulting response of the DUT, $s_y(t)$, is measured by measurement system B. After as much correction as possible, it is estimated to be $s_{ym}[n]$, with systematic uncertainties $u_{sym+}[n]$ and $u_{sym-}[n]$. As with $s_{gm}[n]$, uncertainties due to aliasing, differentiation errors, and noise are to be handled separately. If measurement system B is a part of the DUT, for instance if the DUT is a digitizer, then determination of $u_{sym+}[n]$ and $u_{sym-}[n]$ is problematic; a linear DUT response, $s_y(t)$, may not exist, so that uncertainties of measuring it cannot be defined. In such a case, we recommend setting $u_{sym-}[n] = u_{sym+}[n] = 0$, and to note having done that with the results.

For mathematical simplicity in later calculations, it is best if $s_y(t)$ is measured using the same number of points M and sampling period T that are used to measure $s_g(t)$, thus assuring that $Y_m(f)$ and $G_m(f)$ are known at the same set of discrete frequencies, f_k. However, that requires that M and T are set such that $s_g(t)$ and $s_y(t)$ are both sampled with acceptably low aliasing[1,4], that the required resolution of the final frequency response estimate (Δf= 1/MT) is achieved, and that both $s_g(t)$ and $s_y(t)$ are virtually completely settled at both the beginning and end of the data records, to avoid spectral leakage errors [5]. Therefore, it may be necessary to sample $s_g(t)$ and $s_y(t)$ with different sampling periods and record lengths, and then use interpolation and decimation to assure that $H_m(f)$ and $Y_m(f)$ are determined at the same set of discrete frequencies, f_k. For simplicity, we assume here that $s_{gm}[n]$ and $s_{ym}[n]$ have the same values of M and T.

The response of the DUT, $s_{ym}[n]$, is differentiated, Fourier-transformed, and normalized by T, yielding $Y_m(f_k)$:

$$Y_m(f_k) = T\sum_{0}^{M-1} y_m[n]\cdot\exp(-j2\pi kn/M), \qquad (6)$$

where $y_m[n]$ is the discrete-time differentiation of $s_{ym}[n]$. Again using the envelope-modulation method, the systematic magnitude and phase uncertainties of $Y_m(f_k)$ are determined: $U_{\|Ym}(f_k)$ and $U_{\phi Ym}(f_k)$, respectively.

$G_m(f_k)$ can then be extracted from $Y_m(f_k)$ using deconvolution methods, to yield the estimate of H(f)

$$H_m(f_k) = \frac{Y_m(f_k)\,R(f_k)}{G_m(f_k)}, \qquad (7)$$

where $R(f_k)$ is a regularization operator that is determined according to the nature of $G_m(f_k)$ and $Y_m(f_k)$ [7,13]. $R(f_k)$ is essentially unity within the passband of $G_m(f_k)$, which should include the frequencies of interest in determining $H_m(f_k)$, if the criteria in Section II are followed regarding selection of the step-like signal generator.

IV. FREQUENCY RESPONSE UNCERTAINTIES

The systematic magnitude and phase uncertainties of $H_m(f_k)$, which are $U_{\|Hm}(f_k)$ and $U_{\phi Hm}(f_k)$, respectively, are determined by processing $U_{\|Gm}(f_k)$, $U_{\phi Gm}(f_k)$, $U_{\|Ym}(f_k)$, and $U_{\phi Ym}(f_k)$, in correspondence with the deconvolution calculation in (7). Regarding magnitude uncertainties, if we try to find the uncertainty on $|H_m(f_k)|$ by including magnitude uncertainties in (7), we get an expression like

$$|H_m(f_k)| + U_{\|Hm}(f_k) \approx \frac{|Y_m(f_k)| + U_{\|Ym}(f_k)}{|G_m(f_k)| - U_{\|Gm}(f_k)}\,|R(f_k)|. \qquad (8)$$

If we constrain our interest to the area well within the passband of $G_m(f_k)$, where $|G_m(f_k)| >> U_{\|Gm}(f_k)$, we can use a Taylor series approximation for the fraction in the righthand side of (8). Combining the uncertainties $U_{\|Gm}(f_k)$ and $U_{\|Ym}(f_k)$ via root-sum-of-squares yields

$$U_{\|Hm}(f_k) = |H_m(f_k)|\left(\left(\frac{U_{\|Ym}(f_k)}{|Y_m(f_k)|}\right)^2 + \left(\frac{U_{\|Gm}(f_k)}{|G_m(f_k)|}\right)^2\right)^{(1/2)}. \qquad (9)$$

Note that where the assumption $|G_m(f_k)| >> U_{\|Gm}(f_k)$ is not valid, determining $U_{\|Hm}(f_k)$ is more complicated.

If phase uncertainties are similarly included in (7), calculations with phasors yield the systematic phase uncertainty for $H_m(f_k)$

$$U_{\phi Hm}(f_k) = \{[U_{\phi Ym}(f_k)]^2 + [U_{\phi Gm}(f_k)]^2\}^{(1/2)}. \qquad (10)$$

As implied earlier, in the case where the DUT is a digitizer or oscilloscope, the terms $U_{\|Ym}(f_k)$ and $U_{\phi Ym}(f_k)$ are neglected in the equations above.

To determine the total uncertainties of $H_m(f_k)$, the systematic uncertainties $U_{\|Gm}(f_k)$, $U_{\phi Gm}(f_k)$, $U_{\|Ym}(f_k)$, and $U_{\phi Ym}(f_k)$ can be augmented, prior to the calculations (9) and (10), with the other significant frequency-domain uncertainties of estimating G(f) and Y(f). These other uncertainties may be due to aliasing, derivative estimation errors, and noise, and should have been determined separately as suggested above. The uncertainties can be combined by methods such as in [14].

V. DISCUSSION

We have proposed a method of determining the systematic magnitude and phase uncertainties, $U_{\|Hm}(f_k)$ and $U_{\phi Hm}(f_k)$, respectively, of a DUT frequency response estimate, $H_m(f_k)$, that is calculated from the response of the DUT to an uncertain, step-like input signal. This method may be applied to the estimation of the frequency responses of DUTs such as filters, amplifiers, attenuators, digitizers, oscilloscopes, and ADCs. It is worth reiterating that in the case of the latter three types of DUTs, where measurement system B is a part of the DUT, determining the uncertainties $u_{sym+}[n]$ and $u_{sym-}[n]$ may be problematic, in that an actual linear response $s_y(t)$ may not exist. In that case, we recommend setting $u_{sym-}[n] = u_{sym+}[n] = 0$, and noting that with the test results ($H_m(f_k)$, $U_{\|Hm}(f_k)$, and $U_{\phi Hm}(f_k)$).

A final note is that the concepts presented here are extensible to impulse-like signals [15,16].

APPENDIX

This appendix contains the mathematical proof of the part of the envelope-modulation method that was empirically derived in [6]. Assume we have the situation shown in Fig. 1a, with a perfect step signal that is the input to the network G(f); the output is best estimated to be $s_{gm}[n]$. The negative of the measurement error, $-e_{sg}[n]$, defined in (2), is enclosed by properly determined uncertainties $u_{sgm+}[n]$ and $u_{sgm-}[n]$ (Fig. 1b). The point-by-point uncertainty average, $u_{av}[n]$, and the halved difference, $u_{hd}[n]$, are

$$u_{av}[n] = (u_{sgm+}[n] + u_{sgm-}[n])/2, \qquad (A.1)$$

$$u_{hd}[n] = (u_{sgm+}[n] - u_{sgm-}[n])/2. \qquad (A.2)$$

The unproved, empirically-derived assumption of the envelope-modulation method is[6]: the magnitude of the Fourier transform of $e_{sg}[n]$ is always less than or equal to the sum of $|U_{hd}[0]|$, the magnitude of the DC component of the Fourier transform of $u_{hd}[n]$, plus the magnitude of the Fourier transform of $u_{av}[n]$

$$|E_{sg}[k]| \le |U_{hd}[0]| + |U_{av}[k]|. \tag{A.3}$$

This assumption is proved below.

Note that $u_{av}[n]$ marks the centerline of the uncertainty envelope, while $u_{hd}[n]$ is always nonnegative and is half of the width of the envelope at sample n. Let r[n] be the difference between the negative of the error, $-e_{sg}[n]$, and $u_{av}[n]$

$$r[n] = -e_{sg}[n] - u_{av}[n]. \tag{A.4}$$

Using (A.4) in (3), and then (A.1) and (A.2), yields

$$|r[n]| \le u_{hd}[n]. \tag{A.5}$$

The magnitude of the DFT of the error is

$$\begin{aligned}|E_{sg}[k]| &= \left|\sum_{n=0}^{M-1} (e_{sg}[n])\cdot\exp(-j2\pi kn/M)\right| \\ &= \left|\sum_{n=0}^{M-1} (-r[n]-u_{av}[n])\cdot\exp(-j2\pi kn/M)\right|.\end{aligned} \tag{A.6}$$

Using the Triangle Inequality with (A.6), we see that

$$\begin{aligned}|E_{sg}[k]| \le &\left|\sum_{n=0}^{M-1} r[n]\cdot\exp(-j2\pi kn/M)\right| \\ &+\left|\sum_{n=0}^{M-1} u_{av}[n]\cdot\exp(-j2\pi kn/M)\right|.\end{aligned} \tag{A.7}$$

It can also be shown using the Triangle Inequality that

$$\begin{aligned}&\left|\sum_{n=0}^{M-1} r[n]\cdot\exp(-j2\pi kn/M)\right| \\ &\le \sum_{n=0}^{M-1} |r[n]\cdot\exp(-j2\pi kn/M)| = \sum_{n=0}^{M-1} |r[n]|\end{aligned} \tag{A.8}$$

and using (A.5) gives

$$\sum_{n=0}^{M-1} |r[n]| \le \sum_{n=0}^{M-1} u_{hd}[n] = U_{hd}[0]. \tag{A.9}$$

Combining (A.7), (A.8), and (A.9) proves (A.3)

$$|E_{sg}[k]| \le |U_{hd}[0]| + |U_{av}[k]|. \tag{A.10}$$

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